Proceedings of the 17th World Congress
The International Federation of Automatic Control
Seoul, Korea, July 6-11, 2008

Preserving System Performance During Feedback Failure

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Abstract: The problem of maintaining acceptable performance of a perturbed control system under conditions of feedback failure is considered. The objective is to maximize the time during which performance remains within desirable bounds without feedback, given that the parameters of the controlled system are within a specified neighborhood of their nominal values. It is shown that there is an optimal open-loop controller that achieves this objective. The performance of this controller can be approximated by a bang-bang controller.

1. INTRODUCTION

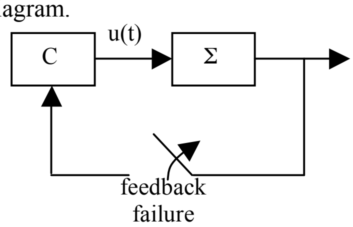
Feedback is often used to ameliorate the adverse effects of perturbations and uncertainties on the performance of engineering systems. However, discontinuations in the service of the feedback channel cannot be completely avoided. For example, in applications such the control and guidance of space vehicles, disruptions of the line-of-sight may cause extended disruptions in feedback signal reception. In other applications, economic or other factors may dictate an operational policy wherein the feedback channel is opened only when performance degrades beyond an acceptable level. One such application is in networked control systems, where feedback is often used only intermittently, so as to reduce network traffic (Nair et al. 2007, Zhivogyladov and Middleton 2003 and Montestruque and Antsaklis 2004).

To accommodate these and other applications, it would be beneficial to develop a controller that maximizes the duration of time during which a system can operate without feedback and not exceed acceptable error bounds. Of course, the parameters of the controlled system are never perfectly known, and this fact has to be taken into consideration when attempting to develop an appropriate control strategy.

In more specific terms, we concentrate on the case of a linear time-invariant system Σ with given initial conditions with state output. We assume that the feedback signal is disconnected at the time $t = 0$. Let Σ_0 be the nominal version of Σ, and let Σ_ε be the system that results when the parameters of Σ experience a perturbation ε from their nominal values. The exact value of the perturbation ε is not known, but it is known that ε does not exceed a specified bound d. After possibly having applied an appropriate shift transformation on the signals, we assume that the desired nominal output of Σ is the zero signal. A maximal deviation of magnitude $M > 0$ is permitted from the nominal output signal. Our objective is to find an input signal $u(t)$ that drives the system Σ_ε in such a way as to guarantee that the deviation of the output signal from zero stays below M for as long as possible, irrespective of the (unknown) deviation ε from nominality. In somewhat more formal terms, we are seeking a signal $u(t)$ and a maximal time t_f such that

$$|\Sigma_\varepsilon u(t)| \leq M \text{ for all } 0 \leq t \leq t_f \text{ and all } \varepsilon \leq d. \tag{1}$$

The signal $u(t)$ is generated by a controller, according to the following diagram.



Here, the controller C acts as a precompensator during feedback failure, generating the signal $u(t)$ that keeps the output of Σ within its bound for the longest possible time. To the best of our knowledge, there are no reports in the literature of earlier examinations of this problem.

Finding the signal $u(t)$ helps maintain proper operation of Σ for as long as possible after feedback failure, providing maximal time for repair. If the feedback is disconnected for cost reasons, then the input signal $u(t)$ can help minimize operational costs by maximizing the time during which the system Σ can operate without feedback. Of course, at the time t_f, feedback must be restored to prevent further increase of the error.

We show in section 2 that the problem of calculating the optimal signal $u(t)$ is a max-min problem; in section 3 we prove that this problem has a solution; and in section 4 we show that the optimal signal $u(t)$ can be replaced by a bang-bang signal, with only a negligible effect on system performance. The fact that the optimal signal can be replaced by a bang-bang signal is significant in applications, since the calculation of optimal bang-bang signals is rather simple: all one needs to do is find the switching times. The paper concludes in section 5 with an example. Finally, we mention that the methodology developed in this paper can be extended to the treatment of partial feedback failure.

