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A Model Predictive Control Approach to Predict Sliding Surface

Gh. Montaseri*, M.J. Yazdanpanah*

**Control & Intelligent Processing Center of Excellence, School of Electrical and Computer Engineering, University of Tehran, Tehran, Iran, (e-mail:g.montaseri @ece.ut.ac.ir, yazdan@ut.ac.ir).*

Abstract: In this paper, a predictive control scheme for a class of nonlinear systems is proposed which combines the model predictive control (MPC) and sliding mode control (SMC).We call this new algorithm sliding mode model predictive control (SMMPC). In this algorithm the pre-designed switching surface is predicted via MPC strategy. First the system nonlinearity is handled by converting the state-dependent state-space representation into the linear time varying representation, and then this model is descritised. Finally the control sequence may be found by solving an open-loop optimal control problem in which the cost function weights the norm of pre-designed sliding surface and control law. Simulation results illustrate that the closed-loop system has desired properties such as robustness and chattering elimination.

1. INTRODUCTION

The model predictive control (MPC), also known as receding horizon control or moving horizon control, is becoming popular in the recent years and is used extensively in industry. It was first introduced by Richalet et al (Richalet et al., 1978). The technique has been further developed by Clarke (Clarke et al., 1987), Bitmead (Bitmead et al., 1991) and Ronald (Ronald, 1992). Extensions to the nonlinear case are due to Abu (Abu el Ata-Doss, 1992).

On the other hand, sliding mode control (SMC) is an effective robust control strategy since it appeared in 1950s, however, Chattering is its undesired phenomenon which can excite the high frequency oscillation of controlled system. Hence the application of the SMC is limited.

To solve this problem we propose an approach that uses an advantageous combination of SMC and MPC. In this approach closed-loop system obtains desired performance by choosing appropriate value for weighting matrices as tuning knobs in MPC cost function and chattering is eliminated due to an inexplicit use of “sign” term in the proposed control signal. In addition robustness is guaranteed because it is an inherent property of SMC.

In the proposed predictive control, first sliding surface is designed off-line to make the sliding mode asymptotically stable and also control law to steer the states to reach the sliding surface. Second the state-dependent state-space representation converts to the linear time varying representation, then the linear time varying state-space model is discretised using Euler integration method (Dutka, 2003). Finally, by predicting system states, the sliding surface is predicted and the control sequence can be found by solving an open-loop optimal control problem which the cost function weights the norm of pre-designed sliding surface and control law.

The remainder of this paper is organized as follow: Section 2 describes the nonlinear predictive control (Dutka, 2003). The new sliding mode model predictive algorithm is proposed in section 3. Section 4 gives the simulation results.

2. NONLINEAR PREDICTIVE CONTROL

Consider the following nonlinear continuous time system:

$$\dot{x}(t) = f(x(t)) + g(x(t))u(t) \quad , \quad y(t) = h(x(t)) \tag{1}$$

Suppose the system (1) is arranged into state and control dependent linear form of the state-space model, so the nonlinearity is handled by converting the state-dependent state-space representation into the linear time varying representation:

$$\begin{aligned} \dot{x}(t) &= A(x(t))x(t) + B(x(t))u(t) \\ y(t) &= C(x(t))x(t) \end{aligned} \tag{2}$$

Where the matrices elements may be constant or state-dependent. Since in MPC algorithm, N (prediction horizon) step ahead control signal is generated in each sampling time, system (2) should be discretised by using Euler integration method:

$$\begin{aligned} x_{n+1} &= A(x_n)x_n + B(x_n)u_n \\ y_n &= C(x_n)x_n \end{aligned} \tag{3}$$

In the model predictive control strategy, the vector of current and future control is calculated in each sampling time. The first element or current control is used as the plant input manipulation and the remaining parts are employed to predict future trajectory of the states of the system in the next sampling time, it is because of the fact that deriving the nonlinear predictive control algorithm is based on the assumption that the future trajectory of the states must be known.

For state-space model (3), the MPC cost function is defined as:

$$J_n = \sum_{i=1}^{N_y} (ref_{n+i} - y_{n+i})^T q_i (ref_{n+i} - y_{n+i}) + \sum_{i=1}^{N_u} (u_{n+i-1})^T r_i (u_{n+i-1}) \quad (4)$$

Where ref_n is a vector of set point of size N_y. $q_i, i = 1,..., N_y$ and $r_j, j = 1,..., N_u$ are symmetric weighting matrices, N_y and N_u are positive integer numbers, greater or equal to one.

