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Reachability analysis of uncertain nonlinear systems using guaranteed set integration

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Abstract: In this paper we show how to compute the reachable space for uncertain nonlinear continuous dynamical systems by using guaranteed set integration. We introduce two ways to do so. The first one is a *full interval* method which handles whole domains for set computation and relies on Taylor series and interval analysis. The second one relies on the theory of monotone dynamical systems and can be used with *cooperative* systems only but makes it possible to bracket the uncertain nonlinear system between two nonlinear dynamical systems where there is no uncertainty. In most cases, the bracketing systems derived are piecewise differentiable functions, hence cannot be directly integrated via interval Taylor models. Our contribution resides then in the use of hybrid automata to model the bounding systems. We give examples which show the potentials of both approaches in presence of parameter and input uncertainties.

Keywords: Continuous-time systems. Hybrid systems. Interval analysis. Non-linear systems. Reachability. Uncertain systems.

1. INTRODUCTION

Hybrid systems are complex dynamical systems which involve the interaction of discrete and continuous dynamics. They are often components of safety-critical systems, it is then necessary to have a thorough and guaranteed insight on the properties of the system, such as performance, safety or stability. The verification of these properties can be achieved through reachability analyses, some of which require an explicit computation of the hybrid state space, i.e. the set of all trajectories of the hybrid system starting from a possible initial set and under all admissible disturbances and variations in parameter values. A key issue when computing the reachable space of a the hybrid dynamical system lays in the calculation of the continuous reachable space for each mode, which boils down to the computation of reachable space for uncertain continuous dynamical systems. The reachable space is then defined as follows

$$\mathcal{R}([t_0, t]; \mathbb{X}_0) = \left\{ \begin{array}{c} \mathbf{x}(\tau),\, t_0 \leq \tau \leq t \,| \\ (\dot{\mathbf{x}}(\tau) = \mathbf{f}(\mathbf{x}, \mathbf{p}, \tau)) \\ \wedge(\mathbf{x}(t_0) \in \mathbb{X}_0 \subseteq [\mathbf{x}_0]) \\ \wedge(\mathbf{p} \in \mathbb{P} \subseteq [\mathbf{p}]) \end{array} \right\} \tag{1}$$

Several methods have been developed recently for the explicit computation of the reachable space. When the continuous dynamics are linear, these methods compute over-approximations of the reachable sets by combining time discretization, numerical integration and computational geometry. They use various representations for the reachable sets such as polytopes [Chutinan and Krogh, 2003], zonotopes [Girard, 2005] or ellipsoids [Kurzhanski and Varaiya, 2005]. Some other methods proceed with hybrid abstractions [Lefebvre and Guéguen, 2006]. When the continuous dynamics are modelled with a non-linear differential equation, the computation of the reachable set becomes much harder which forms one of the main obstacle in safety verification of hybrid systems [Lefebvre and Guéguen, 2006]. Most computationnal methods rely on an hybridization of the continuous-time models, i.e. the use of simple piecewise affine approximations of the analysed system on cells defined on the state space [Asarin et al., 2007]. Unfortunately, these reachability computations are tractable only for systems where the dimension of the continuous state component is small.

In this paper, we will show that the computation of reachable space for uncertain nonlinear dynamical systems can be achieved via guaranteed set integration, i.e. a guaranteed computation of the flow pipes. Hence, we will investigate two methods capable of dealing with nonlinear dynamics. The first one is a *full interval* method which handles whole domains for set computation and relies on Taylor series and interval analysis (see the review by Nedialkov et al. [1999]). Interval Tayor models have already been used for computing the reachable space of nonlinear continuous dynamical systems in the context of the verification of hybrid systems [Henzinger et al., 2000] but no parameter uncertainty were considered. Nevertheless, we will show that interval Taylor methods can be used with undertain systems in some cases. The second one relies on *comparison theorems* and the theory of quasi-monotone dynamical systems, mainly developed by Hirsch after the seminal work of Müller, Kamke and Krasnoselskij (see [Hirsch and Smith, 2005] and the references therein). Our method can be used with *cooperative* systems only but makes it possible to bracket the uncertain system between two dynamical systems where there is no uncertainty. In most cases, the bracketing systems derived are piecewise differentiable functions, and hence cannot be directly integrated via interval Taylor models. Our contribution resides then in the use of hybrid automata to model the bracketing systems. This means that the reachable space of an uncertain system is computed by running in a guaranteed way, two hybrid systems involving no uncertinaty which characterize the boundaries of the reachable space. We will show how to build these hybrid systems and how to run them in a guaranteed way. The organisation of the paper is as follows. Section 2 introduces guaranteed set integration. Section 3 recall how to compute a reachable space via set integration. Section 4 shows how to compute the reachable of cooperative uncertain continuous systems using hybrid automata. Examples are given in section 5.

