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Computation of Stability Margins for Uncertain Linear Fractional-order Systems using Interval Constraint Propagation

P. S. V. Nataraj * **Rambabu Kalla** ** **Manoj M. Deshpande** ***

* *Systems and Control Engineering Group, IIT Bombay, India (Tel: +91-22-25767887; e-mail: nataraj@sc.iitb.ac.in).*
** *Systems and Control Engineering Group, IIT Bombay, India (Tel: +91-22-25764884; e-mail: ram@sc.iitb.ac.in)*
*** *Systems and Control Engineering Group, IIT Bombay, India (Tel: +91-22-25764885; e-mail: mmdeshpande@sc.iitb.ac.in)*

Abstract: The present paper proposes an algorithm for finding the stability margins and cross over frequencies for an uncertain fractional-order system using interval constraint propagation technique. It is first shown that the problem of finding the stability margins and crossover frequencies can be formulated as a interval constraint satisfaction problem and then solved using branch and prune algorithm. The algorithm guarantees that the stability margins and the crossover frequencies are computed to prescribed accuracy and that theses values are reliable in the face of all kinds of computational errors. The other advantage of the method is that the stability margins and crossover frequencies can be computed without the need of frequency response plots or any kind of approximations. Two examples of uncertain fractional-order systems are taken from the literature and their stability margins and crossover frequencies are computed using the proposed algorithm.

Keywords: Constraint satisfaction problems, Control system analysis, Fractional-order systems, Interval analysis, Parameter uncertainty, Reliable computation, Robust control.

1. INTRODUCTION

The gain and phase margins are popularly used as stability specifications in classical methods of analysis and synthesis of linear control systems. These specifications basically relate to the maximum allowable variation in the open loop gain or phase of the system to conserve closed loop stability. Theses margins may be extracted from the frequency response plots or computed directly.

For fractional order systems, available methods and tools (such as those available in Grace et al. (2002)) cannot be readily applied, owing to the non-rational nature of these transfer functions. Further for *uncertain* fractional-order systems, no methods are currently available in the literature. In this paper, we therefore present an algorithm to compute the stability margins and cross over frequencies for uncertain fractional-order systems using interval constraint propagation. The proposed method algorithm determines the stability margins directly without the need of the frequency response plots, nor does it involve any approximations.

The proposed algorithm is applicable to a larger class of uncertain fractional-order transfer functions, whose magnitude and phase functions are bounded and continuous in frequency. Subject to these assumptions, there is no restriction on the structure or form of the transfer function. Thus, the proposed algorithm is applicable to quite general nonlinear parametric dependencies, where the coefficients and powers of the fractional-order systems can be *nonlinear* functions of the uncertain parameters. The proposed algorithm has several key features - the computed margins and crossover frequencies are reliable and accurate (to a user specified accuracy), multiple margins and crossover frequencies can be detected, and no approximation of the fractional-order transfer function is required.

The rest of the paper is organised as follows: Section-2 deals with the background of fractional-order systems and interval arithmetic, and interval constraint processing. The methodology followed in proposed algorithm is presented in the Section-3. In Section-4, we give the proposed algorithm for the computation of stability margins and crossover frequencies. Section-5 demonstrates the proposed algorithm on two examples taken from the literature and Section-6 gives the conclusions.

2. BACKGROUND

2.1 Fractional-order Systems

A fractional-order transfer function is of the form

$$P(s) = \frac{b_m s^{\beta_m} + ... + b_1 s^{\beta_1} + b_0 s^{\beta_0}}{a_n s^{\alpha_m} + ... + a_1 s^{\alpha_1} + a_0 s^{\alpha_0}} \tag{1}$$

as given by Podlubny (1998), where α_i, β_j are positive *real* numbers and a_i, b_j, $i = 0, 1, ..., n$, $j = 0, 1, ..., m$ are arbitrary real coefficients. In the time domain, this transfer function can be represented by a *fractional-order* differential equation. Several researchers have shown that certain phenomena, especially those governed by non-integer order physical laws, can be described more accurately by fractional-order differential equations; for example transmission lines given in Wang (1987), electrical noises given in Mandelbrot (1967), and diffusion of heat into a semi-infinite solid given in Petras (1999), etc.