Current and future values of the control input u_n and future values of states x_n and outputs y_n, construct the following vectors:

$$\begin{aligned} X_{n+1,N_y} &= [x^T_{n+1},..., x^T_{n+N_y}]^T \\ U_{n,N_u} &= [u^T_n,..., u^T_{n+N_u-1}]^T \\ Y_{n+1,N_y} &= [y^T_{n+1},..., y^T_{n+N_y}]^T \\ Ref_{n+1,N_y} &= [ref^T_{n+1},..., ref^T_{n+N_y}]^T \end{aligned} \quad (5)$$

Vector form of the cost function (4) with notation (5) is:

$$J_n = (Ref_{n+1,N_y} - Y_{n+1,N_y})^T Q\, (Ref_{n+1,N_y} - Y_{n+1,N_y}) + U^T_{n,N_u} R\, U_{n,N_u} \quad (6)$$

Where $Q = diag(q_1,..., q_{N_y})$ and $R = diag(r_1,..., r_{N_u})$. Next the future states prediction may be obtained from the following equation ($j = 1,..., N_y$ and $w = \min(j, N_u)$):

$$\begin{aligned} x_{n+j} = &[A_{n+j-1}A_{n+j-2}...A_n]x_n \\ &+ [A_{n+j-1}A_{n+j-2}...A_{n+1}]B_n u_n \\ &+ [A_{n+j-1}A_{n+j-2}...A_{n+2}]B_{n+1}u_{n+1} + ... \\ &+ [A_{n+j-1}A_{n+j-2}...A_{n+N_y}]B_{n-1+w}u_{n-1+w} \end{aligned} \quad (7)$$

Hence to obtain the state prediction at time instant $n + j$, the matrices $A_n...A_{n+j-1}$ and $B_n...B_{n-1+\min(j,N_u)}$ should be calculated. Now consider the following notation:

$$\left[\coprod_{k=L}^{m} A_{n+k}\right] \equiv \begin{cases} A_{n+m}A_{n+m-1}...A_{n+L} & \text{if} \quad L \le m \\ I & \text{if} \quad L > m \end{cases} \quad (8)$$

Then (7) is simplified by introducing (8):

$$\begin{aligned} x_{n+j} = &\left[\coprod_{k=0}^{j-1} A_{n+k}\right] x_n + \left[\coprod_{k=1}^{j-1} A_{n+k}\right] B_n u_n + \\ &+ \left[\coprod_{k=2}^{j-1} A_{n+k}\right] B_{n+1} u_{n+1} + ... + \\ &+ \left[\coprod_{k=N_u}^{j-1} A_{n+k}\right] B_{n-1+\min(j,N_u)} u_{n-1+\min(j,N_u)} \end{aligned} \quad (9)$$

From (5) and (9) the future states prediction vector is obtained:

$$X_{n+1,N_y} = \Omega_{n,N_y} A_n x_n + \Psi_{n,N_y} U_{n,N_u} \quad (10)$$

Where

$$\Omega_{n,N_y} = \left[\left[\coprod_{k=1}^{0} A_{n+k}\right]^T \left[\coprod_{k=1}^{1} A_{n+k}\right]^T ... \left[\coprod_{k=1}^{N_y-1} A_{n+k}\right]^T \right]^T$$

$$\Psi_{n,N_y} = \begin{bmatrix} \left[\coprod_{k=1}^{0} A_{n+k}\right] B_n & 0 & ... & 0 \\ \left[\coprod_{k=1}^{1} A_{n+k}\right] B_n & \left[\coprod_{k=2}^{1} A_{n+k}\right] B_{n+1} & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots \\ \left[\coprod_{k=1}^{N_y-1} A_{n+k}\right] B_n & \left[\coprod_{k=2}^{N_y-1} A_{n+k}\right] B_{n+1} & \cdots & \left[\coprod_{k=N_u}^{N_y-1} A_{n+k}\right] B_{n+N_u-1} \end{bmatrix}$$