2. GUARANTEED SET INTEGRATION

2.1 Interval analysis

Interval analysis was initially developed to account for the quantification errors introduced by the floating point representation of real numbers with computers and was extended to validated numerics (see [Jaulin et al., 2001] and the references therein). A real interval $[a] = [\underline{a},\ \bar{a}]$ is a connected and closed subset of $\mathbb{R}$. The set of all real intervals of $\mathbb{R}$ is denoted by $\mathbb{IR}$. Real arithmetic operations are extended to intervals. Consider $\mathbf{g} : \mathbb{R}^n \longmapsto \mathbb{R}^m$; the range of this function over an interval vector [a] is given by $\mathbf{g}([\mathbf{a}]) = \{\mathbf{g}(\mathbf{u}) \mid \mathbf{u} \in [\mathbf{a}]\}$. The interval function $[\mathbf{g}] : \mathbb{IR}^n \longmapsto \mathbb{IR}^m$ is an inclusion function for $\mathbf{g}$ if $\forall [\mathbf{a}] \in \mathbb{IR}^n,\ \mathbf{g}([\mathbf{a}]) \subseteq [\mathbf{g}]([\mathbf{a}])$. An inclusion function of $\mathbf{g}$ can be obtained by replacing each occurrence of a real variable by the corresponding interval and each standard function by its interval counterpart. The resulting function is called the natural inclusion function, which performance depends on the formal expression for $\mathbf{g}$. Given a bounded set $\mathbb{E}$ of complex shape, one usually defines a box or a paving, i.e. a union of non-overlaping boxes, $\overline{\mathbb{E}}$ which contains the set $\mathbb{E}$: this is known as an *outer* approximation of it. Likewise, one also defines an *inner* approximation $\underline{\mathbb{E}}$ which is contained in the set $\mathbb{E}$. Hence, we have the following properties

$$(\underline{\mathbb{E}} \subseteq \mathbb{E} \subseteq \overline{\mathbb{E}}) \wedge (vol(\underline{\mathbb{E}}) \leq vol(\mathbb{E}) \leq vol(\overline{\mathbb{E}})) \qquad (2)$$

where $vol(.)$ is the volume of a set.

2.2 Guaranteed set integration using interval Taylor models

Consider the following differential equation:

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}, \mathbf{u}, \mathbf{p}, t), \\ \mathbf{x}(t_0) \in \mathbb{X}_0 \subseteq [\mathbf{x}_0] \subset \mathbb{D},\ \mathbf{u} \in \mathbb{U},\ \mathbf{p} \in \mathbb{P}_0 \subseteq [\mathbf{p}] \end{cases} \qquad (3)$$

with $t_0 \geq 0$. The function $\mathbf{f}$, possibly nonlinear, is assumed to be at least $k-$times continuously differentiable in a domain $\mathbb{D} \subseteq \mathbb{R}^n$. The objective is to compute interval vectors $[\mathbf{x}_j], j = 1, \ldots, n_T$, that are guaranteed to contain the solution of (3) at $t_1, t_2, \ldots, t_{n_T}$. Effective methods for solving such a problem are based on Taylor expansions, see [Nedialkov et al., 1999] and the references therein. These methods are usually one-step methods which proceed with two phases:

(1) they first verify existence and uniqueness of the solution using the fixed point theorem and the Picard-Lindelöf operator, compute an a priori enclosure $[\tilde{\mathbf{x}}_j]$ such that $\mathbf{x}(t) \in [\tilde{\mathbf{x}}_j]$ for all $t \in [t_j, t_{j+1}]$ and adapt integration step size h_j if necessary in order to keep the relative width of the solution's enclosure smaller than a given threshold ;
(2) then they compute a tighter enclosure $[\mathbf{x}_{j+1}]$ of the solution of (3) at t_{j+1} as

$$[\mathbf{x}_{j+1}] = [\mathbf{x}_j] + \sum_{i=1}^{k-1} h_j^i \mathbf{f}^{[i]}([\mathbf{x}_j]) + h_j^k \mathbf{f}^{[k]}([\tilde{\mathbf{x}}_j]) \qquad (4)$$

which corresponds to a Taylor expansion of order k where $[\tilde{\mathbf{x}}_j]$ is used to compute the remainder term. The coefficients $\mathbf{f}^{[i]}$ are the Taylor coefficients of the solution $\mathbf{x}(t)$ which can be computed either numerically by automatic differentiation or analytically via formal methods.

The enclosures thus obtained are said *validated* which is in contrast with conventional numerical integration techniques which derive approximations with unknown global error and where the accumulation of both truncation and round-off errors may cause the computed solution to deviate widely from the real one. When using interval Taylor models it is then possible to control the global truncation error since it is directly connected to the width of the solution enclosure. Unfortunately, the *wrapping* effect, i.e. the overestimation due to the bracketing of a set of any shape by a box makes the explicit scheme (4) width-increasing and thus not suitable for numerical implementation. To solve such a drawback, one uses usually mean value forms, matrices preconditioning and linear transform [Nedialkov et al., 1999]. A more general scheme has been developed in [Nedialkov et al., 2001] where the interval method is founded on the Hermite-Obreshkoff expansion series where the sought enclosure appears both implicitly and explicitly. In practice, apart for some particular cases such as affine uncertain stable systems, the above techniques derive usefull enclosures only if the ODE under study involves no uncertain variable. Indeed, when the widths of the initial state or the parameter interval vectors are large, or when one proceeds with numerical integration over a long period of time, the enclosure $[\mathbf{x}_{j+1}]$ usually becomes very pessimistic and thus useless, notwithstanding all the techniques used to circumvent the wrapping effect in interval computations. In the next subsection, we indicate how to solve this problem when the system under study is a monotone dynamical system.

