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Certainty Equivalence in Nonlinear Output Regulation with Unmeasurable Regulated Error ⋆

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Abstract: In the present paper we consider a general nonlinear output regulation problem in which the regulated error is unmeasurable. It is assumed that the interconnection of the controlled plant with the exosystem observed through the measured output satisfies some appropriate observability conditions that allow the design of an asymptotic observer. Then, the contribution of this paper consists in showing that in the latter scenario, a design based on certainty equivalence is effective for determining a controller that achieves semiglobal output regulation.

1. INTRODUCTION

A central problem in automatic control consists in controlling the output of a system so as to achieve asymptotic tracking of prescribed reference signals and asymptotic rejection of disturbances. In the context of time-invariant finite-dimensional systems, it can be assumed that the reference signals and the disturbances, which together are called exogenous inputs, are solutions of an autonomous system referred as exosystem. The problem with this setup has been coined as the output regulation problem (or servomechanism problem).

A general solution to the output regulation problem for linear systems has been presented in the works by Davison [1976] and by Francis and Wonham [1976]. A solution for nonlinear systems near an equilibrium point has been provided by Isidori and Byrnes [1990], Huang and Rugh [1990], and Huang and Lin [1994]. Under appropriate assumptions, the local results in Isidori and Byrnes [1990] and Huang and Lin [1994] have been extended to obtain solutions to the so called semiglobal output regulation problem which corresponds to considering arbitrary large compact sets of initial states (see Khalil [1994] , Isidori [1997], and Serrani et al. [2000]). Recently, some of the semiglobal results have been generalized to the case of nonlinear systems that do not necessarily posses equilibria, setting the so called "non-equilibrium theory" of output regulation (see Byrnes and Isidori [2003], Delli Priscoli et al. [2006], and Marconi et al. [2007]).

In most works on output regulation the regulated error is assumed to be part of the measurable output, but some contributions that are not based on such assumption are available in the literature. In the case of linear systems, when only tracking is pursued, design methods are presented in Hara and Sugie [1988] and in Sugie and Vidyasagar [1989], whereas when tracking and disturbance rejection with respect to constant exogenus inputs are pursued, design methods are available in Imai [1997] and Imai et al. [1998]. In Serrani [2006] and in Fiorentini et al. [2006] the authors consider output regulation problems for linear systems in which the measurement of the regulated error is corrupted by a harmonic disturbance. Similar scenarios for nonlinear passive systems and for Euler-Lagrange systems are investigated in Pisu et al. [2006], in Zarikian and Serrani [2007], and in Pisu and Serrani [2007].

In this paper, we consider a general nonlinear output regulation problem in a non equilibrium setting in which it is *not* assumed that the regulated error is measurable; on the other hand, it is assumed that the interconnection of the controlled plant with the exosystem observed through the measured output is diffeomorphic to a system into Gauthier-Kupka's observability canonical form (see Gauthier and Kupka [2001]); consequently, it is known how to design a high-gain observer for such interconnection. The contribution of this paper consists in showing that in the latter scenario, a design based on certainty-equivalence is effective for determining a controller that achieves semiglobal output regulation. The latter result generalizes in some directions what presented in Serrani [2006] as it will be better explained in the concluding section.

The rest of the paper is organized as follows. In Section 2 the output regulation problem is formulated and the main assumptions are stated; in Section 3 the certainty-equivalence regulator is introduced and its effectiveness is formally proved. Concluding remarks end the paper.

Notation. For $x \in \mathbb{R}^n$, $|x|$ denotes the Eucledian norm of x and, for $A \subseteq \mathbb{R}^n$, $|x|_A = \inf_{y\in A} |x - y|$ denotes the distance of x from A; $\text{int}(A)$ denotes the interior of A, and $\text{cl}(A)$ denotes its closure. For $B \in \mathbb{R}^{n\times n}$, $|B|$ denotes the 2-norm of B.

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Let (1) denote the number of the differential equation

$$\dot{x} = f(x, \mu) \quad (1)$$

where $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a function that is locally Lipschitz with respect to x, and where μ represents a parameter. We denote by $\psi_{(1)} : E \to \mathbb{R}^n$, $(t, x_0, \mu) \to \psi_{(1)}(t, x_0, \mu)$, the flow generated by (1). $\psi_{(1)}$ is defined on $E = \{(t, x_0, \mu) : x_0 \in \mathbb{R}^n, \mu \in \mathbb{R}^m, t \in (-t^-_{(1)}(x_0, \mu), t^+_{(1)}(x_0, \mu))\}$ where $t^-_{(1)}, t^+_{(1)} : \mathbb{R}^n \times \mathbb{R}^m \to (0, \infty]$, and $(-t^-_{(1)}(x_0, \mu), t^+_{(1)}(x_0, \mu))$ is the maximal interval of existence of the solution $\psi_{(1)}(\cdot, x_0, \mu)$.