From (3) it is clear that:

$$y_{n+j} = C_{n+j} x_{n+j} \quad (11)$$

By Combining (5) and (11), the following relationship between vectors X_{n+1,N_y} and Y_{n+1,N_y} is obtained:

$$Y_{n+1,N_y} = \Theta_{n,N_y} X_{n+1,N_y} \quad (12)$$

Where

$$\Theta_{n,N_y} = diag(C_{n+1}, C_{n+2},..., C_{n+N_y})$$

Finally replacing X_{n+1,N_y} in (12) by (10), the output prediction is obtained:

$$Y_{n+1,N_y} = \Phi_{n,N_y} A_n x_n + T_{n,N_y} U_{n,N_u} \quad (13)$$

Where

$$\Phi_{n,N_y} = \Theta_{n,N_y}\Omega_{n,N_y}, \quad T_{n,N_y} = \Theta_{n,N_y}\Psi_{n,N_y}$$

Substituting Y_{n+1,N_y} in the MPC cost function by (13), the control vector is calculated by minimizing (6):

$$U_{n,N_u} = (T^T_{n,N_y} Q\ T_{n,N_y} + R)^{-1} T^T_{n,N_y} Q \times (Ref_{n+1,N_y} - \Phi_{n,N_y} A_n x_n) \quad (14)$$

Note that as it was mentioned before, all elements of (14) are going to be used. The first element for the input manipulation of the controlled object, and the others for predicting the future states, to calculate the matrices Q and T in the next sampling time.

3. A NOVEL SLIDING MODE MODEL PREDICTIVE CONTROL (SMMPC)

Consider the system (3) again. Firstly designing sliding surface $s_n = s(x_n)$ to make the sliding mode asymptotically stable and finding variable structure control law $uvsc(x_n)$ to steer the states reach the sliding surface from the initial state in finite horizon.

For SMMPC the cost function is defined as (Zhou, 2001):

$$J_{SMMPC} = \sum_{i=1}^{N_s} s_{n+i}^T q_i s_{n+i} + \sum_{i=1}^{N_u} \{(u_{n+i-1} - uvsc_{n+i-1})^T r_i \times (u_{n+i-1} - uvsc_{n+i-1})\} \quad (15)$$

q_i and r_i are weighting matrices, N_s is the sliding surface prediction horizon and N_u is the control horizon. Minimizing cost function (15) means when the states reach the sliding surface, the control signal is equal to the pre-designed variable structure control law.

By introducing the following vectors:

$$\begin{aligned} X_{n+1,N_s} &= [x_{n+1}^T, \dots, x_{n+N_s}^T]^T \\ U_{n,N_u} &= [u_n^T, \dots, u_{n+N_u-1}^T]^T \\ S_{n+1,N_s} &= [s_{n+1}^T, \dots, s_{n+N_s}^T]^T \\ Uvsc_{n,N_u} &= [uvsc_n^T, \dots, uvsc_{n+N_u-1}^T]^T \end{aligned} \quad (16)$$

Now it is possible to write the cost function (15) in the following vector form:

$$J_{SMMPC} = S_{n+1,N_s}^T Q\, S_{n+1,N_s} + (U_{n,N_u} - Uvsc_{n,N_u})^T \times R(U_{n,N_u} - Uvsc_{n,N_u}) \quad (17)$$

Where $Q = diag(q_1, \dots, q_{N_s})$ and $R = diag(r_1, \dots, r_{N_u})$.

Before minimizing the cost function (17), vectors S_{n+1,N_s} and $Uvsc_{n,N_u}$ should be written in state and control dependent linear form. We do this in continue.

Suppose the pre-designed sliding surface can be arranged into the state-dependent linear form as:

$$s_n = c_n x_n \quad (18)$$

By repeating the sliding surface in the future sampling times, the sliding surface prediction vector is constructed:

$$S_{n+1,N_s} = \Theta_{n,N_s} X_{n+1,N_s} \quad (19)$$

Where

$$\Theta_{n,N_s} = diag(c_{n+1}, c_{n+2}, \dots, c_{n+N_s})$$

X_{n+1,N_s} can be expressed as (20) in the similar way of (10):

$$X_{n+1,N_s} = \Omega_{n,N_s} A_n x_n + \Psi_{n,N_s} U_{n,N_u} \quad (20)$$

Where Ψ_{n,N_s} and Ω_{n,N_s} are obtained by replacing N_y by N_s in (10).