2.3 Guaranteed set integration with the theory of monotone dynamical systems

A monotone dynamical system is just a dynamical system on an ordered metric space which has the property that ordered initial states lead to ordered subsequent states. The application of monotone methods and comparison arguments in differential equations started in the early 1920s. A comprehensive monograph on this topic is the one by [Hirsch and Smith, 2005].

Definition: the dynamical system is *cooperative* over $\mathbb{D}$, if all the off-diagonal terms of its Jacobian matrix are non negative over $\mathbb{D}$, i.e.

$$\forall i \neq j,\ t \geq 0,\ \mathbf{x} \in \mathbb{D},\ \frac{\partial f_i(\mathbf{x}, t)}{\partial x_j} \geq 0 \qquad (5)$$

Property: If a system of ODE is *cooperative* then the dynamical system is monotone and it is possible to compute an inclusion function for the solution of the ODE.

Theorem 1. [Walter and Kieffer, 2003]: Let us consider two cooperative systems

$$\dot{\overline{\mathbf{x}}} = \overline{\mathbf{f}}(\overline{\mathbf{x}}, \overline{\mathbf{p}}, \underline{\mathbf{p}}, \mathbf{u}, t) \tag{6}$$

$$\dot{\underline{\mathbf{x}}} = \underline{\mathbf{f}}(\underline{\mathbf{x}}, \overline{\mathbf{p}}, \underline{\mathbf{p}}, \mathbf{u}, t) \tag{7}$$

which satisfy the condition

$$\begin{array}{l} \forall \mathbf{p} \in [\underline{\mathbf{p}}, \overline{\mathbf{p}}],\ \forall \mathbf{x} \in \mathbb{D},\ \mathbf{u} \in \mathbb{U},\ \forall t \geq t_0, \\ \underline{\mathbf{f}}(\underline{\mathbf{x}}, \overline{\mathbf{p}}, \underline{\mathbf{p}}, \mathbf{u}, t) \leq \mathbf{f}(\mathbf{x}, \mathbf{p}, \mathbf{u}, t) \leq \overline{\mathbf{f}}(\overline{\mathbf{x}}, \overline{\mathbf{p}}, \underline{\mathbf{p}}, \mathbf{u}, t) \end{array} \tag{8}$$

Moreover, if there exist two initial conditions such that

$$\forall \mathbf{p} \in [\underline{\mathbf{p}}, \overline{\mathbf{p}}],\ \underline{\mathbf{x}}(\underline{\mathbf{p}}, \overline{\mathbf{p}}, t_0) \leq \mathbf{x}(\mathbf{p}, t_0) \leq \overline{\mathbf{x}}(\underline{\mathbf{p}}, \overline{\mathbf{p}}, t_0) \tag{9}$$

then the solution of (3) satisfies

$$\forall t \geq t_0,\ [\mathbf{x}(t)] \subseteq [\underline{\mathbf{x}}(t), \overline{\mathbf{x}}(t)] \tag{10}$$

Now, the bracketing systems (6)–(7) involve *degenerated* intervals only, therefore interval Taylor models can be used for the guaranteed numerical evaluation of $\underline{\mathbf{x}}(t)$ and $\overline{\mathbf{x}}(t)$ since the *wrapping* effect in interval computations can be efficiently controlled by the methods we introduced previously. The main difficulty is to obtain suitable bracketing systems in the general case. However, when the components of $\mathbf{f}$ are monotonic with respect to each parameter, it is quite easy to define these systems [Kieffer and Walter, 2006], while avoiding possible divergence that may occur when both upper and lower components of a parameter appear simultaneously in the same expression of the components of the bracketing systems [Ramdani et al., 2006].

Rule 1. Use of monotonicity property: Here we adapt the idea introduced in [Kieffer and Walter, 2006]. Assume that function f is differentiable w.r.t $\mathbf{p}$. Define $\overline{\delta}^i(p_k)$ as follows

$$\overline{\delta}^i(p_k) = \begin{cases} \overline{p}_k & \text{if } \frac{\partial f_i}{\partial p_k} \geq 0 \\ \underline{p}_k & \text{if } \frac{\partial f_i}{\partial p_k} < 0 \end{cases} \tag{11}$$

where inequalities must hold for all $\mathbf{p}$ in $[\underline{\mathbf{p}}, \overline{\mathbf{p}}]$, all $\mathbf{x}$ in $\mathbb{D}$, all $\mathbf{u}$ in $\mathbb{U}$ and all $t \geq t_0$; and $\overline{\delta}(\mathbf{p}) = [\overline{\delta}^i(p_1), ..., \overline{\delta}^i(p_k), ...]^T$. In a similar way, define $\underline{\delta}^i(p_k)$ as follows

$$\underline{\delta}^i(p_k) = \begin{cases} \underline{p}_k & \text{if } \frac{\partial f_i}{\partial p_k} \geq 0 \\ \overline{p}_k & \text{if } \frac{\partial f_i}{\partial p_k} < 0 \end{cases} \tag{12}$$

and $\underline{\delta}(\mathbf{p}) = [\underline{\delta}^i(p_1), ..., \underline{\delta}^i(p_k), ...]^T$.