For an uncertain fractional-order system the coefficients and the powers in the transfer function need not be fixed real numbers but are *intervals* of real numbers.

2.2 Interval Arithmetic

The key idea behind interval arithmetic as given by Moore (1966) is the approximation of real numbers by intervals to quantify the errors introduced with finite precision arithmetic. In addition, interval computations provide an appropriate framework to deal with uncertain data.

A closed interval $\mathbf{x} = [\underline{x}, \overline{x}]$, with $\underline{x}, \overline{x} \in \mathcal{R}$ can be regarded as the set of real numbers $\{r | \underline{x} \leq r \leq \overline{x}\}$, or as an approximation of some real numbers lying within that set. Instead of using a single floating-point number to approximate a real, interval arithmetic encloses the real number within a closed interval having (in general) floating-point bounds. An interval vector $\mathbf{x} = (\mathbf{x}_1, ..., \mathbf{x}_n)^T$ with components $\mathbf{x}_k = [\underline{x}_k, \overline{x}_k]$ is called as a box $\mathbf{x}$. $I(\mathbf{x})$ is the set of all boxes contained in $\mathbf{x}$. The general results of interval arithmetic like natural inclusion function, hull, union, projection, width of an interval $w(\mathbf{x})$ etc., can be found in the book by Moore (1966).

2.3 Interval Constraint Processing

Numerical constraint satisfaction problems accept as input only problems represented by exact numerical values and correspondingly produce only crisp solutions as output. This limitation can be removed by implementing generalized constraint propagation schemes based on interval arithmetic instead of conventional arithmetic. By using intervals instead of exact values, we may express inexact numerical constraints in a well-defined way and compute necessary conditions for consistency in inconsistent situations.

An interval constraint satisfaction problem (ICSP) as given by Hyvonen (1992) is composed of

1. A set of real valued variables, e.g., $v = \{v_1, ..., v_n\}$;

2. A set of interval domains of the variables, e.g., $\mathbf{x} = \{\mathbf{x_1}, ...\mathbf{x_n}\}$;

3. A set of constraints, e.g., $c = \{c_1, ..., c_m\}$ over the given set of variables.

The problem is to find in the initial box $\mathbf{x_1} \times ... \times \mathbf{x_n}$ all the consistent values with respect to all constraints.

A variable $v_i \leftarrow \mathbf{x_i}$ is consistent if and only if each interpretation $v_i \leftarrow x \in \mathbf{x_i}$, can be satisfied with respect to all constraints by some extension:

$\forall x \in \mathbf{x_i} \; \exists \; \{v_1 \leftarrow x_1 \in \mathbf{x_1}, ..., v_i \leftarrow x, ..., v_m \leftarrow x_m \in \mathbf{x_m}\}$: $c_1, ..., c_m$ are satisfied.

The set of variables of the constraint c_i is denoted by $\mathbb{V}_{c_i}$

There are two steps in solving an ICSP, constraint propagation and constraint branching. The basic idea of constraint propagation algorithms (also called filtering or narrowing or consistency algorithms or narrowing operators) consists of removing, from the domains associated to the constraint variables, inconsistent values that can never be part of the solution. This process reduces significantly the search tree and possibly the computational effort to find a solution if one exists or to demonstrate that there is no solution. In general, the results are propagated through the whole constraint set, and the process is repeated until a stable set is obtained. Research in the area of solving interval constraint satisfaction problems (see, for example, Benhamou et al. (1999)) is devoted to finding correct and (near) optimal interval propagation techniques that can be efficiently implemented. A constraint narrowing algorithm transforms the domains of those variables involved in it into tighter intervals such that:

1. Resulting intervals are always included in the original ones (*contractance* property).

2. All values in the original intervals verifying the associated constraint of the narrowing operator, belong to the resulting intervals (*soundness* or *correctness*).