Fix $\mu \in \mathbb{R}^m$; then, for $A \subseteq \mathbb{R}^n$ such that $t^+_{(1)}(x_0, \mu)) = \infty \ \forall x_0 \in A$, we denote by $\omega(A)$ the *ω-limit set of A* (under the flow $\psi_{(1)}(\cdot, \cdot, \mu)$) (see [Hale et al., 2002, pp. 7-8] and also [Byrnes et al., 2005, p. 317] for a definition of ω-limit set of a set).

Given a locally Lipschitz function $V : A \to \mathbb{R}$, with A open subset of $\mathbb{R}^n$, we denote by $DV^+_{(1)}$ the function $DV^+_{(1)} : A \times \mathbb{R}^m \to \mathbb{R}$ defined by

$$DV^+_{(1)}(x, \mu) = \limsup_{h \to 0^+} \frac{1}{h} \left[V(\psi_{(1)}(h, x, \mu)) - V(x) \right] . \quad (2)$$

Note that $DV^+_{(1)}(x, \mu)$ is equal to the upper Dini derivative of $g(t) = V(\psi_{(1)}(t, x, \mu))$ evaluated at $t = 0$. If V is continuously differentiable on A, then

$$DV^+_{(1)}(x, \mu) = \frac{\partial V}{\partial x}(x) f(x, \mu) \quad \forall (x, \mu) \in A \times \mathbb{R}^m , \quad (3)$$

and in this case we denote $DV^+_{(1)}(x, \mu)$ with $\dot{V}_{(1)}(x, \mu)$.

2. PROBLEM FORMULATION AND MAIN ASSUMPTIONS

Consider the problem of output regulation for the smooth system

$$\begin{aligned} \dot{x} &= f(x, u, w) \\ e &= h(x, w) \\ y &= k(x, w) \end{aligned} \quad (4)$$

in which $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}$ is the control input, $w \in \mathbb{R}^d$ is the exogenus input, $e \in \mathbb{R}$ is the regulated error, and $y \in \mathbb{R}$ is the measured output. The initial state of (4) $x(0)$ is unknown but ranges in a known arbitrary compact set $X \subseteq \mathbb{R}^n$. The exogenus input w is supposed to be generated by the smooth exosystem

$$\dot{w} = s(w) \quad (5)$$

whose initial state $w(0)$ is unknown but ranges in a known arbitrary compact *invariant* set $W \subseteq \mathbb{R}^d$.

Note that it is *not* assumed that $e = y$ i.e. e is measurable; as a result, the problem formulation departs from the standard framework of output regulation and includes, as a special case, the situation in which the measurement of the error is corrupted by a disturbance generated by the exosystem.

The regulator is modeled by equations of the form

$$\begin{aligned} \dot{\chi} &= \varphi(\chi, y) \\ u &= \rho(\chi, y) \end{aligned} \quad (6)$$

with φ and ρ locally Lipschitz, and with initial condition $\chi(0) \in \mathbb{R}^p$ ranging in a fixed (but otherwise arbitrary) compact set $\Delta \subset \mathbb{R}^p$.

Let

$$E = \{(x, w, \chi) : e = 0\} . \quad (7)$$

Controller (6) solves the problem of *semiglobal output regulation* if

- the positive orbit of $X \times W \times \Delta$ under the flow of (4), (5), and (6) is bounded;
- E (uniformly) attracts $X \times W \times \Delta$ under the flow of (4), (5), and (6) (see [Hale et al., 2002, p. 8] and also [Byrnes et al., 2005, p. 317] for a definition of (uniform) attractivity of a set with respect to another set).

We assume that when both x and w are measurable, a memoryless solution $u = u^*(x, w)$ to the proposed semiglobal output regulation problem is available; more precisely we make the following assumption.