If system (3) is cooperative over $\mathbb{D}$ then the enclosing systems (6)–(7) can be obtained as follows

$$\dot{\overline{\mathbf{x}}}(t) = \mathbf{f}(\overline{\mathbf{x}}, \overline{\delta}(\mathbf{p}), \mathbf{u}, t); \quad \overline{\mathbf{x}}(t_0) = \overline{\mathbf{x}}_0 \tag{13}$$

$$\dot{\underline{\mathbf{x}}}(t) = \mathbf{f}(\underline{\mathbf{x}}, \underline{\delta}(\mathbf{p}), \mathbf{u}, t); \quad \underline{\mathbf{x}}(t_0) = \underline{\mathbf{x}}_0 \tag{14}$$

2.4 How tight are the enclosures

An important issue when deriving enclosures for the solution of (3) as suggested in theorem 1 and by using rule 1 is how tight are the enclosures given by (13)-(14).

Now, the derived enclosures for the reachable space (1) are *tight* if they give interval state vectors at each time t which can be actually reached by system (3) with the given initial and parameter intervals. Obviously, this is true if systems (13)-(14) are feasible which in turn is true if there exists $\mathbf{p}_1$ and $\mathbf{p}_2$ in $[\underline{\mathbf{p}}, \overline{\mathbf{p}}]$ such that

$$\dot{\overline{\mathbf{x}}}(t) = \mathbf{f}(\overline{\mathbf{x}}, \mathbf{p}_1, \mathbf{u}, t); \quad \overline{\mathbf{x}}(t_0) = \overline{\mathbf{x}}_0 \tag{15}$$

and

$$\dot{\underline{\mathbf{x}}}(t) = \mathbf{f}(\underline{\mathbf{x}}, \mathbf{p}_2, \mathbf{u}, t); \quad \underline{\mathbf{x}}(t_0) = \underline{\mathbf{x}}_0 \tag{16}$$

In general, the derived enclosures will be *not* tight. However, when (i) the system under study is cooperative over $\mathbb{D}$ and (ii) the bracketing systems are built using (13)-(14) then the enclosures derived via the theory of monotone dynamical systems will be at least tighter than the ones obtained via interval Taylor models. Furthermore, if (iii) conditions (15)-(16) hold too as well as conditions (i) and (ii) then the enclosures derived via the theory of monotone dynamical systems should be tight.

3. ENCLOSURES FOR THE REACHABLE SPACE

In this section, we will show how to compute the reachable space by using set integration

For $j = 0, \ldots, n_T - 1$, define

$$\forall t \in [t_j, t_{j+1}],$$
$$[\mathbf{x}](t) = [\mathbf{x}_j] + \sum_{i=1}^{k-1} (t - t_j)^i \mathbf{f}^{[i]}([\mathbf{x}_j]) + (t - t_j)^k \mathbf{f}^{[k]}([\psi_j]) \tag{17}$$

Proposition 1.

$$\begin{array}{l} j = 0, \ldots, n_T - 1, \\ if\ [\psi_j] \supseteq [\tilde{\mathbf{x}}_j] \Rightarrow \forall t \in [t_j, t_{j+1}],\ \mathbf{x}(t) \in [\mathbf{x}](t) \end{array} \tag{18}$$

Proof 1. It suffices to write a Taylor series expansion at time t_j and use $[\tilde{\mathbf{x}}_j]$ as defined in section 2.2 for evaluating the remainder term (see [Nedialkov et al., 1999]).

Define $\overline{\mathcal{R}}$ as an over-approximation of a reachable space $\mathcal{R}$, as follows

$$\forall t, t' \in [t_0, t_{n_T}],\ \overline{\mathcal{R}}([t, t']; [\mathbf{x}](t)) \supseteq \mathcal{R}([t, t']; [\mathbf{x}](t)) \tag{19}$$

Proposition 2. The over-approximation $\overline{\mathcal{R}}$ is given by

$$\begin{array}{l} j = 0, \ldots, n_T - 1, \\ \forall t \in [t_j, t_{j+1}],\ \overline{\mathcal{R}}([t_j, t]; [\mathbf{x}_j]) = \cup_{\tau \in [t_j, t]} [\mathbf{x}](\tau) \end{array} \tag{20}$$

and satisfies

$$\begin{array}{l} j = 0, \ldots, n_T - 1, \\ \forall t \in [t_j, t_{j+1}], \overline{\mathcal{R}}([t_j, t]; [\mathbf{x}_j]) \subseteq [\tilde{\mathbf{x}}_j] \end{array} \tag{21}$$

where $[\tilde{\mathbf{x}}_j]$ is defined in section 2.2.

Proof 2. Obvious from (18).

Define $\overline{\mathcal{R}}([t_0, t_0]; [\mathbf{x}_0]) = [\mathbf{x}_0]$.

Proposition 3. An over-approximation of the reachable space (1) is given by

$$\begin{array}{l} j = 1, \ldots, n_T - 1,\ \forall t \in [t_j, t_{j+1}], \\ \overline{\mathcal{R}}([t_0, t]; [\mathbf{x}_0]) = \overline{\mathcal{R}}([t_0, t_j]; [\mathbf{x}_0]) \cup \overline{\mathcal{R}}([t_j, t]; [\mathbf{x}_j]) \end{array} \tag{22}$$

and satisfies

$$\begin{array}{l} j = 1, \ldots, n_T - 1, \\ \forall t \in [t_j, t_{j+1}], \overline{\mathcal{R}}([t_0, t]; [\mathbf{x}_0]) \subseteq \cup_{i \in \{0, j\}} [\tilde{\mathbf{x}}_i] \end{array} \tag{23}$$

Proof 3. Obvious from (18) and proposition 2

It is clear that thanks to (17), (23) and (22), one can derive explicit formulas which characterize the time-history of the boundaries of the reachable space. In practice however, one can use instead of (22) the over-approximation (23) obtained by using the *a priori* solutions $[\tilde{\mathbf{x}}_j]$ only.