2. NOTATIONS AND PROBLEM FORMULATION

Consider a linear time invariant continuous-time system given by a realization of the form

$$\Sigma : \dot{x}(t) = A'x(t) + B'u(t), \qquad x(0) = x_0. \tag{2}$$

Here, A' and B' are constant real matrices of dimensions $n \times n$ and $n \times m$, respectively. The state $x(t)$ of the system is available

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10.3182/20080706-5-KR-1001.3272

as output; the initial state x_0 is the state of Σ at the time feedback was lost, and thus is known. The input function $u(t)$ is of dimension m.

An important aspect of our discussion is the fact that there are uncertainties about the entries of the matrices A' and B'. To describe these uncertainties, we use the standard ℓ^∞-norm $||\bullet||$ given, for a $q\times r$ matrix G by

$||G|| := \max_{i=1,...,q;j=1,...,r} |G_{ij}|$,

where G_{ij} is the i, j entry of the matrix G. Similarly, for a vector valued function $u(t)$ having the components $u_1(t), ..., u_m(t)$, we set $||u(t)|| = \max_{i=1,...,m} |u_i(t)|$ at each time t. Now, given a real number $d > 0$, let Δ_A be the set of all real $n\times n$ matrices α satisfying $||\alpha|| \le d$, and let Δ_B be the set of all real $n\times m$ matrices β satisfying $||\beta|| \le d$. In other words, Δ_A and Δ_B are the set of all corresponding size matrices whose entries are in the range $[-d, d]$. Then, we set

$$A' := A + D_A,\ B' := B + D_B, \text{ and } D := (D_A,D_B), \quad (3)$$

where A and B are the nominal values of the matrices A' and B' of (2), respectively, while $D_A \in \Delta_A$ and $D_B \in \Delta_B$ are the perturbation matrices that represent uncertainties. The only information available about our system Σ are the nominal matrices A and B and the uncertainty magnitude d; the entries of the matrices D_A and D_B are not given. We use the notation $D := (D_A,D_B)$ and $\Delta := \Delta_A\times\Delta_B$, so that $D \in \Delta$. We refer to Δ as the *uncertainty range*. We note that controllability of the pair (A',B') is not required for the results presented in this paper. However, when the feedback *is* connected, controllability would be required to stabilize the system and reduce errors.

Recalling the bound $M > 0$ of (1) and taking into consideration the fact that the output of our system Σ is its state $x(t)$, our performance requirement becomes

$$x^T(t)x(t) \le M \quad (4)$$

where x^T is the transpose of x. The initial state satisfies $x_0^Tx_0 \le M$, so that performance was within the desirable range when the feedback channel was disconnected.

It is convenient to define a weighted inner product over our space of input functions, as follows. Given two m-dimensional vector valued functions $a(t), b(t)$, set

$$<a, b> = \int_0^\infty e^{-\alpha t} a(t)^T b(t)dt, \qquad ||a||_{2,\alpha} := \sqrt{<a, a>} \quad (5)$$

where α is a positive real number, the integral is taken in the Lebesgue sense, and $||a||_{2,\alpha}$ is the corresponding norm. The weight function $e^{-\alpha t}$ comes to allow us to include all bounded input functions in the domain over which the inner product (5) is well defined. We denote by $L_2^{\alpha,m}$ the Hilbert space of all m-dimensional Lebesgue measurable functions with the inner product (5).

Needless to say, all engineering systems are subject to input amplitude restrictions determined by the largest amplitude signal a system can tolerate. For our system Σ, we assume that its input amplitude bound is $K > 0$, so all input functions $u(t)$ of Σ must satisfy $||u(t)|| \le K$ for all t, and thus are members of the Hilbert space $L_2^{\alpha,m}$. It is convenient to introduce the set of input functions

$$U := \{u \in L_2^{\alpha,m} : ||u(t)|| \le K \text{ for all } t \ge 0\}, \quad (6)$$

which describes all permissible input functions of Σ. In these terms, our objective is to find an input function $u \in U$ that drives Σ so as to satisfy the state amplitude bound (4) for the longest possible time.