By Replacing (20) in (19):

$$S_{n+1,N_s} = \zeta_{n,N_s} A_n x_n + \gamma_{n,N_s} U_{n,N_u} \quad (21)$$

Where

$$\varepsilon_{n,N_s} = \Theta_{n,N_s} \Omega_{n,N_s} \quad , \quad \gamma_{n,N_s} = \Theta_{n,N_s} \Psi_{n,N_s}$$

In continue we do the same procedure for $Uvsc_{n,N_u}$ in (17).

Consider the general form of variable structure control law:

$$uvsc_n = \alpha_n + \beta_n sign(s_n) \quad (22)$$

Where α_n and β_n are state-dependent components. Suppose (22) can be written as:

$$uvsc_n = e_n x_n \quad (23)$$

By repeating (23) in the future sampling times, the sliding mode variable structure control vector is constructed as:

$$Uvsc_{n,N_u} = E_{n+1,N_u} X_{n+1,N_s} + D_{n,N_u} x_n \quad (24)$$

Where

$$E_{n+1,N_u} = \begin{bmatrix} 0 & 0 & 0 & \cdots & \cdots & 0 \\ e_{n+1} & 0 & 0 & \cdots & \cdots & 0 \\ 0 & e_{n+2} & 0 & 0 & \cdots & 0 \\ 0 & 0 & e_{n+3} & 0 & 0 & 0 \\ \vdots & \vdots & 0 & \ddots & \vdots & 0 \\ 0 & 0 & 0 & \cdots & e_{n+N_u} & 0 \end{bmatrix}$$

$$D_{n,N_u} = [e_n, 0, 0, \cdots, \cdots, 0]^T$$

By replacing X_{n+1,N_s} from (20) in (24):

$$Uvsc_{n,N_u} = (\xi_{n,N} A_n + D_{n,N_u}) x_n + \varepsilon_{n,N_u} U_{n,N_u} \quad (25)$$

Where

$$\xi_{n,N} = E_{n+1,N_u} \Omega_{n,N_s} \quad , \varepsilon_{n,N_u} = E_{n+1,N_u} \Psi_{n,N_s}$$

Finally by replacing (21) and (25) in the cost function (17) and solving the minimization problem, the control input vector is obtained:

$$\begin{aligned} U_{n,N_u} = {} & [(I - \varepsilon_{n,N_u})^T R (I - \varepsilon_{n,N_u}) + \gamma_{n,N_s}^T Q\, \gamma_{n,N_s}]^{-1} \\ & \times [(I - \varepsilon_{n,N_u})^T R\, (\xi_{n,N} A_n + D_{n,N_u}) \\ & - \gamma_{n,N_s} Q\, \xi_{n,N_s} A_n]\, x_n \end{aligned} \quad (26)$$

By changing the matrices Q and R the performance is tuned. In order to make the cost function (17) richer, we add the new term as follow:

$$\begin{aligned} J_{SMMPC} = {} & S_{n+1,N_s}^T Q\, S_{n+1,N_s} + (U_{n,N_u} - Uvsc_{n,N_u})^T \times \\ & R \times (U_{n,N_u} - Uvsc_{n,N_u}) + \\ & (U_{n,N_u} - Upre_{n,N_u})^T P\, (U_{n,N_u} - Upre_{n,N_u}) \end{aligned} \quad (27)$$

Where $Upre_{n,N_u}$ is the control signal which is obtained in the previous sampling time and P is another tuning knob like R and Q. The control input according to (27) is:

$$U_{n,N_u} = \{(I-\varepsilon_{n,N_u})^T R (I-\varepsilon_{n,N_u}) + \gamma^T_{n,N_s} Q \gamma_{n,N_s} + P\}^{-1} \times \{[(I-\varepsilon_{n,N_u})^T R(\xi_{n,N} A_n + D_{n,N_u}) - \gamma_{n,N_s} Q \xi_{n,N_s} A_n] x_n + P \; Upre_{n-1,N_u}\} \quad (28)$$

In the next section results of the proposed algorithm are shown.