Assumption 1. There exist a smooth function $u^*(x, w)$ such that system

$$\begin{aligned} \dot{x} &= f(x, u^*(x, w), w) \\ \dot{w} &= s(w) \end{aligned} \quad (8)$$

restricted to the locally invariant cylinder $\mathbb{R}^n \times W$ satisfies the following

- the positive orbit of $X \times W$ under the flow of (8) is bounded;
- let $\mathcal{A} = \omega(X \times W)$; then
$$\mathcal{A} \subseteq \{(x, w) : h(x, w) = 0\} ; \quad (9)$$
- $\mathcal{A}$ is locally asymptotically stable with a domain of attraction $\mathcal{D} \supset X \times W$.

In addition, we make an "observability" assumption; specifically, we assume that the interconnection of the controlled plant with the exosystem observed through the measured output can be trasformed into a system in Gauthier-Kupka's observability canonical form (see [Gauthier and Kupka, 2001, p. 22]).

Assumption 2. There exists a smooth global diffeomorphism

$$z = \phi(x, w) \quad z \in \mathbb{R}^{\tilde{n}} \quad (10)$$

where $\tilde{n} = n + d$ that carries system

$$\begin{aligned} \dot{x} &= f(x, u, w) \\ \dot{w} &= s(w) \\ y &= k(x, w) \end{aligned} \quad (11)$$

into the following Gauthier-Kupka's observability canonical form

$$\dot{z} = \begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \vdots \\ \dot{z}_{\tilde{n}-1} \\ \dot{z}_{\tilde{n}} \end{pmatrix} = \begin{pmatrix} F_1(z_1, z_2, u) \\ F_2(z_1, z_2, z_3, u) \\ \vdots \\ F_{\tilde{n}-1}(z_1, z_2, \ldots, z_{\tilde{n}}, u) \\ F_{\tilde{n}}(z_1, z_2, \ldots, z_{\tilde{n}}, u) \end{pmatrix} = F(z, u)$$

$$y = K(z_1) . \quad (12)$$

with F_i's such that

$$\frac{\partial F_i}{\partial z_{i+1}}(z_1, z_2, \ldots, z_{i+1}, u) \neq 0$$
$$\forall (z_1, z_2, \ldots, z_{i+1}, u) \in \mathbb{R}^{i+2} \quad i = 1, \ldots, \tilde{n} - 1 , \quad (13)$$

and with K such that

$$\frac{\partial K}{\partial z_1}(z_1) \neq 0 \quad \forall z_1 \in \mathbb{R} . \quad (14)$$

Remark 1. An explicit construction of a *local* change of coordinates that puts system (11) into Gauthier-Kupka's observability canonical form can be found in [Gauthier and Kupka, 2001, pp. 22-23]; for the construction of a *global* change of coordinates the interested reader is referred to [Marconi et al., 2004, Lemma 2], and, in the special case of input-affine systems, to [Isidori, 1995, pp. 460–464].

Denote with

$$e = H(z) \tag{15}$$

the regulated-error map in the z coordinates, and let Z, $\tilde{\mathcal{A}}$, and $\tilde{\mathcal{D}}$ be the images through ϕ of $X \times W$, $\mathcal{A}$, and $\mathcal{D}$ respectively; then, from (9)

$$H(z) = 0 \quad \forall z \in \tilde{\mathcal{A}} \, . \tag{16}$$

3. CERTAINTY-EQUIVALENCE REGULATOR

In this section a certainty-equivalence compensator that solves the given output regulation problem is proposed. In this regard observe that Assumption 1 states that if we can measure both x and w, then a memory-less controller $u^*(x, w)$ that solves the output regulation problem under consideration is available; moreover, if the trajectories of the interconnection of (4) and (5) stay in a compact set, and if the control u is bounded, then Assumption 2 implies that an asymptotic state observer for the latter interconnection can be designed (see [Gauthier and Kupka, 2001, pp. 95–101]). The proposed regulator is thus obtained replacing x and w in $u^*(x, w)$ with estimates provided by the asymptotic observer .

In order to introduce the regulator first we need to define a bound $l > 0$ on the amplitude of the control input u, and we need to introduce a compact and convex set $\Theta \subset \mathbb{R}^{\tilde{n}}$ which will be shown to contain the trajectories of system (12) that start from $Z \subset \mathbb{R}^{\tilde{n}}$ when the latter is controlled by the proposed regulator.