4. COMPUTING THE REACHABLE SPACE USING HYBRID AUTOMATA AS BRACKETING SYSTEMS

In this section, we address the case of uncertain monotone dynamical systems for which the sign of the partial derivatives $\frac{\partial f_i}{\partial p_k}$ may vary over the time period under study; therefore rule 1 cannot be directly used. Since it is still possible to use the rule over each time interval where this sign is constant, the idea retained in the sequel is to regard both upper and lower bounding systems as piecewise nonlinear ODEs and thus as hybrid dynamical systems. They can be modeled by an *hybrid automaton* where the hybrid state encompasses both a *discrete time* component and *continuous time* state variables associated to it [Alur et al., 1995]. The hybrid automaton which will model the systems which bracket (3) is defined by :

$$H = (\mathbb{Q}, \mathbb{E}, \mathbb{D}, \mathbb{U}, \mathbb{F}, \mathbb{T}, \mathbb{R}) \tag{24}$$

where:

(1) $\mathbb{Q}$ is a finite set of the discrete components of the hybrid states called modes or locations. For each location corresponds two continuous-time systems which provide the locally upper and lower solutions of (3). These systems are built using the monotonicity property, i.e. rule 1 and hence equations (13)-(14).
(2) $\mathbb{E} \subseteq \mathbb{Q} \times \mathbb{Q}$ is the set of the transitions. It contains all the possible commutations between the locally upper (resp. lower) continuous systems which bracket (3).
(3) $\mathbb{D}$ is the state space of (3).
(4) $\mathbb{U}$ is the definition domain for the input of (3).
(5) $\mathbb{F} = \overline{\mathbb{F}} \cup \underline{\mathbb{F}}$ where $\overline{\mathbb{F}} = \{\overline{\mathbf{f}}_q, q \in \mathbb{Q}\}$ and $\underline{\mathbb{F}} = \{\underline{\mathbf{f}}_q, q \in \mathbb{Q}\}$ are the collections of the field vectors defined by the upper and the lower systems which enclose locally the state flow generated by (3).
$$\forall q \in \mathbb{Q}, \overline{\mathbf{f}}_q : \mathbb{D} \times \mathbb{U} \longrightarrow \mathbb{R}^n \tag{25}$$
$$\forall q \in \mathbb{Q}, \underline{\mathbf{f}}_q : \mathbb{D} \times \mathbb{U} \longrightarrow \mathbb{R}^n \tag{26}$$
(6) $\mathbb{T} = \{t_e, e \in \mathbb{E}\}$ is the collection of switching time instants. Define $g_{i,k}(.) = \frac{\partial f_i}{\partial p_k}(.)$. We assume that functions $g_{i,k}$ are continuous. The set $\mathbb{T}$ is defined as
$$\mathbb{T} = \left\{ \begin{array}{c} t_e \in [t_0, t_{n_T}] \mid \\ \exists\, k = 1, ..., n_p, \, \exists i = 1, ..., n, \\ \exists \mathbf{p} \in [\mathbf{p}] \mid g_{i,k}(\mathbf{x}, \mathbf{u}, \mathbf{p}, t_e) = 0 \end{array} \right\} \tag{27}$$
That is to say that if the monotonicity of $\mathbf{f}$ with respect to one of the parameters changes at t_e, a transition $e = (q, q') \in \mathbb{E}$ occurs and the bracketing systems change too from $\{\overline{\mathbf{f}}_q, \underline{\mathbf{f}}_q\}$ to $\{\overline{\mathbf{f}}_{q'}, \underline{\mathbf{f}}_{q'}\}$.
(7) $\mathbb{R} = \{\mathbf{R}_e, e \in \mathbb{E}\}$ is the collection of reset functions. They initialize the field vectors $\overline{\mathbf{f}}_{q'}$ (resp.$\underline{\mathbf{f}}_{q'}$) after the activation of a transition $e = (q, q')$: $\overline{\mathbf{x}}_{q'}(t_0) = \mathbf{R}_e(\overline{\mathbf{x}}_q(t_e))$ and $\underline{\mathbf{x}}_{q'}(t_0) = \mathbf{R}_e(\underline{\mathbf{x}}_q(t_e)$. Since system (3) is monotone, reset functions are only needed to instanciate the bounds for the parameter vector in (13)-(14).

Now, in order to build $\{\overline{\mathbf{f}}_q$ and $\underline{\mathbf{f}}_q\}$ using rule 1 and hence (13)-(14), we will split the experiment time period $[t_0, t_{n_T}]$ into a succession of integration time intervals $[t_j, t_{j+1}]$ where $t_{j+1} = t_j + h_j$ and where integration time steps h_j are either chosen a priori or adapted on-line. Denote $\mathbb{I}_M$, the set of time intervals $[t_j,\, t_{j+1}]$ over which no switching occurs, i.e., all the components of the field vectors $\mathbf{f}$ of (3) are monotonic with respect to each parameter.

$$\mathbb{I}_M = \{[t_j,\, t_{j+1}] \subset [0, t_{n_T}] \,|\, \forall e \in \mathbb{E},\, t_e \notin [t_j,\, t_{j+1}]\} \tag{28}$$