4. SIMULATION RESULTS

In order to illustrate the proposed algorithm, consider the following system (Khalil, 2003):

$$\begin{cases} \dot{x}_1 = x_2 + \theta_1 x_1 \sin(x_2) \\ \dot{x}_2 = \theta_2 x_2^2 + x_1 + u(t) \end{cases} , |\theta_1| < a \quad , \quad |\theta_2| < b \quad (27)$$

Where θ_1 and θ_2 are uncertain parameters and only their upper band (a and b) are known. Sliding surface and sliding mode variable structure control signal for (27) according to uncertainties θ_1 and θ_2, are designed:

$$\begin{aligned} s &= x_2 + (1+a)x_1 \\ uvsc &= -x_1 - (1+a)x_2 - \beta(x)\,\mathrm{sgn}(s) \\ \beta(x) &= a(1+a)|x_1| + bx_2^2 \end{aligned} \quad (28)$$

Suppose a=1 and b=2. System (1) is descritised with the sampling time $T_s = 0.01\,\mathrm{sec}$. In SMMPC algorithm, the sliding surface horizon and control horizon are equal to 7 and weighting matrices are constant and chosen as:

$$Q = \mathrm{diag}(0.1 \times [8500, 4000, 2000, 1000, 1000, 1000, 1000])$$

$$R = \mathrm{diag}(0.1 \times [1.2, 0.1, 0.1, 0.1, 0.1, 0.1, 0.1])$$

$$S = \mathrm{diag}(0.1 \times [2, 0, 0, 0, 0, 0, 0])$$

The results of the SMMPC algorithm are shown in Fig. 1. In order to make comparison, we plot the discrete SMC results simultaneously.

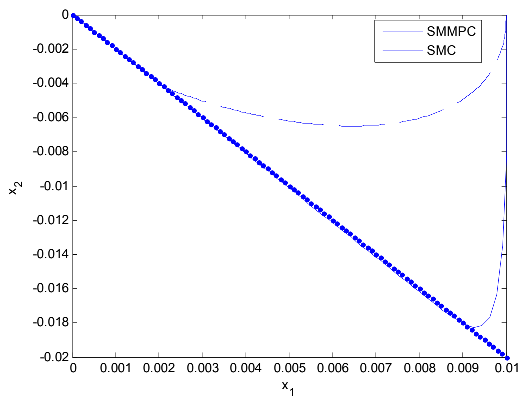

Fig. 1.a. State-space trajectory (Q, R and P are constant)

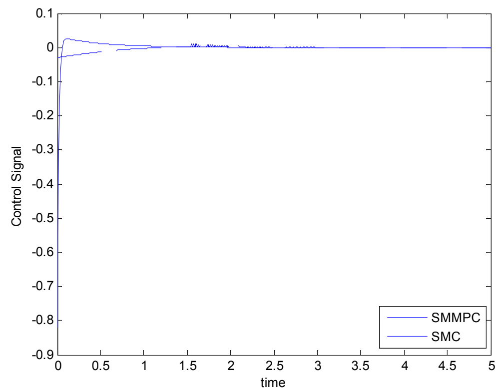

Fig. 1.b. Plot of control signal (Q, R and P are constant)

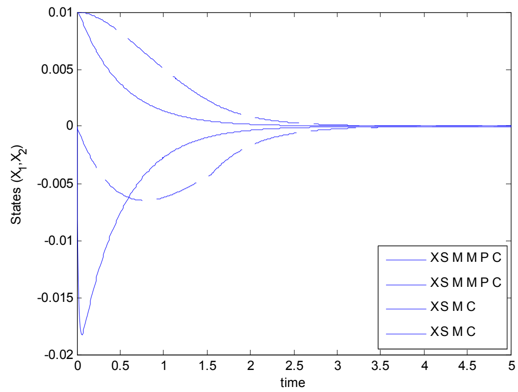

Fig. 1.c. Plot of States (Q, R and P are constant)

From the above simulation, we can see that the SMMPC algorithm can predict the sliding surface very fast and states regulate well, but the energy of control signal is 1.1021 in SMMPC and is 0.0284 in SMC. To decrease this energy in SMMPC, we do not select the constant value for weighting matrices and they change according to states distance from the sliding surface. At first Q is selected greater to force the states reach the sliding surface as fast as possible and after reaching the sliding surface, Q is decreased and we tune these matrices again to get the desired performance. The related simulation results are shown in Fig. 2.