Denote by

$$(x, w) = \phi^{-1}(z) \tag{17}$$

the inverse map of (10), and define

$$\tilde{u}^*(z) = u^*(\phi^{-1}(z)) \, . \tag{18}$$

Consider the system

$$\dot{z} = F(z, \tilde{u}^*(z)) = \tilde{F}(z) \, , \tag{19}$$

and set

$$\eta(z) = \left(1 + \frac{1}{|z|_{\partial\tilde{\mathcal{D}}}}\right) |z|_{\tilde{\mathcal{A}}} \, . \tag{20}$$

By Assumption 1 and a simple adaptation of [Marconi et al., 2007, Theorem 4] it follows that there exists a continuous function $V : \tilde{\mathcal{D}} \to \mathbb{R}$ with the following properties

P1. there exist class $\mathcal{K}_\infty$ functions $\underline{a}$, $\overline{a}$ such that

$$\underline{a}(\eta(z)) \le V(z) \le \overline{a}(\eta(z)) \quad \forall \, z \in \tilde{\mathcal{D}} \, ; \tag{21}$$

P2. there exists $c > 0$ such that

$$D^+ V_{(19)}(z) \le -cV(z) \qquad \forall \, z \in \tilde{\mathcal{D}} \, ;$$

P3. for all $\alpha > 0$ there exists $L_\alpha > 0$ such that for all z_1, $z_2 \in \tilde{\mathcal{D}}$ such that $\eta(z_1) \le \alpha$, $\eta(z_2) \le \alpha$ the following holds

$$|V(z_1) - V(z_2)| \le L_\alpha |z_1 - z_2| \, .$$

Pick $b \ge 0$ such that

$$\Omega_b = \{z \in \tilde{\mathcal{D}} : V(z) \le b\} \supseteq Z \, . \tag{22}$$

Such b exists since V satisfies property P1. Choose l that satisfies

$$l \ge \max_{z \in \Omega_{b+1}} |\tilde{u}^*(z)| + 1 \, . \tag{23}$$

and define U as

$$U = \{u \in \mathbb{R} : |u| \le l\} \, . \tag{24}$$

Pick a compact and convex set Θ such that $\Theta \supseteq \Omega_{b+1}$; then, by [Gauthier and Kupka, 2001, p. 96] there exists a smooth $F^{gl} : \mathbb{R}^{\tilde{n}} \times \mathbb{R} \to \mathbb{R}^{\tilde{n}}$ with

$$F^{gl}(z, u) = \begin{pmatrix} F_1^{gl}(z_1, z_2, u) \\ F_2^{gl}(z_1, z_2, z_3, u) \\ \vdots \\ F_{\tilde{n}-1}^{gl}(z_1, z_2, \ldots, z_{\tilde{n}}, u) \\ F_{\tilde{n}}^{gl}(z_1, z_2, \ldots, z_{\tilde{n}}, u) \end{pmatrix} \, , \tag{25}$$

and there exists a smooth $K^{gl} : \mathbb{R} \to \mathbb{R}$ such that the following properties hold

P4. $F^{gl}(z, u) = F(z, u)$ and $K^{gl}(z_1) = K(z_1) \quad \forall (z, u) \in \Theta \times U$;

P5. denote by $\underline{z}_i'$ the vector $(z_1', \ldots, z_i') \in \mathbb{R}^i$ and by $\underline{z}_i''$ the vector $(z_1'', \ldots, z_i'') \in \mathbb{R}^i$, $i = 1, \ldots, \tilde{n}$; then, $\exists L > 0$ such that

$$|F_i^{gl}(\underline{z}_i', z_{i+1}', u) - F_i^{gl}(\underline{z}_i'', z_{i+1}'', u)| \le \frac{L}{\sqrt{\tilde{n}}} |\underline{z}_i' - \underline{z}_i''|$$
$$\forall (\underline{z}_i', z_{i+1}', \underline{z}_i'', z_{i+1}'', u) \in \mathbb{R}^{2i+3} \quad i = 1, \ldots, \tilde{n} - 1 \tag{26}$$

and

$$|F_{\tilde{n}}^{gl}(\underline{z}_{\tilde{n}}', u) - F_{\tilde{n}}^{gl}(\underline{z}_{\tilde{n}}'', u)| \le \frac{L}{\sqrt{\tilde{n}}} |\underline{z}_{\tilde{n}}' - \underline{z}_{\tilde{n}}''|$$
$$\forall (\underline{z}_{\tilde{n}}', \underline{z}_{\tilde{n}}'', u) \in \mathbb{R}^{2\tilde{n}+1} \tag{27}$$