2.1 Problem Statement

The state trajectory $x(t)$ of the system Σ depends, of course, on the perturbation matrices D_A and D_B, as well as on the input function u. It is convenient, for the moment, to include these variables in explicit form in the function x, namely, to write $x(t,D,u)$ instead of $x(t)$, where $D = (D_A,D_B)$. Rewrite now (4) in the form

$$x^T(t,D,u)x(t,D,u) \le M. \quad (7)$$

Next, the time duration during which the square-magnitude $x^T(t)x(t)$ stays below or at the bound M can be represented by

$$T(M,D,u) := \inf \{t \ge 0 : x^T(t)x(t) > M\}, \quad (8)$$

where $T(M,D,u) := \infty$ if $x^T(t)x(t) \le M$ for all $t \ge 0$. As the initial state satisfies $x_0^Tx_0 \le M$, we have $T(M,D,u) \ge 0$. Our objective is to select the input function u so as to obtain the largest possible duration $T(M,D,u)$.

Among the variables of the state $x(\bullet)$, the entries of the pair of matrices D are unknown and unpredictable. As there is no feedback available, the control input function u cannot depend on D. We must guaranty that the bound (7) is valid for all possible D. As a result, we must consider the "worst case" with respect to the pair of matrices D, and this leads us to the quantity

$$T^*(M,u) := \inf_{D\in\Delta} T(M,D,u). \quad (9)$$

Then, inequality (7) is valid for all $t \in [0, T^*(M,u)]$, irrespective of the entries of D. The duration $T^*(M,u)$ still depends on the input function u, and we can choose any input function in the set U of (6). The best choice will, of course, be an input function u that maximizes $T^*(M,u)$, yielding the maximal duration

$$t_f^* := \sup_{u\in U} T^*(M,u). \quad (10)$$

Assuming that such an input function exists, let us denote it by u^*, so that $t_f^* = T^*(M,u^*)$. In this notation, our objectives can be formally phrased as follows.

Problem 1: (i) Determine whether or not an input function $u^* \in U$ exists, and (ii) if there is such a function u^*, describe a method for its computation. ♦

As we can see from (9) and (10), the calculation of the input function u^* involves the solution a max-min optimization problem. In the next section, we show that an optimal solution u^* exists within our framework. Then, in section 4, we show that this optimal input function can be replaced by a bang-bang function, without appreciably affecting performance. Bang-bang functions are relatively easy to compute and work with in engineering environments.

3. EXISTENCE OF AN OPTIMAL INPUT FUNCTION

In this section, we prove the existence of an optimal input function $u^*(t)$. The proof proceeds in two phases: first, we show that the set U of (6) has a certain compactness feature; then, we show that the function $T^*(M,u)$ has an appropriate continuity property. The existence of the optimal input function $u^*(t)$ within U follows then from the well known fact that a continuous functional over a compact set achieves is maximum within the set. We start by reviewing a few notions from analysis (e.g., Liusternik and Sobolev 1961).

Definition 1: Let H be a Hilbert space with inner product $<\bullet,\bullet>$. (**i**) A sequence $\{x_n\}$ in H *converges weakly* to an element $x \in H$ if $\lim_{n\to\infty} <x_n,y> = <x,y>$ for every element $y \in H$. (**ii**) A subset W of H is *weakly compact* if every sequence of elements of W has a subsequence that converges weakly to an element of W. (**iii**) A sequence $\{z_n\} \subset H$ is *strongly convergent* if there is an element $z \in H$ such that $\lim_{n\to\infty} <z_n-z,z_n-z> = 0$. (**iv**) A set S is *strongly closed* if every strongly convergent sequence of elements of S has its limit in S. ♦

We are ready now for the first phase of our proof regarding the existence of an optimal input function.