3. The subset interval relation is conserved by the transformation (*monotonicity*).

Well known examples of constraint narrowing operators are hull and box consistency (Benhamou et al. (1999)) and kBConsistency operators (Lhomme (1993)). In our problem, we make use of efficient implementation of hull consistency known as HC4 as the the narrowing operator. This is described below.

HC4 filter: The HC4 filter was proposed by Benhamou et al. (1999). Inputs to the HC4 filter are the constraint in user form (i.e. without decomposing it in several equations) and the set of interval domains (box). The algorithm efficiently computes an interval extension of the equation, narrowing intervals of the variables involved. Inside the HC4 filter the input equation is represented as an attribute tree where the root node is a p-ary relation symbol, and terms in the equation form sub-trees rooted at nodes containing either a variable, a constant or an operation symbol.

The HC4 filter works in two phases called forward evaluation and backward propagation. The forward phase is a tree traversal going from the leaves to the root, evaluating at each node the natural interval extension of that subterm of the constraint. The backward phase traverses the tree from the root to the leaves, projecting on each node the effects of interval narrowing already performed on its parent node. In the backward propagation phase, an interval may become empty. When this happens the constraint is inconsistent with respect to the initial domains.

Until now we have been dealing with the first step in solving the ICSP, i.e., constraint propagation. Constraint propagation algorithms alone are not sufficient for solving an ICSP, that is to say, they do not eliminate all the non-solution elements from the domains. As a consequence, it is necessary to employ some additional strategy to solve it. One complementary method is the so-called constraint branching that divides the variable domains to construct new sub-problems, i.e., branches in the search tree on which constraint propagation is reactivated. The process is also called as splitting or sub-division process.

In the present paper, we address the problem of finding the stability margins and crossover frequencies as an ICSP.

In the next section we describe the method used to finding the stability margins and crossover frequencies.

3. METHODOLOGY

Consider a linear fractional-order system represented by the transfer function $p(s,q)$, where s is the Laplace variable and q is the vector of system parameters. Let ω denote the frequency. Define the magnitude (in decibels, dB) and phase (in degrees) functions of $p(s,q)$ as

$$f_{mag}(\omega,q) = 20log_{10}|p(s=j\omega,q)|$$
$$f_{phase}(\omega,q) = \angle p(s=j\omega,q)$$
$$f(\omega,q) = 180 + f_{phase}(\omega,q)$$

Suppose that there is parametric uncertainty in the system such that the parameter vector q varies over a bounding box $\mathbf{q}^0 \in I(\mathcal{R}^l)$. The parametric uncertainty gives rise to an uncertain linear fractional-order system. Define the set of Ω_{pcf} of the phase cross over frequencies and the set $\mathcal{GM}$ of gain margins as

$$\Omega_{pcf} = \{\omega : f(\omega,q) = 0, q \in \mathbf{q}^0\}$$
$$\mathcal{GM} = \{-f_{mag}(\omega,q) : q \in \mathbf{q}^0, \omega \in \Omega_{pcf}\}$$

As we know that the gain margin is evaluated at the phase cross over frequency, i.e., it is function evaluation at the phase crossover frequency. So we can have the expression for phase crossover frequency as the only constraint and evaluate the gain margin at the computed phase crossover frequency. The other method is to pose expression for $\mathcal{GM}$ as one more constraint with initial search domain for $\mathcal{GM}$ as $(-\infty,+\infty)$, where $-\infty$ and $+\infty$ are the smallest and the largest machine representable numbers in IEEE standard for binary floating-point arithmetic. The later is adopted in the present work.

