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H_∞ Tracking with Preview by Output Feedback for Linear Systems with Impulsive Effects

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Abstract: In this paper we study H_∞ tracking problems with preview by output feedback for linear systems with impulsive effects and the sampled-observation on the finite and infinite time interval. We consider the problems that the reference signals are previewed in a fixed time interval and known *a priori* over a whole time interval, and present feedback control laws for the H_∞ tracking problems. Our theory can be also applied into the sampled-control system with the control input realized through a zero-order hold and the sampled-observation.

1. INTRODUCTION

It is well known that, for the design of tracking control systems, the preview information of reference signals is very useful for improving the performance of the closed-loop systems, and recently much work has been done for preview control systems. Considering the effect of modelling uncertainties or disturbance is also very important on preview control theory. U. Shaked et al. have studied the H_∞ tracking theory with preview for continuous- and discrete-time systems by the game theoretic approach ([1][2][3][4][8][9]).

Control theory for linear systems with impulsive effects (or linear jump systems), which contain linear continuous and discrete time systems, can be widely applied, for example, to mechanical systems, ecosystems, chemical processes, finantial engineering and so on. It has been reseached in detail by A. Ichikawa and H. Katayama ([6]). Their theory can be also applied into the sampled-data control systems with the control input realized through a zero-order hold and the sampled-observation ([5]).

In this paper we study the H_∞ tracking problems with preview by output feedback for linear systems with impulsive effects (or linear jump systems). Our systems are described by the ordinary differential equations with impulsive effects and the sampled-observation. We consider two different tracking problems according to the preview lengths and give the control strategies for them respectively. Our theory can be applied into the control systems with the control input realized through a zero-order hold and the sampled-observation. Our theory can be also easily reduced to the case that only the preview information of discrete reference signals are available.

2. PROBLEM FORMULATION

Consider the following linear system with impulsive effects.

$$\begin{aligned}
&\dot{x}(t) = A(t)x(t) + B_1(t)w(t) + B_3(t)r_c(t),\ t \neq k\tau, \\
&\qquad x(0) = x_0 \\
&x(k\tau^+) = A_d(k)x(k\tau) + B_{2d}(k)u(k) + B_{3d}(k)r_d(k) \\
&z_c(t) = C_1(t)x(t) + D_{13}(t)r_c(t),\ t \neq k\tau \qquad (1)\\
&z_d(k) = C_{1d}(k)x(k\tau) + D_{12d}(k)u(k) + D_{13d}(k)r_d(k) \\
&y(k) = C_2(k)x(k\tau) + D_{21}(k)w_d(k) + v(k)
\end{aligned}$$

where $x \in \mathbf{R^n}$ is the state, $w \in \mathbf{R^p}$ and $w_d \in \mathbf{R^{p_d}}$ are the exogenous disturbances, $v \in \mathbf{R^k}$ is the measurement noise, $u \in \mathbf{R^m}$ is the control input, $y \in \mathbf{R^k}$ is the measured output, $z_c \in \mathbf{R^{k_c}}$ and $z_d \in \mathbf{R^{k_d}}$ are the controlled outputs, $r_c(t) \in \mathbf{R^{r_c}}$ and $r_d(k) \in \mathbf{R^{r_d}}$ are known or mesurable reference signals, x_0 is an unknown initial state. We assume that all matrices are of compatible dimensions. Throughout this paper the dependence of the system matrices on t or k will be omitted for the sake of notation simplification.

The H_∞ tracking problems we address in this paper for the system (1) are to design control law $u(\cdot){\in}l_2[0,N]$ over the finite horizon $[0,T]$, $N\tau < T < (N+1)\tau$ using the information available on the known parts of the reference signals $r_c(t)$ and $r_d(k)$ and minimizing the sum of the energy of $z_c(t)$ and $z_d(k)$, for the worst case of the initial condition x_0, the disturbances $w(t){\in}\tilde{L}_2([0,T];\mathbf{R^p})$ and $w_d(k){\in}\tilde{l}_2([0,N];\mathbf{R^{p_d}})$. We denote by $\tilde{L}_2([0,T];\mathbf{R^k})$ and $\tilde{l}_2([0,N];\mathbf{R^{k_d}})$ the space of nonanticipative signals. Considering the average of the performance indices over the statistics of the unknown parts of r_c and r_d, we define the following two performance indices.

$$\begin{aligned}
&J_T(x_0,u,w,r_c,r_d) = -\gamma^2 x_0' R^{-1} x_0 \\
&\quad + \left\{ \sum_{k=0}^{N-1} \int_{k\tau^+}^{(k+1)\tau} + \int_{N\tau^+}^{T} \right\} \mathbf{E}_{\bar{R}_s}\{\|z_c(s)\|^2\} ds \\
&\qquad + \sum_{k=0}^{N} \mathbf{E}_{\bar{R}_k}\{\|z_d(k)\|^2\} - \gamma^2 \|w\|_2^2 \qquad (2)
\end{aligned}$$

and

$$J_{vT}(x_0,u,w,w_d,v,r_c,r_d)$$

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$$:= J_T(x_0, u, w, r_c, r_d) - \gamma^2 \left[\|w_d\|_2^2 + \|v\|_2^2\right] \tag{3}$$

where $N\tau < T < (N+1)\tau$, $R = R' > O$ is a given weighting matrix for the initial state, $\mathbf{E}_{\bar{R}_s}$ and $\mathbf{E}_{\bar{R}_k}$ mean expectations over $\bar{R}_{s+h\tau}$ and $\bar{R}_{k+h}$, h is the preview length of $r_c(t)$ and $r_d(k)$, and $\bar{R}_s$ and $\bar{R}_j$ denote the future information on r_c and r_d at time s and $j\tau$ respectively, i.e., $\bar{R}_s := \{r_c(l); s < l \le T\}$ and $\bar{R}_j := \{r_i; j < i \le N\}$.