Lemma 1: The set U of (6) is weakly compact in the topology of the Hilbert space $L_2^{\alpha,m}$.

Proof: By (6), the set U is a bounded set. Recall Alaoglu's theorem, which states that every bounded sequence in Hilbert space contains a weakly convergent subsequence (e.g., Halmos 1982, pgs. 14 and 180). Hence, every sequence of elements of U has a subsequence that is weakly convergent to an element of $L_2^{\alpha,m}$. To prove weak compactness, we need to show that this element is a member of U. In fact, we will show that U is weakly closed, namely, that every weakly convergent sequence of elements of U has its limit in U. We will utilize Mazur's theorem, which states that a bounded and strongly closed convex set in Hilbert space is also weakly closed (e.g., Halmos 1982, pg. 180).

In preparation for applying Mazur's theorem, note first that U is convex. Indeed, given two Lebesgue measurable functions $v, w \in U$, we have, by the definition of U, that $||v(t)|| \le K$ and $||w(t)|| \le K$ for all t. Then, for a number $0 \le a \le 1$, the function $z(t) := av(t) + (1-a)w(t)$ is clearly Lebesgue measurable, and $||z(t)|| \le a||v(t)|| + (1-a)||w(t)|| \le K$. Whence, $w(t) \in U$, and U is a convex set.

To show that U is also strongly closed, let $u^n \in U$, $n = 1, 2, ...$, be a strongly convergent sequence of functions with the limit u, namely, $\lim_{n\to\infty} <u - u^n, u - u^n> = 0$. Assume, by contradiction, that $u \notin U$. Being the limit of a sequence of Lebesgue measurable functions, u is Lebesgue measurable as well. But then, considering (6), the relation $u \notin U$ implies that there is a Lebesgue measurable subset δ of the time axis such that $||u(t)|| \ge K + \varepsilon$ for all $t \in \delta$, where $\varepsilon > 0$ and δ has non-zero measure. As $u(t)$ is a vector of dimension m, it further follows that there is an integer $1 \le i \le m$ and a measurable subset $\delta_i \subset \delta$ of non-zero measure, such that the i-th component $u_i(t)$ of $u(t)$ satisfies

$$|u_i(t)| - K \ge \varepsilon \text{ for all } t \in \delta_i. \tag{11}$$

Now, calculating the norm of the difference $u - u^n$, we get

$$<u - u^n, u - u^n> = \int_0^\infty e^{-\alpha t}[u(t)-u^n(t)]^T[u(t)-u^n(t)]dt \ge$$
$$\int_{\delta_i} e^{-\alpha t}[u(t) - u^n(t)]^T[u(t) - u^n(t)]dt \ge$$
$$\int_{\delta_i} e^{-\alpha t}(u_i(t) - u_i^n(t))^2 dt, \tag{12}$$

where $u_i^n(t)$ is the i-th component of the function $u^n(t)$. Then, since $u^n \in U$, we have that $||u^n(t)|| \le K$ for all t, so that $|u_i^n(t)| \le K$ for all t as well. Thus, $|u_i(t) - u_i^n(t)| \ge |u_i(t)| - |u_i^n(t)| \ge |u_i(t)| - K$. Using (11), this entails that $|u_i(t) - u_i^n(t)| \ge \varepsilon$ for all $t \in \delta_i$. Substituting into (12) yields

$$<u - u^n, u - u^n> \ge \int_{\delta_i} e^{-\alpha t}(u_i(t) - u_i^n(t))^2 dt \ge \int_{\delta_i} e^{-\alpha t}\varepsilon^2 dt > 0,$$

for all $n = 1, 2, ...$, contradicting the fact that the sequence $\{u^n\}$ is strongly convergent. Thus, $u \in U$, and the Lemma's assertion follows by Mazur's theorem. ♦

In the present note, we focus on cases when the controlled system Σ is *nominally unstable*, namely, on cases when the nominal matrix A has an eigenvalue with positive real part. For such systems, the state trajectory $x(t)$ must escape the bound M for at least one perturbation matrix D, as follows. For each t, we denote $||x(t)|| = \max_{i=1,\dots,n} |x_i(t)|$.