Further, define the set Ω_{gcf} of gain crossover frequencies and the set $\mathcal{PM}$ of phase margins as

$$\Omega_{gcf} = \{\omega : f_{mag}(\omega,q) = 0\ dB, q \in \mathbf{q}^0\}$$
$$\mathcal{PM} = \{f(\omega,q) : q \in \mathbf{q}^0\ \omega \in \Omega_{gcf}\}$$

As we know that the phase margin is evaluated at the gain cross over frequency, i.e., it is function evaluation at the gain crossover frequency. So we can have the expression for gain crossover frequency as the only constraint and evaluate the phase margin at the computed gain crossover frequency. The other method is to pose expression for $\mathcal{PM}$ as one more constraint with initial search domain for $\mathcal{PM}$ as $[-180,+180]$, The later is adopted in the present work.

The problem of finding the stability margins and crossover frequencies is formulated as an ICSP. There are actually two sub-problems :

1. Finding the Ω_{pcf} and $\mathcal{GM}$

2. Finding the Ω_{gcf} and $\mathcal{PM}$.

Each sub-problem is an ICSP in itself.

4. PROPOSED ALGORITHM

4.1 Gain margins and Phase Crossover frequencies

The algorithm for computation of gain margins and phase crossover frequencies is as follows:

Inputs: Expression for the magnitude function, phase function and phase crossover function, the parameter vector q, the initial uncertain interval vector $\mathbf{q}^0$, and accuracy tolerance parameter ϵ for the phase crossover frequency and gain margin.

Output: The set $\mathcal{GM}$ of all gain margins along with the set Ω_{pcf} of all the phase crossover frequencies, computed to the prescribed accuracy tolerance ϵ.

BEGIN Algorithm

Initialization part

(1) Form the variable set comprising $v = \{q, GM, \Omega\}$ for the ICSP.

(2) Construct the initial search domain (box) from $\mathbf{\Omega^0}$ that encloses the phase crossover frequency set Ω_{pcf}, $\mathbf{GM^0}$ that encloses gain margin set $\mathcal{GM}$ and $\mathbf{q^0}$. So the initial search domain or box $\mathbf{x^0} = (\mathbf{\Omega^0}, \mathbf{GM^0}, \mathbf{q^0})$

(3) From the expression for the magnitude and phase functions, construct the natural inclusion functions for the phase crossover frequency and gain margin which are the constraints c_1 and c_2 of the ICSP: $c \leftarrow \{c_1, c_2\}$

(4) Initialize the solution list $\mathcal{L}^{sol} \leftarrow \{\}$ and the working list $\mathcal{L} \leftarrow \mathbf{x}^0$

Iterative part

(5) **WHILE** $\mathcal{L} \neq \{\}$ **DO**

(6) Extract $\mathbf{x}$ from $\mathcal{L}$

(7) $s \leftarrow c$

(8) **WHILE** $s \neq \{\}$ **AND** $\mathbf{x} \neq \{\}$

(9) Extract c_i from s

(10) Narrow the box $\mathbf{x}$ using HC4 filter explained in the Section 2.3 to $\mathbf{x}^{'}$.

(11) **IF** $\mathbf{x} \neq \mathbf{x}^{'}$ **THEN**

(12) $s \leftarrow s \cup \{c_j \mid \exists v_k \in \mathbb{V}_{c_j} \wedge \mathbf{x_k} \neq \mathbf{x_k}^{'}\}$ (Add to the set s all the constraints containing the variables whose search domains are narrowed)

(13) $\mathbf{x} \leftarrow \mathbf{x}^{'}$

(14) **ELSE**

(15) $s \leftarrow s \setminus \{c_i\}$

(16) **ENDIF**

(17) **ENDWHILE**

(18) **IF** $\mathbf{x} \neq \{\}$ **THEN**

(19) **IF** $w(\mathbf{x}) \leq \epsilon$ **THEN**

(20) $\mathcal{L}^{sol} \leftarrow \mathcal{L}^{sol} \cup \mathbf{x}$

(21) $\mathcal{L} \leftarrow \mathcal{L} \setminus \{\mathbf{x}\}$

(22) **ELSE**

(23) Subdivide the box $\mathbf{x}$ along the variable whose width is the largest into $\mathbf{x}^1$ and $\mathbf{x}^2$.