P6. $\exists \, \alpha, \beta \in \mathbb{R}$ with $0 < \alpha < \beta$ such that

$$\alpha \le \left| \frac{\partial F_i^{gl}}{\partial z_{i+1}}(z_1, z_2, \ldots, z_{i+1}, u) \right| \le \beta$$
$$\forall (z_1, z_2, \ldots, z_{i+1}, u) \in \mathbb{R}^{i+2} \quad i = 1, \ldots, \tilde{n} - 1, \tag{28}$$

and

$$\alpha \le \left| \frac{\partial K^{gl}}{\partial z_1}(z_1) \right| \le \beta \quad \forall z_1 \in \mathbb{R} \, . \tag{29}$$

Consider then the system

$$\begin{aligned} \dot{z} &= F^{gl}(z, \sigma_l(u)) \\ y &= K^{gl}(z_1) \, , \end{aligned} \tag{30}$$

where σ_l is a saturation function defined by

$$\sigma_l(r) = \begin{cases} r & \text{if } |r| \le l \\ \mathrm{sgn(r)} l & \text{if } |r| > l \, . \end{cases}$$

Observe that as long as $z(t) \in \Theta$ and $u(t) \in U$ systems (12) and (30) are identical; moreover, note that an asymptotic observer for system (30) is given by

$$\dot{\hat{z}} = F^{gl}(\hat{z}, \sigma_l(u)) + G(y - K^{gl}(\hat{z}_1)) \tag{31}$$

where $G = D_g N$, $D_g = \mathrm{diag}(g, g^2, \ldots, g^{\tilde{n}})$, and $g \in \mathbb{R}$ and $N \in \mathbb{R}^{\tilde{n}}$ are design parameters that can be chosen so that global exponential converge of (31) is achieved (see [Gauthier and Kupka, 2001, pp. 95–101]). Then, the certainty-equivalence regulator proposed here is described by the following equations

$$\begin{aligned} \dot{\hat{z}} &= F^{gl}(\hat{z}, u) + G(y - K^{gl}(\hat{z}_1)) \\ u &= \sigma_l(\tilde{u}^*(\hat{z})) \, . \end{aligned} \tag{32}$$

The initial state of (32) $\hat{z}(0)$ is assumed to range on an arbitrary compact set $\hat{Z} \subset \mathbb{R}^{\tilde{n}}$. Then, the following main result holds.

Proposition 1. There exist $N \in \mathbb{R}^{\tilde{n}}$ and $g > 0$ such that (32) solves the given output regulation problem.

Proof. Consider the interconnection of (4), (5), and (32) and change coordinates (x, w) into $z = \phi(x, w)$ to obtain

$$\dot{z} = F(z, \sigma_l(\tilde{u}^*(\hat{z}))) \tag{33}$$

$$\dot{\hat{z}} = F^{gl}(\hat{z}, \sigma_l(\tilde{u}^*(\hat{z}))) + G(K(z_1) - K^{gl}(\hat{z}_1)) \tag{34}$$

$$e = H(z) \, . \tag{35}$$

Note that $z(0)$ ranges on the compact set Z previously introduced.

Consider the system

$$\begin{aligned} \dot{z} &= F^{gl}(z, \sigma_l(\tilde{u}^*(\hat{z}))) \\ \dot{\hat{z}} &= F^{gl}(\hat{z}, \sigma_l(\tilde{u}^*(\hat{z}))) + G(K^{gl}(z_1) - K^{gl}(\hat{z}_1)) \, , \end{aligned} \tag{36}$$

with $(z(0) \times \hat{z}(0)) \in Z \times \hat{Z}$, and observe that system (33) and (34) is identical to (36) on the set $\Theta \times \mathbb{R}^{\tilde{n}}$. In system (36) change coordinate $\hat{z}$ into

$$\epsilon = \hat{z} - z \, . \tag{37}$$

Proceeding as in [Gauthier and Kupka, 2001, pp. 99-100] write the resulting system in the form

$$\dot{z} = F^{gl}(z, \sigma_l(\tilde{u}^*(z + \epsilon))) \tag{38}$$

$$\dot{\epsilon} = (A(z, \epsilon) - GC(z, \epsilon))\epsilon + \bar{F}(z, \epsilon) \tag{39}$$

where A, C, and $\bar{F}$ satisfy the following properties

P7. there exists $\lambda > 0$, $N \in \mathbb{R}^{\tilde{n}}$, and $S \in \mathbb{R}^{\tilde{n} \times \tilde{n}}$, with S symmetric and positive definite, such that

$$(A(z, \epsilon) - NC(z, \epsilon))^T S + S(A(z, \epsilon) - NC(z, \epsilon)) \leq -\lambda I \quad \forall (z, \epsilon) \in \mathbb{R}^{\tilde{n}} \times \mathbb{R}^{\tilde{n}} \, ; \tag{40}$$