Lemma 2: Assume that the system Σ of (2) is nominally unstable and has a non-zero initial state. Then, for each input function $u(t) \in U$ and for every uncertainty range Δ, there is a perturbation matrix $D \in \Delta$ for which $T(M,D,u) < \infty$.

Proof: Let x_0 be the initial condition of the system Σ, and, referring to (3), denote by $x^0(t) := e^{(A+D_A)t}x_0$ the zero input response of Σ for the perturbation matrix $D_A \in \Delta_A$. Then, $\dot{x}^0(t) = (A+D_A)x^0(t)$. We show first that there is a matrix $D_A \in \Delta_A$ for which the norm $||x^0(t)||$ approaches infinity as $t \to \infty$. Indeed, by assumption, the nominal matrix A has at least one eigenvalue with positive real part. Consequently, there is a similarity transformation $A^+ := PAP^{-1}$ that brings A into the block diagonal form

$$A^+ = \begin{pmatrix} A_s & 0 \\ 0 & A_u \end{pmatrix},$$

where A_u is an $n_u \times n_u$ matrix all of whose eigenvalues have strictly positive real parts, and A_s is an $n_s \times n_s$ matrix whose eigenvalues have non-positive real parts (possibly, $n_s = 0$). Now, define the vector $z^0(t) := Px^0(t)$. Then, $z^0(t)$ satisfies the differential equation $\dot{z}^0(t) = A^+z^0(t)$. We claim that, for every real number $\varepsilon > 0$, there is an $n \times n$ matrix E that satisfies the following:

(i) The equation $\dot{z}(t) = (A^+ + E)z(t)$ has a divergent solution, where $z(0) = z^0(0)$; and

(ii) $||E|| \le \varepsilon$.

To prove this claim note that $z(0) = Px(0) = Px_0 \ne 0$, since the initial state x_0 of Σ is not zero by the Lemma's assumption and the matrix P is non-singular. Partition $z(t)$:

$$z(t) = \begin{pmatrix} p(t) \\ q(t) \end{pmatrix},\ z(0) =: \begin{pmatrix} p_0 \\ q_0 \end{pmatrix},$$

where $p(t)$ has n_s components and $q(t)$ has n_u components. Then, for $E = 0$, we have

$$\dot{z}(t) = \begin{pmatrix} \dot{p}(t) \\ \dot{q}(t) \end{pmatrix} = \begin{pmatrix} A_s & 0 \\ 0 & A_u \end{pmatrix}\begin{pmatrix} p(t) \\ q(t) \end{pmatrix}, \text{ so that}$$
$$\begin{pmatrix} p(t) \\ q(t) \end{pmatrix} = \begin{pmatrix} \exp(A_s t) & 0 \\ 0 & \exp(A_u t) \end{pmatrix}\begin{pmatrix} p_0 \\ q_0 \end{pmatrix}.$$

Now, if $q_0 \ne 0$, then $||q(t)|| \to \infty$ as $t \to \infty$, since all eigenvalues of A_u have positive real parts. Hence $||z(t)|| \to \infty$ as $t \to \infty$ in this case, and our claim is valid for $E = 0$.