(24) $\mathcal{L} \leftarrow \mathcal{L} \cup \mathbf{x}^1 \cup \mathbf{x}^2$

(25) **ENDIF**

(26) **ENDIF**

(27) **ENDWHILE**

Termination part

(28) Construct the sets

$$\Omega_{pcf} \leftarrow \bigcup_{\mathcal{L}^{sol}} \mathbf{\Omega}_{(i)}; \;\; \mathcal{GM} \leftarrow \bigcup_{\mathcal{L}^{sol}} \mathbf{GM}_{(i)}$$

Output the sets Ω_{pcf} and $\mathcal{GM}$ and EXIT.

END Algorithm

The list $\mathcal{L}^{sol}$ contains the set of the gain margins $\mathcal{GM}$ and the phase crossover frequencies Ω_{pcf} in the given search domain.

4.2 Theoretical properties

The properties of the algorithm concerning the reliability, accuracy, enclosure of the actual results, ability to locate multiple margins and crossover frequencies, etc., readily follow from the basics of interval analysis as given in Hansen and Walster (2005).

The maximum error in the computed crossover frequencies or margins cannot be more than the accuracy tolerance ϵ. The procedure ensures that we do not miss out any crossover frequency and helps in identifying multiple crossover frequencies in the given search domain.

4.3 Phase margins

To compute the set $\mathcal{PM}$ of phase margins and set Ω_{gcf} of gain crossover frequencies, we can use the above proposed algorithm with obvious changes. Then, all the results of the previous subsection carry over to this case.

5. ILLUSTRATIVE EXAMPLES

We demonstrate the proposed method on two examples. The examples are executed on an Intel(R) Core2 2.4GHz computer with 2GB of RAM, running Linux Fedora Core-7. The interval solver *RealPaver* developed by Granvilliers and Benhamou (2006) is used to solve both the examples.

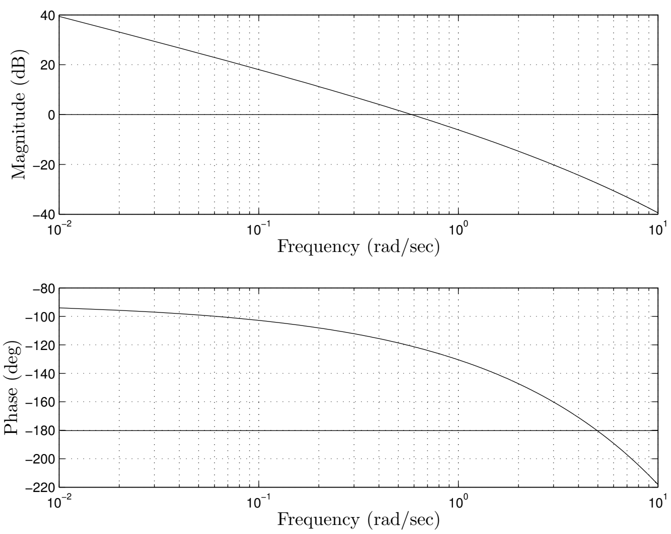

Fig. 1. Bode Plots for the non-inductive cable system in Example 3.1. From these plots, the margins and crossover frequencies are extracted to verify the results of the proposed algorithm

Example 1. Consider the fractional-order model of a non-inductive cable system described by Bonnet and Jonathan (2000)

$$p(s) = \frac{e^{-a\sqrt{s}}}{s}, \;\; a = 1$$

The range and domain accuracy tolerances are specified as 0.01.