P8. $|\bar{F}(z, \epsilon)| \leq L|\epsilon| \;\; \forall (z, \epsilon) \in \mathbb{R}^{\tilde{n}} \times \mathbb{R}^{\tilde{n}}$.

Fix N as in property P7, and note that $\epsilon(0) = \hat{z}(0) - z(0)$ ranges on a certain compact set Γ. Change coordinate ϵ into

$$\tilde{\epsilon} = D_g^{-1} \epsilon \, , \tag{41}$$

so to obtain

$$\dot{z} = F^{gl}(z, \sigma_l(\tilde{u}^*(z + D_g \tilde{\epsilon})) \tag{42}$$

$$\dot{\tilde{\epsilon}} = g(A(z, D_g \tilde{\epsilon}) - NC(z, D_g \tilde{\epsilon}))\tilde{\epsilon} + D_g^{-1} \bar{F}(z, \tilde{D}_g \tilde{\epsilon}) \, . \tag{43}$$

Observe that if $g > 1$, $\epsilon(0) \in \Gamma \Rightarrow \tilde{\epsilon}(0) \in \Gamma$; then, in what follows we assume that $g > 1$ and $\tilde{\epsilon}(0) \in \Gamma$.

Set

$$\hat{F}(z, \tilde{\epsilon}, g) = F^{gl}(z, \sigma_l(\tilde{u}^*(z + D_g \tilde{\epsilon})) \, . \tag{44}$$

For all $(z, \tilde{\epsilon}, g) \in \Omega_{b+1} \times \mathbb{R}^{\tilde{n}} \times \mathbb{R}$ using a result from [Yoshizawa, 1975, p. 3] and property P3 of V, it is easy to obtain

$$\begin{aligned} D^+ V_{(42)}(z, \tilde{\epsilon}, g) &= \limsup_{h \to 0^+} \frac{1}{h} \left[V(\psi_{(42)}(h, z, \tilde{\epsilon}, g)) - V(z) \right] \\ &\leq \limsup_{h \to 0^+} \frac{1}{h} \left| V(z + h\hat{F}(z, \tilde{\epsilon}, g)) - V(z) \right| \\ &\leq \hat{L} |\hat{F}(z, \tilde{\epsilon}, g)| \end{aligned} \tag{45}$$

for some $\hat{L} > 0$. Thus, there exists $M > 0$ such that

$$D^+ V_{(42)}(z, \tilde{\epsilon}, g) \leq M \quad \forall (z, \tilde{\epsilon}, g) \in \Omega_{b+1} \times \mathbb{R}^{\tilde{n}} \times \mathbb{R} \, . \tag{46}$$

Then, it follows that there exists $T > 0$ independent of g such that, picked $(z(0), \tilde{\epsilon}(0)) \in Z \times \Gamma$, the corresponding solution $(z(t), \tilde{\epsilon}(t))$ of (42) and (43) is defined on $[0, T]$ and

$$z(t) \in \Omega_{b+\frac{1}{2}} \quad \forall t \in [0, T] \, . \tag{47}$$

In fact, from (46) we obtain

$$V(z(t)) - V(z(0)) \leq Mt \, . \tag{48}$$

Thus, setting $T = 1/(2M)$, (47) holds since $V(z(0)) \leq b$. Set $Q(\tilde{\epsilon}) = \tilde{\epsilon}^T S \tilde{\epsilon}$; mimicking the Proof of Theorem 2.2 in [Gauthier and Kupka, 2001, p. 99] it is easy to obtain that

$$\dot{Q}_{(43)}(z, \tilde{\epsilon}, g) \leq -(g\lambda - 2L|S|)|\tilde{\epsilon}|^2 \, . \tag{49}$$