Otherwise, we have $q_0 = 0$. Then, since $z(0) \ne 0$, we must have $p_0 \ne 0$. Note that the matrix A_u, having no zero eigenvalues by construction, is invertible. Let $e > 0$ be a real number, and consider the similarity transformation induced by the matrix

$$Q := \begin{pmatrix} I & 0 \\ eI & I \end{pmatrix}, \text{ where } Q^{-1} := \begin{pmatrix} I & 0 \\ -eI & I \end{pmatrix}.$$

Define the function $y(t) := Qz(t)$, and partition

$$y(t) = \begin{pmatrix} y_s(t) \\ y_u(t) \end{pmatrix}.$$

As $p_0 \neq 0$ and $q_0 = 0$, we have that $y_u(0) = ep_0 \neq 0$. Applying the similarity transformation, we get the matrix

$$A' := Q\begin{pmatrix} A_s & 0 \\ 0 & A_u \end{pmatrix}Q^{-1} = \begin{pmatrix} A_s & 0 \\ e(A_s-A_u) & A_u \end{pmatrix}.$$

Adding to A' the perturbation matrix

$$D'_A := \begin{pmatrix} 0 & 0 \\ -e(A_s-A_u) & 0 \end{pmatrix},$$

we obtain the differential equation

$$\dot{y}(t) = \begin{pmatrix} A_s & 0 \\ 0 & A_u \end{pmatrix}y(t).$$

Thus, $y_u(t)$ satisfies the equation $\dot{y}_u(t) = A_u y_u(t)$, or $y_u(t) = \exp(A_u)y_u(0)$. In view of the fact that $y_u(0) \neq 0$ and all eigenvalues of A_u have strictly positive real parts, we obtain that $||y_u(t)|| \to \infty$ as $t \to \infty$. Thus, $||y(t)|| \to \infty$ as $t \to \infty$, and, considering that the matrix Q is invertible, we conclude that $||z(t)|| \to \infty$ as well.

Returning to the original coordinate system, we need to apply the perturbation $E := Q^{-1}D'_A Q$ to the matrix A^+ to achieve the same effect. Considering the forms of Q and Q^{-1}, it follows that $e > 0$ can be selected to satisfy

$$e||Q^{-1}\begin{pmatrix} 0 & 0 \\ -(A_s-A_u) & 0 \end{pmatrix}Q|| \leq \varepsilon,$$

and E fulfils our claim in the zero input case.

Finally, consider the effect of an input function $u(t) \in U$. Upon including the input in the differential equation and denoting the solution by $z'(t)$, we obtain

$$\dot{z}'(t) = (A^+ + E)z'(t) + B^+u(t), \tag{13}$$

where $z'(t) = Px(t)$ and $B^+ := PB$. Then,

$$||z'(t)|| = ||Px(t)|| \leq ||P||_\infty ||x(t)|| \tag{14}$$

where $||\bullet||_\infty$ denotes the matrix norm induced by the ℓ^∞-norm on $x(t)$. Since P is non-singular, $||x(t)||$ approaches infinity when $t \to \infty$, if the same is true for $z'(t)$.

Now, if $||z'(t)|| \to \infty$ as $t \to \infty$ for the current input function $u(t)$, then the Lemma assertion is satisfied by $D := (P^{-1}EP,0)$. Otherwise, if $||z'(t)||$ is bounded for all t, recall that the solution of (13) has the form

$$z'(t) = z(t) + \exp[(A^+ + E)t]\int_0^t \exp[-(A^+ + E)\tau]B^+u(\tau)d\tau,$$

where $\lim_{t\to\infty} z(t) = \infty$ for the current D_A. Defining

$$\phi(t) := \exp[(A^+ + E)t]\int_0^t \exp[-(A^+ + E)\tau]B^+u(\tau)d\tau,$$

we can write $z'(t) = z(t) + \phi(t)$. Now, choose a real number $\delta > 0$ for which $||\delta B^+|| \leq \varepsilon$, and consider the perturbed matrix $B' := B^+ + \delta B^+ = (1 + \delta)B^+$. With the matrix B', we obtain the solution $z''(t) = z(t) + (1+\delta)\phi(t) = z'(t) + \delta\phi(t) = z'(t) + \delta[z'(t) - z(t)] = (1+\delta)z'(t) - \delta z(t)$,