We consider two different tracking problems according to the information structures (preview lengths) of r_c and r_d as follows.

Case a) $\mathbf{H_\infty}$ Fixed-Preview Tracking:
In this case, it is assumed that at the current time t ($k\tau^+ \le t \le (k+1)\tau$), $r_c(s)$ is known for $s \le \min(T, s+h\tau)$ and at the time $k\tau$, $r_d(i)$ is known for $i \le \min(N, k+h)$, where h is the preview length.

Case b) $\mathbf{H_\infty}$ Tracking of Noncausal $\{r_c(t)$ and $r_d(k)\}$:
In this case, the signals $\{r_c(t)\}$ and $\{r_d(k)\}$ are assumed to be known *a priori* for the whole time intervals $t \in [0^+, T]$ and $k \in [0, N]$.

In order to solve these problems, we formulate the following differential game problems for the system (1), the performance indices (2) and (3).

The $\mathbf{H_\infty}$ Tracking Problem by State Feedback:
Find $\{u^*\}$, $\{w^*\}$ and x_0^* satisfying the following (saddle point) condition:

$$\begin{aligned} &J_T(x_0, u^*, w, r_c, r_d) \\ &\quad \le J_T(x_0^*, u^*, w^*, r_c, r_d) \\ &\qquad \le J_T(x_0^*, u, w^*, r_c, r_d) \end{aligned}$$

where the control strategies $u^*(k)$, $0 \le k \le N$, are based on the current state $x(k)$ and the information $R_{s+h\tau} := \{r_c(l); 0 < l \le s+h\tau\}$ and $R_{k+h} := \{r_d(i); 0 < i \le k+h\}$ $(0 \le h \le N)$.

The $\mathbf{H_\infty}$ Tracking Problem by Output Feedback:
Find $\{u^*\}$, $\{w^*\}$, $\{w_d^*\}$, $\{v^*\}$ and x_0^* satisfying the following (saddle point) condition:

$$\begin{aligned} &J_{vT}(x_0, u^*, w, w_d, v, r_c, r_d) \\ &\quad \le J_{vT}(x_0^*, u^*, w^*, w_d^*, v^*, r_c, r_d) \\ &\qquad \le J_{vT}(x_0^*, u, w^*, w_d^*, v^*, r_c, r_d) \end{aligned}$$

where the control strategies $u^*(k)$, $0 \le k \le N$, are based on the observable output $y(k)$ and the information $R_{s+h\tau}$ and R_{k+h} with $0 \le h \le N$.

3. H$_\infty$ TRACKING CONTROLLERS BY STATE FEEDBACK

In this section we present the theory of H$_\infty$ tracking with preview by state feedback. We consider the system (1) and assume the following standard conditions.

A1: $D_{12d}'D_{12d} > O,\ D_{12d}'C_{1d} = O,\ D_{12d}'D_{13d} = O$

Now we consider the following Riccati equation with jump parts.

$$\begin{aligned} &\dot{X} + A'X + XA + C_1'C_1 \\ &\quad + \frac{1}{\gamma^2} X B_1 B_1' X = O,\ t \ne k\tau \end{aligned} \tag{4}$$

$$\begin{aligned} &X(k\tau^-) - [A_d'X(k\tau)A_d + C_{1d}'C_{1d} - A_d'X(k\tau)B_d \\ &\quad \times T_2^{-1}(k)B_d'X(k\tau)A_d] = O,\ k = 0, 1, \cdots \end{aligned} \tag{5}$$

where $T_2(k) = D_{12d}'D_{12d} + B_d'X(k\tau)B_d$.

We obtain the following saddle point strategy for our game problem.

Proposition 3.1. ([7]) Consider the system (1) and the performance index (2), and suppose **A1**. Then **the $\mathbf{H_\infty}$ Tracking Problem** is solvable **by State Feedback** if and only if there exists a matrix $X(t) > O$ satisfying the conditions $X(0^-) < \gamma^2 R^{-1}$ and $X(T) = O$ such that the Riccati equation (4)(5) holds over $[0, T]$. A saddle point strategy is given by

$$\begin{aligned} x_0^* &= [\gamma^2 R^{-1} - X(0^-)]^{-1}\theta(0) \\ w^* &= \frac{1}{\gamma^2} B_1'(Xx + \theta) \\ u^*(k) &= -T_2^{-1}(k)B_d' \\ &\quad \times [X(k\tau)(A_d x(k\tau) + B_{3d} r_d(k)) + \theta_c(k\tau^+)]. \end{aligned}$$

$\theta(t)$, $t \in [0, T]$, satisfies

$$\begin{cases} \dot{\theta}(t) = -\bar{A}'(t)\theta(t) + \bar{B}(t) r_c(t),\ t \ne k\tau \\ \theta(k\tau) = \bar{A}_d'(k)\theta(k\tau^+) + \bar{B}_d(k) r_d(k) \\ \theta(T) = 0 \end{cases} \tag{6}$$

where

$$\begin{aligned} \bar{A} &= A + \frac{1}{\gamma^2} B_1 B_1' X, \\ \bar{B} &= -(XB_3 + C_1'D_{13}), \\ \bar{A}_d(k) &= [I_{\mathbf{n}} - B_d T_2^{-1}(k) B_d' X(k\tau)] A_d, \\ \bar{B}_d(k) &= A_d' X(k\tau)[I_{\mathbf{n}} - B_d T_2^{-1}(k) B_d' X(k\tau)] B_{3d} \\ &\qquad + C_{1d}' D_{13d}, \end{aligned}$$