Since the a priori solution $[\tilde{\mathbf{x}}_j]$ encloses the whole state trajectory over $[t_j,\, t_{j+1}]$, an inner approximation of the set (28) can also be defined without loss of guarantee as follows

$$\underline{\mathbb{I}_M} = \left\{ \begin{array}{c} [t_j,\, t_{j+1}] \subset [0, t_{n_T}] \,| \\ \forall i = 1, ..., n,\, \forall k = 1, ..., n_p, \\ 0 \notin [g]_{i,k}([\tilde{\mathbf{x}}_j], \mathbf{u}, [\mathbf{p}], [t_j, t_{j+1}]) \end{array} \right\} \tag{29}$$

Similarly, define the set $\mathbb{I}_S$ of intervals where a switching occurs, i.e.,

$$\mathbb{I}_S = \{[t_j,\, t_{j+1}] \subset [0, t_{n_T}] \,|\, \exists e \in \mathbb{E},\, t_e \in [t_j,\, t_{j+1}]\} \tag{30}$$

Since we have

$$[t_0, t_{n_T}] = \mathbb{I}_M \cup \mathbb{I}_S \tag{31}$$

then we can write without loss of guarantee

$$\overline{\mathbb{I}_S} = [t_0,\, T] \setminus \underline{\mathbb{I}_M} \tag{32}$$

Now, we can use rule 1 and (13)-(14) over each time interval $[I_m] \in \mathbb{I}_M$ in order to derive $\underline{\mathbf{f}}_m$ and $\overline{\mathbf{f}}_m$ and to bracket all the possible solutions of the uncertain system (3)

$$\begin{array}{c} \forall [I_m] \in \mathbb{I}_M, \forall m \in \mathbb{Q}, \forall \mathbf{p} \in [\mathbf{p}], \forall \mathbf{x} \in \mathbb{D}, \forall \mathbf{u} \in \mathbb{U}, \forall t \in [I_m], \\ \underline{\mathbf{f}}_m(\underline{\mathbf{x}}, \underline{\mathbf{p}}, \overline{\mathbf{p}}, \mathbf{u}, t) \leq \mathbf{f}(\mathbf{x}, \mathbf{p}, \mathbf{u}, t) \leq \overline{\mathbf{f}}_m(\overline{\mathbf{x}}, \underline{\mathbf{p}}, \overline{\mathbf{p}}, \mathbf{u}, t) \end{array} \tag{33}$$

where $\underline{\mathbf{f}}_m \in \underline{\mathbb{F}}$ and $\overline{\mathbf{f}}_m \in \overline{\mathbb{F}}$.

One difficulty remains as the actual time instant, i.e., t_e in (27), when the upper (*resp.* lower) hybrid system reaches one of its switching time instant is unknown a priori. By using a validated interval Taylor model integration method we will be able to solve this problem in an efficient and guaranteed way. It suffices to apply to the system (3) a validated integration method over each time interval $[I_s] \in \mathbb{I}_S$ in order to cross the switching time instant. By doing so, we keep the guarantee property for the enclosures without having to derive the actual time instant where the commutation occurs. Note that since the widths of the intervals $[I_s]$ are equal to an integration time step (i.e., h_j) the *wrapping* effect will be very small in the validated integration method. Eventually, the time intervals $[I_s]$ might also be reduced. Finally, the methodology used for computing the upper (resp. lower) bracketing system is summarized in the algorithm **Hybrid-Bound-Monotone** given below. Note that the upper and lower solutions of (3) can both be computed by using the same algorithm, it suffices to set algorithm **Bnd** to return either the upper or the lower bound of the state vector, accordingly. Algorithm **Hybrid-Bound-Monotone** finds the initial discrete mode q at line 2 (algorithm **Initialize**) and then selects the ODE $\mathbf{f}_q$ which corresponds to this initial discrete state at line 3 (algorithm **Select-ODE**). The latter only implements rule 1. In the *while* loop, it integrates the ODE $\mathbf{f}_q$ until a transition occurs, which is detected at line 6 by algorithm **Switch-Cond**. If this is the case (boolean -*transition*- is *true*), algorithm **Switch-Cond** also returns the new discrete state q'. In order to cross the guard condition with guarantee, one integration step over $[I_s] = [t, t+h]$ is performed for the original uncertain ODE $\mathbf{f}$ with a full interval validated method (algorithm **Interval-Integrate** at line 9).

Algorithm Hybrid-Bound-Monotone
$(in: t_0, t_{n_T}, \mathbf{f}, \mathbb{F}, [\mathbf{x}](t_0), [\mathbf{p}];\ out: \mathbf{Bnd}([\mathbf{x}](t_1)), ..., \mathbf{Bnd}([\mathbf{x}](t_{n_T})), \mathbf{Bnd}([\tilde{\mathbf{x}}](t_0)), ..., \mathbf{Bnd}([\tilde{\mathbf{x}}](t_{n_T-1})))$