so that $||z''(t)|| \geq \big|(1+\delta)||z'(t)|| - \delta||z(t)||\big|$. Using the facts that $\lim_{t\to\infty} ||z'(t)|| < \infty$, that $\lim_{t\to\infty} ||z(t)|| = \infty$, and that $\delta > 0$, we conclude that $\lim_{t\to\infty} ||z''(t)|| = \infty$. Finally, since $z''(t) = Px(t)$ and P is invertible, we obtain from (14) that $\lim_{t\to\infty} ||x(t)|| \geq \lim_{t\to\infty} || z''t)||/||P||_\infty = \infty$. Thus, the Lemma is valid for the perturbation $D := (P^{-1}EP,\delta P^{-1}B^+)$, where $\varepsilon > 0$ and $\delta > 0$ can be selected as small as desired. ♦

In view of Lemma 2, there is always a disturbance matrix D for which the escape time $T(M,D,u)$ is finite. Thus, the smallest of these escape times, $T^*(M,u)$ (see (9)), must be finite. We obtain then the following.

Corollary 1: Assume that the system Σ of (2) is unstable and has a non-zero initial state. Then, for every input function $u(t) \in U$ and for every uncertainty range Δ, one has $T^*(M,u) < \infty$. ♦

We turn now to part (i) of Problem 1, where we need to show that there is an input functional $u^*(t) \in U$ that maximizes $T^*(M,u)$. Considering that the set U is, in a sense, compact, and that a continuous functional always attains its maximum in a compact set, the existence of $u^*(t)$ will follow if we can show that $T^*(M,u)$ is continuous in an appropriate sense. In fact, the following rather weak form of continuity is sufficient for this.

Definition 2. A functional F is *weakly upper semi-continuous* if the following is true for every weakly convergent sequence $z_n \xrightarrow{w} z$: whenever $F(z)$ is bounded, there is for every $\varepsilon > 0$ an integer $N > 0$ such that $F(z_n) - F(z) < \varepsilon$ for $n > N$. ♦

In the next two statements, we show that $T^*(M,u)$ has an appropriate continuity feature.

Lemma 3: For a given perturbation matrix $D \in \Delta$, the function $T(M,D,u)$ of (8) is weakly upper semi-continuous in u.

Proof: Let $x(t,u)$ be the solution of the state equation (2) for the perturbation matrix D and the input function u at the time $t < \infty$. Consider a weakly convergent sequence of input functions $u_1, u_2, ..., \in U$, say $u_n \xrightarrow{w} u$. We claim that the sequence of vectors $x(t,u_1), x(t,u_2), ...$ converges pointwise to the vector $x(t,u)$. Indeed, write

$$x(t,u) = e^{A't}[x_0 + \int_0^t e^{-A'\tau}B'u(\tau)d\tau] = e^{A't}[x_0 + \int_0^\infty \rho(\tau)e^{-A'\tau}B'u(\tau)d\tau]$$

where x_0 is the initial condition, and

$$\rho(\tau) := \begin{cases} 1 & \text{if } \tau \leq t, \\ 0 & \textit{otherwise.} \end{cases}$$

Then, the difference

$$v(t,u) := x(t,u) - e^{A't}x_0 = e^{A't}\int_0^\infty \rho(\tau)e^{-A'\tau}B'u(\tau)d\tau$$

is a linear functional of u. Recalling that weak convergence implies convergence of every linear functional of the sequence, we conclude that $\lim_{n\to\infty} v(t,u_n) = v(t,u)$ for every $t < \infty$. But then, since $x(t,u) = v(t,u) + e^{A't}x_0$, it follows that $\lim_{n\to\infty} x(t,u_n) = x(t,u)$ for every $t < \infty$.