and $\theta_c(t)$ is the 'causal' part of $\theta(\cdot)$ at time t. This θ_c is the expected value of θ over $\bar{R}_s$ and $\bar{R}_k$ and given by

$$\begin{cases} \dot{\theta}_c(s) = -\bar{A}'(s)\theta_c(s) + \bar{B}(s) r_c(s),\ s \ne k\tau \\ \qquad t \le s \le t_f \\ \theta_c(j\tau) = \bar{A}_d'(j)\theta_c(j\tau^+) + \bar{B}_d(j) r_d(j), \\ \qquad k < j \le k_f\ (k_f\tau < t_f < (k_f+1)\tau) \\ \theta_c(t_f) = 0 \end{cases} \tag{7}$$

where, for $k\tau^+ \le t \le (k+1)\tau$,

$$\begin{cases} t_f = t + h\tau \text{ and } k_f = k + h \text{ if } (k+h)\tau < T \\ t_f = T \text{ and } k_f = N \text{ if } (k+h)\tau \ge T. \end{cases}$$

Moreover, the value of the game is

$$\begin{aligned} &J_T(x_0^*, u^*, w^*, r_c, r_d) \\ &\quad = \sum_{k=0}^{N} \mathbf{E}_{\bar{R}_k}\{\|T_2^{-1}(k) B_d' \theta_1(k\tau^+)\|_{T_2(k)}^2\} \\ &\qquad + \bar{J}_c(r_c) + \bar{J}_d(r_d) \end{aligned} \tag{8}$$

where $\theta_1(t) = \theta(t) - \theta_c(t),\ t \in [0, T]$,

$$\begin{aligned} \bar{J}_c(r_c) = &\left\{ \sum_{k=0}^{N-1} \int_{k\tau^+}^{(k+1)\tau} + \int_{N\tau^+}^{T} \right\} \mathbf{R}_{\bar{R}_s}\{\|D_3 r_c\|^2 \\ &\quad + \gamma^{-2}\|B_1'\theta\|^2 + 2\theta' B_3 r_c\} ds, \end{aligned} \tag{9}$$

$$
\begin{aligned}
\bar{J}_d(r_d) &= \gamma^2 \mathbf{E}_{\bar{R}_0}\{\|\theta(0^-)\|^2_{P_0}\} \\
&+\sum_{k=0}^{N} \mathbf{E}_{\bar{R}_k}\{-\|T_2^{-1}(k)B_d'\theta(k\tau^+)\|^2_{T_2(k)} + \|D_{13d}r_d(k)\|^2 \\
&+r_d'(k)B_{3d}'X(k\tau)[I_{\mathbf{n}} - B_dT_2^{-1}(k)B_d'X(k\tau)]B_{3d}r_d(k) \\
&\quad +2\theta'(k\tau^+)[I_{\mathbf{n}} - T_2^{-1}(k)B_d'X(k\tau)]B_{3d}r_d(k)\} \qquad (10)
\end{aligned}
$$

and $P_0 = [R^{-1} - \gamma^{-2}X(0^-)]^{-1}$.

4. H_∞ TRACKING WITH PREVIEW BY OUTPUT FEEDBACK

In this section, utilizing the result in the previous section, we present the solution of the H_∞ tracking problems by output feedback for the system (1) and the design method of output feedback controllers for these problems. For the system (1), we assume the following standard condition in addition to **A1**.

A2: $B_d'D_{21} = O$

Introducing

$$\bar{u}(k) = u(k) + T_2^{-1}(k)B_d'[X(k\tau)B_{3d}r_d(k) + \theta(k\tau^+)]$$

and

$$\bar{w} = w - \gamma^{-2}B_1'(Xx + \theta),$$

and using the Riccati equation (4) with the jump parts (5) and the terminal condition $X(T) = O$, the performance index J_{vT} can be rewrite as

$$
\begin{aligned}
&J_{vT}(x_0, \bar{u}, \bar{w}, w_d, v, r_c, r_d) \\
&= \left\{ \sum_{k=0}^{N-1} \int_{k\tau^+}^{(k+1)\tau} + \int_{N\tau^+}^{T} \right\} \mathbf{E}_{\bar{R}_s}\{-\gamma^2\|\bar{w}\|^2\}ds \\
&+\sum_{k=0}^{N} \mathbf{E}_{\bar{R}_k}\{\|\bar{u}(k) + T_2^{-1}(k)B_d'X(k\tau)A_dx(k\tau)\|^2_{T_2(k)}\} \\
&\quad +\bar{J}_c(r_c) + \bar{J}_d(r_d) - \gamma^2\left[\|w_d\|_2^2 + \|v\|_2^2\right] \\
&\qquad -\gamma^2\mathbf{E}_{\bar{R}_0}\{\|x_0 - \hat{x}_0\|^2_{P_0^{-1}}\} \qquad (11)
\end{aligned}
$$

where $P_0 = [R^{-1} - \gamma^{-2}X(0^-)]^{-1}$ and $\hat{x}_0 = \gamma^{-2}P_0\theta(0)$. Therefore, our output feedback problem can be reduced the problem to find the maximizing x_0, $\bar{w}$, w_d and v, and the minimizing $\bar{u}(k)$ for the performance index (11). Using the above $\bar{u}(k)$ and $\bar{w}$, we can rewrite the system dynamics as follows.