Assume that g is large enough so that $g\lambda - 2L|S| > 0$; then, $\tilde{\epsilon}(t)$ is defined for all $t \in [0, T]$ and

$$|\tilde{\epsilon}(t)| \leq B e^{-a(g)t} |\tilde{\epsilon}(0)| \tag{50}$$

where

$$B = \left(\frac{|S|}{\lambda_{min}(S)} \right)^{\frac{1}{2}} \tag{51}$$

$$a(g) = \frac{g\lambda - 2L|S|}{2|S|} \, . \tag{52}$$

Note that since $g > 1$, the following holds

$$|\epsilon(t)| \leq |D_g| B e^{-a(g)t} |\epsilon(0)| = B g^{\tilde{n}} e^{-a(g)t} |\epsilon(0)| \, . \tag{53}$$

Consequently, for any $r > 0$, there exists $g^* > 1$ such that if $g > g^*$, then for any $\epsilon(0) \in \Gamma$ it occurs that $|\epsilon(T)| < r$; in addition, from the previous arguments it is immediate to derive the following property

P9. $\forall T' > T$ such that

$$z(t) \in \Omega_{b+1} \quad \forall t \in [T, T'] \, , \tag{54}$$

the following is satisfied

$$|\epsilon(t)| < r \quad \forall t \in [T, T'] \, . \tag{55}$$

The latter property will be useful in proving that the trajectories of (38) and (39) are bounded. In order to do so, it is convenient to introduce the following function

$$q(z, \epsilon) = F^{gl}(z, \sigma_l(\tilde{u}^*(z + \epsilon))) - \tilde{F}(z) \, . \tag{56}$$

We wish to show that there exists a class $\mathcal{K}_\infty$ function γ such that

$$|q(z, \epsilon)| \leq \gamma(|\epsilon|) \quad \forall (z, \epsilon) \in \Omega_{b+1} \times \mathbb{R}^{\tilde{n}} \, . \tag{57}$$

To this purpose, set

$$\tilde{q}(z, \epsilon) = F^{gl}(z, \tilde{u}^*(z + \epsilon)) - \tilde{F}(z) \, , \tag{58}$$

and observe that since $\tilde{q}$ is a smooth function that vanishes at $\epsilon = 0$, then there exists a class $\mathcal{K}_\infty$ function $\tilde{\gamma}$ such that

$$|\tilde{q}(z, \epsilon)| \leq \tilde{\gamma}(|\epsilon|) \quad \forall (z, \epsilon) \in \Omega_{b+1} \times \mathbb{R}^{\tilde{n}} \, . \tag{59}$$

For any $z \in \Omega_{b+1}$, let $\mathcal{E}_z$ denote the set of all $\epsilon \in \mathbb{R}^{\tilde{n}}$ such that

$$q(z, \epsilon) = \tilde{q}(z, \epsilon) \, . \tag{60}$$

Then

$$|q(z, \epsilon)| \leq \tilde{\gamma}(|\epsilon|) \quad \forall (z, \epsilon) \in \Omega_{b+1} \times \mathcal{E}_z \, . \tag{61}$$

In view of the specific choice of l (see (23)), $\epsilon = 0$ belongs to the interior of $\mathcal{E}_z$; moreover, there exists $\zeta > 0$ such that

$$|q(z,\epsilon)| \leq \zeta \quad \forall (z,\epsilon) \in \Omega_{b+1} \times \mathbb{R}^{\tilde{n}} \; . \tag{62}$$

Thus, it is possible to find a class $\mathcal{K}_\infty$ function γ that renders (57) fulfilled.

Observe that for all $(z,\epsilon) \in \Omega_{b+1} \times \mathbb{R}^{\tilde{n}}$, using a result from [Yoshizawa, 1975, p. 3] together with properties P1 and P2 of V, and using equation (57), it is easy to obtain

$$\begin{aligned} D^+_{(38)}(z,\epsilon) &= \limsup_{h\to 0^+} \frac{1}{h}\left[V(\psi_{(38)}(h,z,\epsilon)) - V(z)\right] \\ &= \limsup_{h\to 0^+} \frac{1}{h}\left[V(z + h(\tilde{F}(z) + q(z,\epsilon))) - V(z)\right] \\ &\leq \limsup_{h\to 0^+} \frac{1}{h}\Big|V(z + h(\tilde{F}(z) + q(z,\epsilon))) \\ &\quad - V(z + h\tilde{F}(z))\Big| \\ &\quad + \limsup_{h\to 0^+} \frac{1}{h}\left[V(z + h\tilde{F}(z)) - V(z)\right] \\ &\leq \tilde{L}|q(z,\epsilon)| - cV(z) \leq \tilde{L}\gamma(|\epsilon|) - cV(z) \end{aligned} \tag{63}$$

for some $\tilde{L} > 0$. Pick $0 < \delta < b$ and assume that $r > 0$ is small enough so that

$$\tilde{L}\gamma(r) - c\delta < 0 \; . \tag{64}$$

Let

$$\mathcal{S} = \{(z,\epsilon) : \delta \leq V(z) \leq b+1, \; |\epsilon| \leq r\} \; . \tag{65}$$