(1) $t := t_0$;
(2) $q :=$ **Initialize**$(\mathbf{f}, [\mathbf{x}](t_0), [\mathbf{p}])$;
(3) $\mathbf{f}_q :=$ **Select-ODE**$(\mathbf{Bnd}(\mathbb{F}), q)$;
(4) *while* $(t < t_{n_T})$ *do*
(5) $\{h, [\mathbf{x}](t+h), [\tilde{\mathbf{x}}(t)]\} :=$ **Integrate**$(\mathbf{f}_q, \mathbf{Bnd}([\mathbf{x}](t)), t)$;
(6) $\{$*-transition-*$, q'\} :=$ **Switch-Cond**$([\tilde{\mathbf{x}}(t)], \mathbf{f}_q)$;
(7) *if(-transition-)then*
(8) $q := q'$;
(9) $[\mathbf{x}](t+h) :=$ **Interval-Integrate**$(\mathbf{f}, \mathbf{Bnd}([\mathbf{x}](t)), [\mathbf{p}], t, h)$;
(10) $\mathbf{f}_q :=$ **Select-ODE**$(\mathbb{F}, q)$;
(11) *endif*
(12) $t := t + h$;
(13) end

5. APPLICATIONS

In this section, all algorithms are developped in C++ and use the Profil/BIAS C++ class library for interval computations.

Uncertain affine system The system analysed is taken from Girard [2005] where the reachable space were computed with zonotopes while assuming no parameter uncertainty. To the contrary, in this paper we will compute the reachable space while assuming parameter uncertainties. The system is as follows

$$\dot{\mathbf{x}} = \begin{bmatrix} a_1 & a_2 & 0 & 0 & 0 \\ -a_2 & a_1 & 1 & 0 & 0 \\ 0 & 0 & a_3 & a_4 & 0 \\ 0 & 0 & -a_4 & a_3 & 0 \\ 0 & 0 & 0 & 0 & a_5 \end{bmatrix} \mathbf{x} + \mathbf{u} \quad (34)$$

where parameters a_i, $i = 1, ..., 5$ are subject to bounded uncertainties with known bounds: $a_1 = [-1.1, -0.9]$, $a_2 = [-4.1, -3.9]$, $a_3 = [-3.1, -2.9]$, $a_4 = [0.9, 1.1]$ and $a_5 = [-2.1, -1.9]$. Initial domain for state vector is $x_i = [0.8, 1.2]$, $i = 1, ..., 5$ and input $\mathbf{u}$ is taken bounded, i.e. $u_i = [-0.1, 0.1]$ $i = 1, ..., 5$. The reachable space as obtained by interval Taylor models using the extended mean value algorithm introduced by Rihm [1994] with $k = 30$ as order of the Taylor series expansion. The projection of the reachable space onto the $x_1 \times x_2$ subspace is given for two values for the integration time step : $h = 0.01$ and as $h = 0.05$, and are plotted on figure (1). It is clear that the larger the integration time step the larger the overestimation of the reachable space. Even though, interval Taylor models make it possible to compute the reachable space of uncertain affine systems even with fairly large integration time step in a reasonable computation time. However, when evaluated with larger domains for the parameter or state vectors, the size of the enclosures diverges. One way to address this issue would be to partition both parameter and state vectors but then computation time will grow exponentially.

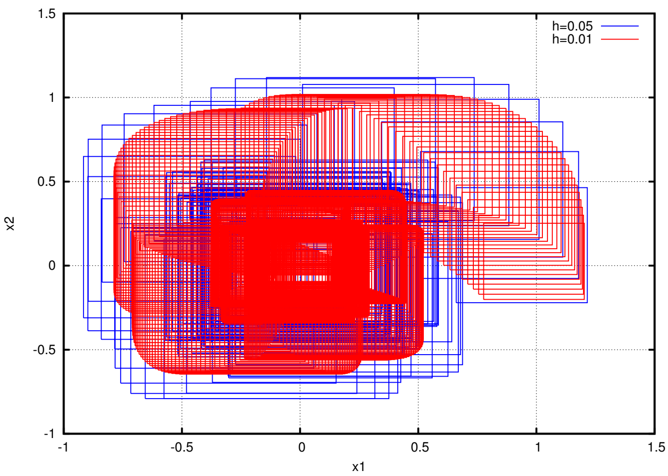

Fig. 1. Reachable space of (34). Projection onto $x_1 \times x_2$ subspace as obtained when integration time step $h = 0.01$ (CPU time 0.63s, PIV 2Ghz) and $h = 0.05$ (CPU time 1.29s).

Monotone uncertain nonlinear system Consider the thermal model of a material sample taken from [Ramdani et al., 2006] submitted to a multi-harmonic signal. State vector $\mathbf{x} \in \mathbb{R}^{13}$ stands for temperature and t denotes time. The state equation is as follows:

$$\begin{cases} \dot{x}_1 &= \alpha_1(x_2 - 2x_1 + u_0 + u(t)) \\ \dot{x}_2 &= 2\alpha_1(x_1 - (1 + \frac{\rho_1}{\rho_2})x_2 + \frac{\rho_1}{\rho_2}x_3) \\ \dot{x}_3 &= 2(p_0 + p_1 x_3)(x_4 - x_3 + p_2\frac{\delta_2}{\rho_2}(x_2 - x_3)) \\ \dot{x}_i &= (p_0 + p_1 x_i)(x_{i+1} - 2x_i + x_{i-1}) \qquad i = 4, \ldots, 9 \\ \dot{x}_{10} &= 2(p_0 + p_1 x_{10})(x_9 - x_{10} + p_2\frac{\delta_2}{\rho_2}(x_{11} - x_{10})) \\ \dot{x}_{11} &= 2\alpha_2(x_{12} - (1 + \frac{\rho_3}{\rho_2})x_{11} + \frac{\rho_3}{\rho_2}x_{10}) \\ \dot{x}_{12} &= \alpha_3(x_{13} - 2x_{12} + x_{11}) \\ \dot{x}_{13} &= 2\alpha_3(x_{12} - (1 + \frac{\rho_3}{\rho_4})x_{13} + \frac{\rho_3}{\rho_4}u_0) \\ u(t) &= \sum_{l=1 \ldots 5} u_l sin(2^{l-1}\omega_0 t + \phi_0) \end{cases} \quad (35)$$