$$
\begin{aligned}
\dot{x} &= \bar{\bar{A}}x + B_1\bar{w} + \bar{r}_c,\ t \neq k\tau,\ x(0) = x_0 \\
x(k\tau^+) &= A_dx(k\tau) + B_d\bar{u}(k) + \bar{r}_d
\end{aligned}
$$

where $\bar{\bar{A}} = A + \frac{1}{\gamma^2}B_1B_1'X$, $\bar{r}_c = B_3r_c + \frac{1}{\gamma^2}B_1B_1'\theta$ and $\bar{r}_d = B_{3d}r_d(k) - B_dT_2^{-1}(k)B_d'[X(k\tau)B_{3d}r_d(k) + \theta(k\tau^+)]$. For this system, we consider the following type of controller.

$$
\begin{aligned}
\dot{\hat{x}} &= \bar{\bar{A}}\hat{x} + \bar{r}_c,\ t \neq k\tau,\ \hat{x}(0) = \hat{x}_0 \\
\hat{x}(k\tau^+) &= A_d\hat{x}(k\tau) + B_d\bar{u}_*(k) + \bar{r}_d \qquad (12) \\
&\quad +L_1[y(k) - C_2\hat{x}],\ \hat{x}(0) = x_0^* \\
\bar{u}_*(k) &= -T_2^{-1}(k)B_d'X(k\tau)A_d\hat{x}(k\tau) \\
&\quad +L_2[y(k) - C_2\hat{x}(k\tau)]
\end{aligned}
$$

where L_1 and L_2 are the controller gains to decide later, using the solutions of the Riccati equations.

Let $e := x - \hat{x}$, we get the error system

$$
\begin{aligned}
\dot{e} &= \bar{\bar{A}}e + B_1\bar{w},\ t \neq k\tau,\ e(0) = x_0 - \hat{x}_0 \\
e(k\tau^+) &= (A_d - L_1C_2)e(k\tau) \qquad (13) \\
&\quad +[-L_1D_{21}\ \ -L_1]\begin{bmatrix} w_d \\ v \end{bmatrix}.
\end{aligned}
$$

Now we define the estimated output as follows.

$$
\begin{aligned}
f(k) &:= T_2^{\frac{1}{2}}(k)\{\bar{u}_*(k) + T_2^{-1}(k)B_d'X(k\tau)A_dx(k\tau)\} \\
&= T_2^{\frac{1}{2}}(k)\{(T_2^{-1}(k)B_d'X(k\tau)A_d \\
&\quad +L_2C_2)e(k\tau) + [L_2D_{21}\ L_2]\begin{bmatrix} w_d \\ v \end{bmatrix}\}
\end{aligned}
$$

For the error system (13) with the estimated output $f(k)$, we consider the problem to find x_0, $\bar{w}$, v and w_d maximizing the performance index

$$
\begin{aligned}
&J_{vT}(x_0, \bar{u}_*, \bar{w}, w_d, v, r_c, r_d) \\
&= \left\{ \sum_{k=0}^{N-1} \int_{k\tau^+}^{(k+1)\tau} + \int_{N\tau^+}^{T} \right\} \mathbf{E}_{\bar{R}_s}\{-\gamma^2\|\bar{w}\|^2\}ds \\
&\quad +\sum_{k=0}^{N} \mathbf{E}_{\bar{R}_k}\{\|f(k)\|^2\} - \gamma^2\mathbf{E}_{\bar{R}_0}\{\|x_0 - \hat{x}_0\|^2_{P_0^{-1}}\} \\
&\qquad +\bar{J}_c(r_c) + \bar{J}_d(r_d) - \gamma^2\left[\|w_d\|_2^2 + \|v\|_2^2\right]. \quad (14)
\end{aligned}
$$

Note that this problem is equivalent to the problem to give x_0, $\bar{w}$ and w_{de} maximizing

$$
\begin{aligned}
&\left\{ \sum_{k=0}^{N-1} \int_{k\tau^+}^{(k+1)\tau} + \int_{N\tau^+}^{T} \right\} \mathbf{E}_{\bar{R}_s}\{-\gamma^2\|\bar{w}\|^2\}ds - \gamma^2\|w_{de}\|_2^2 \\
&\quad +\sum_{k=0}^{N} \mathbf{E}_{\bar{R}_k}\{\|\bar{z}(k)\|^2\} - \gamma^2\mathbf{E}_{\bar{R}_0}\{\|x_0 - \hat{x}_0\|^2_{P_0^{-1}}\}
\end{aligned}
$$

for the system

$$
\begin{aligned}
\dot{e} &= \bar{\bar{A}}e + B_1\bar{w},\ t \neq k\tau,\ e(0) = x_0 - \hat{x}_0 \\
e(k\tau^+) &= \hat{A}_de(k\tau) + \hat{B}_dw_{de} \\
\bar{z}(k) &= \hat{C}e(k\tau) + \hat{D}w_{de}, \\
\hat{A}_d &= A_d - L_1C_2,\ \hat{B}_d = [-L_1D_{21}\ \ -L_1], \\
\hat{C} &= T_2^{\frac{1}{2}}(k)(T_2^{-1}(k)B_d'X(k\tau)A_d + L_2C_2), \\
\hat{D} &= T_2^{\frac{1}{2}}(k)[L_2D_{21}\ L_2],\ w_{de} = \begin{bmatrix} w_d \\ v \end{bmatrix}.
\end{aligned}
$$

Namely our problem can be reduced to the so called output estimation (OE) problem on the standard H_∞ disturbance attenuation theory. In order to solve this problem, we consider the following Riccati equation with the jump parts and the initial condition for it.