From previous equations, it is easy to check that

$$D^+_{(38)}(z,\epsilon) < 0 \quad \forall (z,\epsilon) \in \mathcal{S} \; . \tag{66}$$

We have already shown that the solution $(z(t),\epsilon(t))$ of (38) and (39) is defined on $[0,T]$, that $z(t) \in \text{int}(\Omega_{b+1})$ $\forall t \in [0,T]$, and that $|\epsilon(T)| \leq r$. Then, using (66) and property P9 it is easy to show that $(z(t),\epsilon(t))$ is defined $\forall t \geq 0$, $z(t) \in \Omega_{b+1}$ $\forall t \geq 0$, and $|\epsilon(t)| \leq r \;\; \forall t \geq T$. In addition, (53) implies that

$$\lim_{t\to\infty} \epsilon(t) = 0 \; . \tag{67}$$

By standard arguments based on properties of the ω-limit set of $z(t)$ (see [Isidori, 1999, p.148]), it can be proved that in a finite time $z(t)$ enters Ω_δ. Bearing in mind that l has been fixed as in (23), then it is easy to obtain that for $r > 0$ small enough and $\forall t \geq T$ system (38) and (39) restricted to $\Omega_{b+1} \times \{\epsilon \in \mathbb{R}^{\tilde{n}} : |\epsilon| \leq r\}$ is equivalent to system

$$\dot{z} = \tilde{F}(z) + \tilde{q}(z,\epsilon) \tag{68}$$

$$\dot{\epsilon} = (A(z,\epsilon) - GC(z,\epsilon))\epsilon + \bar{F}(z,\epsilon) \; . \tag{69}$$

Then, applying a simple modification of [Marconi et al., 2007, Lemma 1] to system (68), and taking into account that $\delta > 0$ can be chosen arbitrarily small, it is easy to obtain that

$$\lim_{t\to\infty} |z(t)|_{\tilde{\mathcal{A}}} = 0 \; . \tag{70}$$

Next, observe that $(z(t), z(t)+\epsilon(t))$ is the solution of (36) that starts from $(z(0), z(0)+\epsilon(0))$; however, since it has been shown before that $z(t) \in \Omega_{b+1} \subseteq \Theta$ $\forall t \geq 0$, it occurs that $(z(t), z(t)+\epsilon(t))$ is also the solution of (33) and (34) that starts from the same initial state. Then, from (67) and (70) it is easy to derive that $\omega(Z \times \hat{Z})$ under the flow of (33) and (34) is equal to $\tilde{\mathcal{A}} \times \tilde{\mathcal{A}}$; in addition, by [Hale et al., 2002, Lemma 2.0.1] the latter set (uniformly) attracts $Z \times \hat{Z}$. Since $\tilde{\mathcal{A}} \times \tilde{\mathcal{A}} \subseteq \{(z,\hat{z}) : H(z) = 0\}$ (see (16)), then the proposition is proved. $\triangleleft$

4. CONCLUSION

In this paper we have shown that given a nonlinear output regulation problem in which the regulator error is not measurable, if appropriate observability conditions are satisfied, then semiglobal output regulation can be obtained through a certainty-equivalence controller. The latter result represents an extension in some directions of what presented in Serrani [2006]. In both papers certainty-equivalence controllers are proposed; however, here nonlinear systems are considered instead of linear systems; in addition, in Serrani [2006] it is assumed that what makes the regulated error unmeasurable is exclusively the presence of an unmeasurable harmonic disturbance that is additive at the controller's input whereas in this paper we do not restrict to only such scenario. On the other hand, while the controller presented in Serrani [2006] is able to tackle with parametric uncertainties in the exosystem, the regulator proposed here might not be able to do so. Moreover, both the compensator presented here and the one in Serrani [2006] leave open the question of robustness with respect to uncertainties in the plant.

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