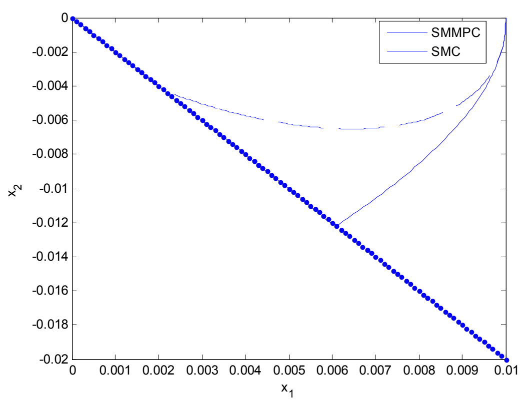

Fig. 2.a. State-space trajectory (Q, R and P are variable)

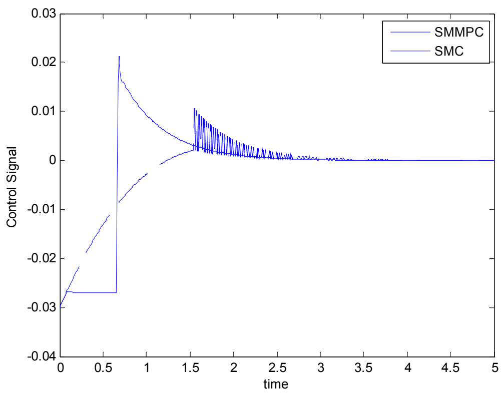

Fig. 2.b. Plot of control input (Q, R and P are variable)

In this case the energy of SMMPC decreases to 0.0572. Fig. 2 shows that there is a trade off between the sliding surface prediction speed and the quality of control signal. As a rule of thumb, variation in Q changes the sliding surface prediction speed and changes in R and P lead to changes in quality of a control signal.

The significant point in control signal is chattering elimination which is obvious in Fig. 1.b and Fig. 2.b. It is duo to an inexplicit presence of "sign" term (that causes chattering phenomenon) in control law which is obtained by (28).

To show that the proposed algorithm profits the robustness properties of sliding mode control method, the upper band of uncertain parameters θ_1 and θ_2 is increased:

$$|\theta_1| < 7 \quad , \quad |\theta_2| < 8$$

The simulation results are shown in Fig. 3:

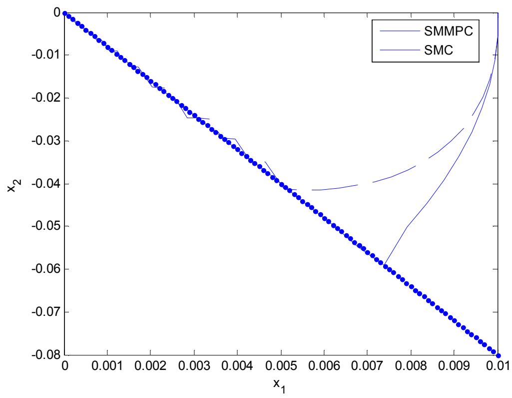

Fig. 3.a. State-space trajectory ($|\theta_1| < 7 \quad , \quad |\theta_2| < 8$)

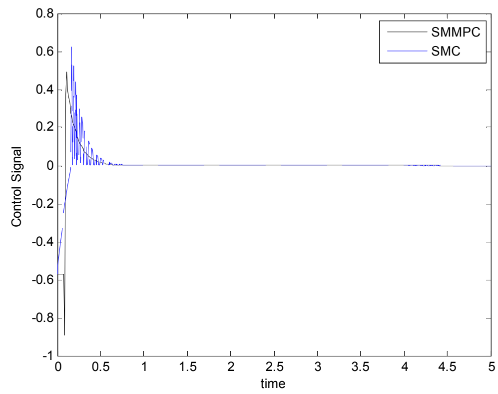

Fig. 3.b. Plot of control input ($|\theta_1| < 7 \quad , \quad |\theta_2| < 8$)

We can see that despite the increase in uncertainty bands of parameters, speed and quality of sliding surface prediction do not change in SMMPC algorithm, while chattering increases in SMC algorithm.

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