$$\dot{Q} + \bar{\bar{A}}'Q + Q\bar{\bar{A}}' + \frac{1}{\gamma^2}QB_1B_1'Q = O,\ t \neq k\tau \qquad (15)$$

$$Q(k\tau^-) = \hat{A}_d' Q(k\tau)\hat{A}_d + \hat{C}'\hat{C} + \hat{R}_d'(k)\hat{T}_d^{-1}(k)\hat{R}_d(k), \ Q(0^-) = \gamma^2 P_0^{-1} \tag{16}$$

where $\hat{R}_d(k) = \hat{B}_d' Q(k\tau)\hat{A}_d + \hat{D}'\hat{C}$ and $\hat{T}_d(k) = \gamma^2 I_{\mathbf{p_d+k}} - \hat{D}'\hat{D} - \hat{B}_d' Q(k\tau)\hat{B}_d$, and we assume

$$\hat{T}_d(k) > O. \tag{17}$$

Then, using the Riccati equation(15)(16) and completing the performance index (14) with respect to $col(w_d', v')$, we have ([5],[6])

$$J_{vT}(x_0, \bar{u}_*, \bar{w}, w_d, v, r_c, r_d) = -\left\{\sum_{k=0}^{N-1}\int_{k\tau^+}^{(k+1)\tau} + \int_{N\tau^+}^{T}\right\}\mathbf{E}_{\bar{R}_s}\{\gamma^2\|\bar{w} - \gamma^{-2}B_1'Qe\|^2\}ds + \bar{J}_c(r_c) + \bar{J}_d(r_d) - e'(T)Q(T)e(T) - \sum_{k=0}^{N}\mathbf{E}_{\bar{R}_k}\{\|\hat{T}_d^{1/2}(k)[\begin{bmatrix} w_d \\ v \end{bmatrix} - \hat{T}_d^{-1}(k)R_d(k)e(k)]\|^2\}.$$

For $\bar{w}$, w_d, v and x_0 in the error system (13), let $\bar{w} = \bar{w}^* = \gamma^{-2}B_1'Qe$, $\begin{bmatrix} w_d \\ v \end{bmatrix} = \begin{bmatrix} w_d^* \\ v^* \end{bmatrix} = \hat{T}_d^{-1}(k)\hat{R}_d(k)e(k)$, $x_0 = x_0^* = \hat{x}_0$. Then we get $e(t) = 0$ over $[0, T]$, because $e(t) = 0$ is an equibrium point of the error system (13) for $t \in [0, T]$. Therefore, since $\begin{bmatrix} w_d^* \\ v^* \end{bmatrix}(k) = 0, \ k \in [0, N]$, the estimated output

$$f(k) = \hat{C}e(k\tau) + \hat{D}\begin{bmatrix} w_d^* \\ v^* \end{bmatrix}(k) = 0$$

for $k \in [0, N]$. As a result, we get

$$J_{vT}(x_0^*, \bar{u}_*, \bar{w}^*, w_d^*, v^*, r_c, r_d) = \bar{J}_c(r_c) + \bar{J}_d(r_d).$$

Now we adopt $\bar{u}^* = \bar{u}_*$ as the optimal input minimizing J_{vT} for the worst case disturbance and the worst case initial state and so

$$J_{vT}(x_0^*, \bar{u}, \bar{w}^*, w_d^*, v^*, r_c, r_d) = \sum_{k=0}^{N}\mathbf{E}_{\bar{R}_k}\{\|\bar{u}(k) + T_2^{-1}(k)B_d'X(k\tau)A_d x(k\tau)\|^2_{T_2(k)}\} + \bar{J}_c(r_c) + \bar{J}_d(r_d)$$
$$\geq \bar{J}_c(r_c) + \bar{J}_d(r_d) = J_{vT}(x_0^*, \bar{u}^*, \bar{w}^*, \bar{w}_d^*, \bar{v}^*, r_c, r_d)$$

because $x_0^* = \hat{x}_0$, $\bar{w}^* = 0$, $w_d^* = 0$, $v^* = 0$ and $e(t) = 0$ for $t \in [0, \ T]$. Moreover, using this optimal input $\bar{u}^* = \bar{u}_*$, the inequality

$$J_{vT}(x_0, \bar{u}^*, \bar{w}, w_d, v, r_c, r_d) = -\left\{\sum_{k=0}^{N-1}\int_{k\tau^+}^{(k+1)\tau} + \int_{N\tau^+}^{T}\right\}\mathbf{E}_{\bar{R}_s}\{\gamma^2\|\bar{w} - \gamma^{-2}B_1'Qe\|^2\}ds + \bar{J}_c(r_c) + \bar{J}_d(r_d) - \|e(T)\|^2_{Q(T)} - \sum_{k=0}^{N}\mathbf{E}_{\bar{R}_k}\{\|\hat{T}_d^{1/2}(k)[\begin{bmatrix} w_d \\ v \end{bmatrix} - \hat{T}_d^{-1}(k)R_d(k)e(k)]\|^2\}$$
$$\leq \bar{J}_c(r_c) + \bar{J}_d(r_d) = J_{vT}(x_0^*, \bar{u}^*, \bar{w}^*, w_d^*, v^*, r_c, r_d)$$

holds. By these inequalities, the following theorem, which gives the solution of H_∞ tracking problem by output feedback, holds.