The proposed algorithm finds the gain margin and phase margin as

$$\mathcal{GM} = [27.5000, 27.5000], \;\; \Omega_{pcf} = [4.9348, 4.9348]$$

in $0.25sec$, and the phase margin and gain crossover frequency as

$$\mathcal{PM} = [58.7500, 58.7500], \;\; \Omega_{gcf} = [0.5828, 0.5828]$$

in less than $1ms$.

The results of the proposed algorithm are compared with those obtained from Bode plots shown in Fig 1. By zooming into the Bode plots, we find the following:

$$\mathcal{GM} = 27.5, \;\; \Omega_{pcf} = 4.95$$
$$\mathcal{PM} = 58.75, \;\; \Omega_{gcf} = 0.58$$

which agree well with those of the proposed algorithm.

Example 2. Consider the fractional-order transfer function of the gas turbine model obtained through identification given in Deshpande (2006) is

$$p(s) = \frac{b_1}{a_1 s^{\alpha 1} + a_2 s^{\alpha 2} + 1}$$

where, the coefficients and powers of s vary over the following interval

$$b1 \in [103.9705, 110.9238]; \;\; a1 \in [0.00734, 0.0130]$$
$$a2 \in [0.1356, 0.1818]; \;\; \alpha 1 \in [1.6062, 1.6807]$$
$$\alpha 2 \in [0.7089, 0.8421]$$

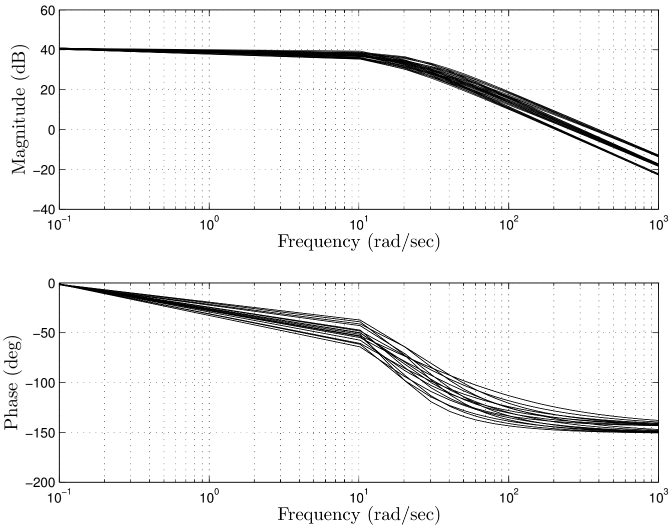

Fig. 2. Bode Plots for the uncertain fractional order gas turbine plant. From these plots, the results of the proposed algorithm can be verified.

The range and domain accuracy tolerances are specified as 0.01.

The proposed algorithm finds the gain margin and phase margin as

$$\mathcal{PM} = [32.59, 46.91],\ \Omega_{gcf} = [199.5262, 407.3803]$$

in 1024 seconds. As there are no phase crossover frequencies in this example, the proposed algorithm returns an empty solution set Ω_{pcf}.

The results are cross-checked with the Bode plots obtained using the gridding method based on 32 plants. The Bode plots are shown in Fig 2. By zooming into the Bode plots, we find that the values computed using the algorithm agree with that of the Bode plot.

6. CONCLUSIONS

The proposed algorithm for stability margins and crossover frequencies meets an important requirement in the area of uncertain fractional-order control systems. The proposed algorithm guarantees that the stability margins and crossover frequencies are computed to a prescribed accuracy, and that these values are reliable in face of all kinds of computational errors. The algorithm does not require any approximation of fractional-order terms.

The algorithm is demonstrated on two examples. Besides being accurate and reliable, the algorithm is also computationally efficient in that it is able to generate in a few minutes, results of required accuracy for up to five uncertain variables. The results are verified with the Bode plots of the both the example problems considered.

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