Parameter vector $\mathbf{p} = [p_0\ p_1\ p_2]^T$ is taken in the set $\mathbb{P}_0$ but the other parameters are assumed perfectly known. $\mathbb{P}_0 = [0.7, 1.23]\mathrm{s}^{-1} \times [0.01, 0.015]\mathrm{s}^{-1}\mathrm{K}^{-1} \times [0.23, 0.64]\mathrm{mW}^{-1}\mathrm{K}^{-1}$ and initial state vector domain is taken as $\mathbb{X}_{0_i} = [90, 110]°\mathrm{C}$. When one uses a *full interval* method, i.e. interval Hermite-Obreschkoff series with variable step control as implemented in the VNODE software [Nedialkov et al., 2001], the computed enclosures diverge as long as parameter vector $\mathbf{p}$ is taken uncertain, even with very small uncertainty. Since system (35) is cooperative and hence monotone, we will use the hybrid bracketing technique introduced in section 4. In order to build the automaton (24) characterizing the bounding systems for (35), we need to study the signs of the partial derivatives $\frac{\partial f_i}{\partial p_k}$. Note that parameters p_0, p_1 and p_2 appear in f_3 and f_{10}, and parameters p_0 and p_1 appear in f_i, $i = 4 \ldots 9$. In addition, the signs of the partial derivatives $\frac{\partial f_i}{\partial p_0}$ and $\frac{\partial f_i}{\partial p_1}$ are similar. Therefore the set $\mathbb{Q}$ of discrete modes contains 2^{10} elements, but not all of them may be activated. Fig.2 shows the evolution of the discrete modes as obtained for the upper and lower hybrid automata used for bracketing the solutions of (35) as generated by the algorithms **Hybrid-Bound-Monotone**, when both initial state vec-

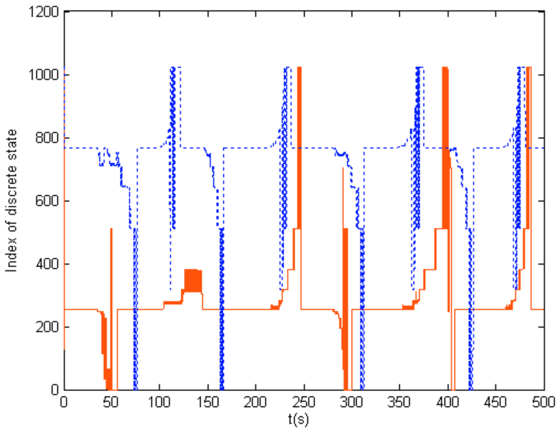

Fig. 2. Discrete mode evolution for the bounding systems automata, with $\mathbb{X}_{0_i} = [9, 11]°\mathrm{C}$ and $\mathbb{P}_0 = [0.73, 1.23]\mathrm{s}^{-1} \times [0.23, 0.64]\mathrm{mW}^{-1}\mathrm{K}^{-1}$. (continuous line: upper system, dash-dot: lower system)

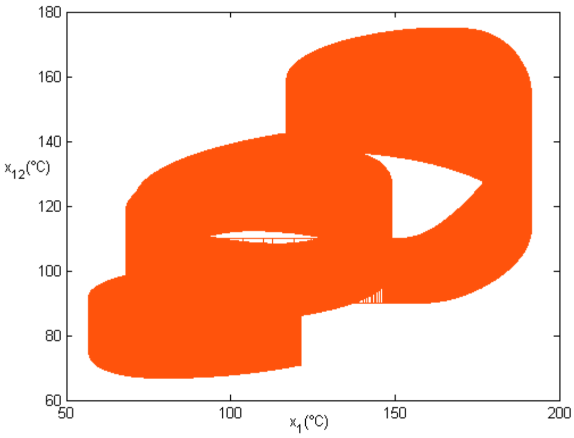

Fig. 3. Reachable space of (35). Projection onto $x_1 \times x_{12}$ subspace. (CPU time 67.8s PIV 2Ghz)

tor and parameter vector are taken uncertain with large uncertainties. Fig.3 shows the projection of the reachable space for (35) onto $x_1 \times x_{12}$ subspace. Obviously, even for very large parameter boxes the hybrid bracketing method does not diverge.

6. CONCLUSION

In this paper we have addressed the issue of computing the reachable space for uncertain nonlinear continuous dynamical systems. We have shown that reachable spaces can be computed via guaranteed set integration. A first approach uses interval Taylor models. It is capable of handling efficiently affine uncertain systems only with quite small uncertainty. A second approach applicable to monotone (cooperative) systems uses hybrid automata with no uncertainty as bounding systems. It can then handle uncertain nonlinear systems with inputs even in presence of large uncertainty in both initial state and parameter vector. Further work will address the extension of the second method to non-monotone systems.

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