Proposition 4.1. Consider the system (1) and the performance index (3), and suppose **A1** and **A2**. Then **the $\mathbf{H}_\infty$ Tracking Problem** is solvable **by Output Feedback** if and only if there exist $X(t) > O$ and $Q(t) > O$ satisfying the conditions $X(0^-) < \gamma^2 R^{-1}$, $X(T) = O$ and $Q(0^-) = \gamma^2 P_0^{-1}$ such that the Riccati equations (4)(5), (15)(16) and the condition (17) hold over $[0, T]$. A saddle point strategy is given by

$$x_0^* = [\gamma^2 R^{-1} - X(0^-)]^{-1}\theta(0),$$
$$w^* = \frac{1}{\gamma^2}B_1'(Xx + \theta), \ w_d^* = 0, \ v^* = 0,$$
$$u_c^*(k) = -T_2^{-1}(k)B_d' \times [X(k\tau)(A_d\hat{x}_c(k\tau) + B_{3d}r_d(k)) + \theta_c(k\tau^+)] + L_2[y(k) - C_2\hat{x}_c(k\tau)]$$

where $\theta(t)$, $t \in [0, T]$ satisfies (6) and $\theta_c(s)$, $s \in [t, t+h]$ satisfies (7). $\hat{x}_c(t)$ is the 'causal' part of (12) at time t. $\hat{x}_c$ is the expected value of $\hat{x}$ over $\bar{R}_s$ and $\bar{R}_k$, but the actual value $\hat{x}_c(t)$ is determined based on the information $y(k)$, $R_{s+h\tau}$ and R_{k+h} with $0 \leq h \leq N$. Moreover, the value of the game is

$$J_{vT}(x_0^*, u_c^*, w^*, w_d^*, v^*, r_c, r_d) = \sum_{k=0}^{N}\mathbf{E}_{\bar{R}_k}\{\|T_2^{-1}(k)B_d'X(k\tau)A_d\hat{x}_1(k\tau) + T_2^{-1}(k)B_d'\theta_1\|^2_{T_2(k)}\} + \bar{J}_c(r_c) + \bar{J}_d(r_d) \tag{18}$$

where $\theta_1(t) = \theta(t) - \theta_c(t)$, $\hat{x}_1(t) = \hat{x}(t) - \hat{x}_c(t)$, $t \in [0, T]$, $\bar{J}_c(r_c)$ and $\bar{J}_d(r_d)$ are given by (9) and (10).

(Proof)

Sufficiency: We have already described the sufficient condition for the solvability of the trackig problem. The optimal input $u_c^*(k)$, $k \in [0, N]$ can be adopted using only the causal parts θ_c and $\hat{x}_c$ of θ and $\hat{x}$, determined on-line based on the information $R_{s+h\tau}$, R_{k+h} and $y(k)$, $k \in [0, N]$.

Necessity: On this game problem, the reference signals $\{r_c(\cdot)\}$ and $\{r_d(\cdot)\}$ are arbitrary. Therefore, by considering the case of $r_c(\cdot) \equiv 0$ and $r_d(\cdot) \equiv 0$, one can easily deduce the necessity for the solvability of our game problem. (Refer to [5][6] and etc.)(QED.)

Remark 4.1. $\bar{J}_c(r_c)$ and $\bar{J}_d(r_d)$, which mean the tracking errors including the preview information vector θ, are equal to zero, if $r_c = 0$, $r_d = 0$ and $\theta(t) = 0$ at all $t \in [0, T]$. Namely, in the case of neither inputting any reference signals nor considering any preview information, these tracking error terms are reduced to zero.

The saddle-point strategies are given by arbitrary $\{r_c\}$ and $\{r_d\}$ and the jump parts (16) of the Riccati equation do not depend on the coefficient matrices B_3, B_{3d}, D_{13} and D_{13d} of the reference signals $\{r_c\}$ and $\{r_d\}$. Therefore we can decied the controller gains L_1 and L_2 by considering the case where $\{r_c\}$ and $\{r_d\}$ are identically zero. (See ([5][6]) for details.) Now let $Z(t) = \gamma^2 Q^{-1}(t)$ with $Z(k\tau) = \gamma^2 Q^{-1}(k\tau^-)$ and $Z(k\tau^+) = \gamma^2 Q^{-1}(k\tau)$ We consider the following Riccati equation with jump parts.

$$\dot{Z} = \bar{\bar{A}}Z + Z\bar{\bar{A}}' + B_1 B_1' \ t \neq k\tau \tag{19}$$

$$Z(k\tau^+) = A_d Z(k\tau) A_d' - (R_{2Z}' T_{2Z} R_{2Z})(k) + (F_{1Z}' V_Z F_{1Z})(k) \tag{20}$$

$$V_Z(k) > aI \text{ for some } a > 0 \tag{21}$$

where

$$\begin{aligned}
R_2(k) &= B_d' X(k\tau) A_d,\\
T_{1Z}(k) &= \gamma^2 I_{\mathbf{m}} - T_2^{-1/2} R_2(k) Z(k\tau) R_2' T_2^{-1/2}(k),\\
T_{2Z}(k) &= \hat{D}_{21}\hat{D}_{21}' + C_2 Z(k\tau) C_2',\\
R_{1Z}(k) &= T_2^{-1/2} R_2(k) Z(k\tau) A_d',\\
R_{2Z}(k) &= C_2 Z(k\tau) A_d',\\
S_Z(k) &= C_2 Z(k\tau) R_2' T_2^{-1/2}(k),\\
V_Z(k) &= [T_{1Z} + S_Z' T_{2Z}^{-1} S_Z](k),\\
F_{1Z}(k) &= [V_Z^{-1}(R_{1Z} - S_Z' T_{2Z}^{-1} R_{2Z})](k),\\
F_{2Z}(k) &= -[T_{2Z}^{-1}(R_{2Z} + S_Z F_{1Z})](k),\\
\hat{D}_{21} &= [D_{21} \ I_{\mathbf{k}}].
\end{aligned}$$

Finally, utilizing this Riccati equation, we get the following theorem.

Theorem 4.1. Consider the system (1) and the performance index (3), and let $\gamma > 0$ be a given scalar. Suppose **A1** and **A2**. Then each of **the H$_\infty$ Tracking Problems** is solvable **by Output Feedback** if and only if there exist $X(t) > O$ and $Z(t) > O$ satisfying (4)(5), (19)-(21) over $[0, T]$ such that $X(0) < \gamma^2 R^{-1}$, $X(T) = O$ and $Z(0^-) = P_0$. Then the following results hold using the gains

$$\begin{aligned}
L_1(k) &= R_{2Z}' T_{2Z}^{-1}(k),\\
L_2(k) &= -T_2^{-1} R_2(k) Z(k\tau) C_2' T_{2Z}^{-1}(k).
\end{aligned}$$

Case a) A suitable control law for **the H$_\infty$ fixed-preview tracking** is given by

$$\begin{aligned}
\dot{\hat{x}}_c &= A\hat{x}_c + B_1 \bar{w}_c^* + B_3 r_c\\
\hat{x}_c(k\tau^+) &= A_d \hat{x}_c(k\tau) + B_d u_{o,a}(k) + B_{3d} r_d(k) \\
&\quad + L_1[y(k) - C_2\hat{x}_c(k)], \ \hat{x}_c(0) = x_{0c}^*\\
u_{o,a}(k) &= K_x \hat{x}_c(k\tau) + K_{r_d} r_d(k) + K_\theta \theta_c(k\tau^+)\\
&\quad + L_2[y(k) - C_2\hat{x}_c(k\tau)]
\end{aligned} \tag{22}$$

where $\bar{w}_c^* = \gamma^{-2} B_1'(X\hat{x}_c + \theta_c)$, $x_{0c}^* = [\gamma^2 R^{-1} - X(0^-)]^{-1}\theta_c(0)$. Moreover, the value of the performance index $J_{vT}(x_0^*, u_{o,a}(k), w^*, w_d^*, v^*, r_c, r_d)$ coincides with (18).

Case b) A suitable control law for **the H$_\infty$ tracking of noncausal $\mathbf{r_c}(\cdot)$ and $\mathbf{r_d}(\cdot)$** is given by

$$\begin{aligned}
\dot{\hat{x}} &= A\hat{x} + B_1 \bar{w}^* + B_3 r_c, \ \bar{w}^* = \gamma^{-2} B_1'(X\hat{x} + \theta)\\
\hat{x}(k\tau^+) &= A_d \hat{x}(k\tau) + B_d u_{o,b}(k) + B_{3d} r_d(k) \\
&\quad + L_1[y(k) - C_2\hat{x}(k)], \ \hat{x}(0) = x_0^*\\
u_{o,b}(k) &= K_x \hat{x}(k\tau) + K_{r_d} r_d(k) + K_\theta \theta(k\tau^+)\\
&\quad + L_2[y(k) - C_2\hat{x}(k\tau)]
\end{aligned} \tag{23}$$

Moreover, since $\theta(t) = \theta_c(t)$ and $\hat{x}(t) = \hat{x}_c(t)$ for all $t \in [0, T]$, the value of the performance index $J_{vT}(x_0^*, u_{o,b}, w^*, w_d^*, v^*, r_c, r_d) = \bar{J}_c(r_c) + \bar{J}_d(r_d)$

5. INFINITE HORIZON CASE

Consider

$$\begin{aligned}
\dot{x} &= Ax + B_2 u, \ t \neq k\tau, \ x(0) = x_0\\
x(k\tau^+) &= A_d x(k\tau) + B_{2d} u_d(k)\\
z_c &= C_1 x, \ t \neq k\tau\\
z_d(k) &= C_{1d} x(k\tau) + D_{12d} u_d(k)
\end{aligned} \tag{24}$$

Definition 5.1. (a) The system (24) (or $([A, A_d],[B_2, B_{2d}])$) is said to be stabilizable if there exist matrices K and K_d such that $(A + B_2 K, \ A_d + B_{2d} K_d)$ is exponentially stable.
(b) The system (24) (or $([C_1, C_{1d}],[A, A_d])$) is said to be detectable if there exist matrices J and J_d such that $(A + JC_1, \ A_d + J_d C_{1d})$ is exponentially stable.
(c) If (a) and (b) hold, the system (24) (or $([A, A_d],[B_2, B_{2d}],[C_1, C_{1d}])$) is said to be stabilizable and detectable.

The sufficient and necessary conditions for the solvability of the H$_\infty$ tracking problems in the finite horizon case are the same as the ones for the solvability of the standard H$_\infty$ control problems and so we can obtain the convergence and stability conditions for the H$_\infty$ tracking problems in the infinite horizon case.

Now we consider the following conditions for the system (1).
J1: $([A, A_d], [B_1, O], [C_1, C_{1d}])$ is stabilizable and detectable.
J2: $([A, A_d], [O, B_{2d}], [O, C_{2d}])$ is stabilizable and detectable.

Theorem 5.1. Consider the ststem (1) and the performance index (3) with $T \to \infty$, and let $\gamma > 0$ be a given scalar. Suppose **A1** and **A2**. Also suppose that R is sufficiently small. Then each of **the H$_\infty$ Tracking Problems** is solvable **by Output Feedback** if and only if there exist τ-periodic stabilizing solutions $X(t) > O$ and $Z(t) > O$ satisfying (4)(5), (19)-(21) over $[0, T]$ such that $X(0) < \gamma^2 R^{-1}$ and $Z(0^-) = P_0$. Then the output feedback controller for each case of the H$_\infty$ tracking problems is given by (22) and (23) respectively Now θ and θ_c are given by

$$\begin{cases} \dot{\theta}(t) = -\bar{A}'(t)\theta(t) + \bar{B}(t) r_c(t), \ t \neq k\tau\\ \theta(k\tau) = \bar{A}_d'(k)\theta(k\tau^+) + \bar{B}_d(k) r_d(k) \end{cases}$$

and

$$\begin{cases} \dot{\theta}_c(s) = -\bar{A}'(s)\theta_c(s) + \bar{B}(s) r_c(s), \ s \neq k\tau\\ \qquad t \leq s \leq t_f\\ \theta_c(j\tau) = \bar{A}_d'(j)\theta_c(j\tau^+) + \bar{B}_d(j) r_d(j),\\ \qquad k < j \leq k_f \ (k_f\tau < t_f < (k_f+1)\tau)\\ \theta_c(t_f) = 0 \end{cases}$$

where, for $k\tau^+ \leq t \leq (k+1)\tau$, $t_f = t + h\tau$ and $k_f = k + h$.

6. A NUMERICAL EXAMPLE

We consider the following numerical example.(cf.[1],[8])

$$\dot{\bar{x}}_s = \bar{A}\bar{x}_s + \bar{B}_1 w_s + \bar{B}_2 \tilde{u},\ \bar{x} \in \mathbf{R}^2, w_s \in \mathbf{R}^1,$$
$$y(k) = C_2 x(k\tau) + D_{21} w_d(k) + v(k)$$

$$\bar{A} = \begin{bmatrix} 0 & 1 \\ -1 & 0.4 \end{bmatrix}, \bar{B}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \bar{B}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$
$$C_2 = [\,0\ 1\ 0\,], D_{21} = [\,0\ 1\,]$$

where the control input $\tilde{u}$ is realized through a zero-order hold i.e., $\tilde{u}(t) = u(k),\ k\tau < t < (k+1)\tau$, and τ is a sampling period. The dynamics of this system can be represented by the following linear system with impulsive effects (or linear jump system).([5][6])

$$\dot{x} = Ax + B_1 w,\ t \neq k\tau,\ x(0) = x_0$$
$$x(k\tau^+) = A_d x(k\tau) + B_d u(k),\ x \in \mathbf{R}^3, w \in \mathbf{R}^1$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0.4 & 1 \\ 0 & 0 & 0 \end{bmatrix}, A_d = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$B_1 = col(1, -1, 0)$ and $B_d = col(0, 0, 1)$. Motivated by the above jump system representation of the sampled-data system, we consider the following jump system with a feedforward term of the reference signal at the jump part and introduce an objective function.

$$\dot{x} = Ax + B_1 w,\ t \neq k\tau,\ x(0) = x_0$$
$$x(k\tau^+) = A_d x(k\tau) + B_d u(k) + B_{3d} r_d(k)$$
$$z_d(k) = C_{1d} x(k\tau) + D_{12d} u(k) + D_{13d} r_d(k)$$
$$y(k) = C_2 x(k\tau) + D_{21} w_d(k) + v(k)$$

where

$$B_{3d} = \begin{bmatrix} 0.1 \\ 0 \\ 0 \end{bmatrix}, C_{1d} = \begin{bmatrix} 0.35 & 1.5 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$
$$D_{12d} = \begin{bmatrix} 0 \\ 0.5 \end{bmatrix}, D_{13d} = \begin{bmatrix} -1.0 \\ 0 \end{bmatrix}$$

$$J_{dT}(x_0, u, w, w_d, v, r_d) = -\gamma^2 x_0' R^{-1} x_0$$
$$-\gamma^2 [\|w\|_2^2 + \|w_d\|_2^2 + \|v\|_2^2]$$
$$+ \sum_{k=0}^{N} \mathbf{E}_{\bar{R}_k} \{\|C_{1d} x(k) - r_d(k)\|^2 + 0.5^2 \|u(k)\|^2\}$$

where $N\tau < T < (N+1)\tau$ and T is assumed to be very large. By the term $B_{3d} r_d(k)$, the tracking performance can be expected to be improve as similar to [1][8].

Let $\gamma = 20$, $\tau = 0.05$ and we design a output feedback law by which the function J_{dT} is minimized. We apply the results of H_∞ tracking for $r_d(k) = 2\sin(2k)$ with various step lengths of preview, and show the simulation results. It is shown that increasing the preview steps form $h = 0$ to $h = 3, 6, 9, 12$, improves the tracking performance. In fact, the square values $\|C_{1d}x(k) + D_{13d} r_d(k)\|^2$ of the tracking errors are shown in **Fig. 1** and it is clear the tracking error decreases as increasing the preview steps by this figure. For the preview length $h = 24$, the tracking performance becomes better, and, for $h = 36$, finally, the tracking error almost tends to zero except for a little vibration.

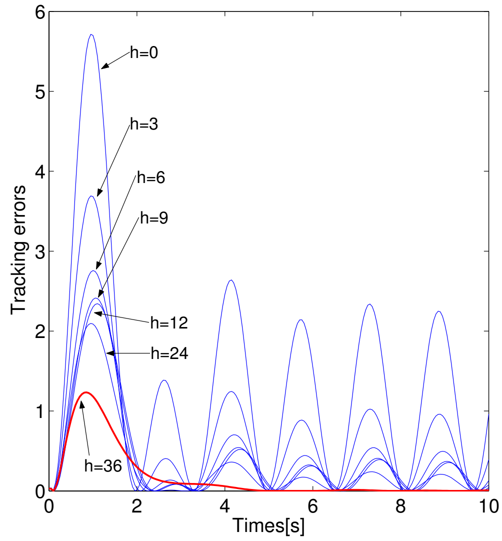

Fig. 1. The error of tracking $r_d(k) = 2\sin(2k)$ for various preview lengths

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