Next, consider the following functional defined over state trajectories:

$$\Theta(x) = \inf \{t \geq 0 : x^T(t)x(t) > M\}, \tag{15}$$

where $\Theta(x) := \infty$ if $x^T(t)x(t) \leq M$ for all $t \geq 0$. Let $x_1(t), x_2(t)$,

... be a sequence of state trajectories that converges to the function $x(t)$ for each $t \geq 0$, and assume that $\Theta(x)$ is bounded. We show that, for any $\varepsilon > 0$, there is an integer $N > 0$ that satisfies the following condition: $\Theta(x_n) - \Theta(x) < \varepsilon$ all integers $n > N$.

Clearly, if there is an integer $N > 0$ for which $\Theta(x_n) \leq \Theta(x)$ for all $n > N$, then our claim is true. So let us examine the case when there is no such N. In such case, there is a divergent sequence of integers $i(1), i(2), \ldots$ such that $\Theta(x_{i(n)}) > \Theta(x)$ for all integers $n > 0$. Set $T^x := \Theta(x)$; since $\Theta(x)$ is bounded by assumption, we have $T^x < \infty$. By (15), the following is true for every real number $\varepsilon > 0$: there is a time $t' \in [T^x, T^x+\varepsilon)$ such that $x^T(t')x(t') > M$.

Now, by assumption, we have that $x_n(t) \rightarrow x(t)$ pointwise for every $t \geq 0$. Consequently, we also have that $\lim_{n\to\infty} x_n^T(t)x_n(t) = x^T(t)x(t)$ for every $t \geq 0$. Therefore, setting $t = t'$, there must be an integer $N > 0$ such that $|x_n^T(t')x_n(t') - x^T(t')x(t')| < [x^T(t')x(t') - M]/2$. For such n, we have $x_n^T(t')x_n(t') = x^T(t')x(t') + [x_n^T(t')x_n(t') - x^T(t')x(t')] \geq x^T(t')x(t') - [x^T(t')x(t') - M]/2 \geq x^T(t')x(t')/2 + M/2 > M$, i.e., $x_n^T(t')x_n(t') > M$. By the last inequality, $\Theta(x_n) \leq t'$; whence $\Theta(x_n) < \Theta(x) + \varepsilon$ for all $n > N$, and $\Theta(x)$ is upper semi-continuous.

We turn now to the functional $T(M,D,u)$ of (8). Denote by $\Sigma_t(D,u)$ the solution $x(t)$ of (2). Then, we clearly have the composition $T(M,D,u) = \Theta(\Sigma_t(D,u))$. For the weakly convergent sequence of input functions $u_n \xrightarrow{w} u$, we have shown pointwise convergence of the sequence $\Sigma_t(D,u_n)$ for every t. Combining this with the upper semi-continuity of Θ just shown, it follows that $T(M,D,u)$ is weakly upper semi-continuous in u. ♦

Lemma 4: Assume that the system Σ of (2) is unstable and has a non-zero initial state. Then, the function $T^*(M,u)$ of (9) is weakly upper semi-continuous in u.

Proof: Our proof is based on the following general fact: Let S and A be two topological spaces, and let f_α be a weakly upper semi-continuous real valued function on S for each element $\alpha \in A$. If $\inf_{\alpha\in A} f_\alpha(x)$ exists at each point $x \in X$, then the function $f(x) := \inf f_\alpha(x)$ is weakly upper semi-continuous on X (e.g., Willard 1970, p. 49). Now, in view of Lemma 3, the function $T(M,D,u)$ is weakly upper semi-continuous on U for each $D \in \Delta$. Furthermore, since Σ is unstable and has a non-zero initial state, it follows by Lemma 2 that $\inf_{D\in\Delta} T(M,D,u) < \infty$, so that the infimum exists for every $u \in U$. Thus, by the fact quoted at the beginning of this proof, $T^*(M,u) := \inf_{D\in\Delta} T(M,D,u)$ is weakly upper semi-continuous in u. ♦

We are ready now to state the main result of this section, namely, the existence of an input function that maximizes the time during which our perturbed system's state remains within a specified error bound. This resolves Problem 1(i).

Theorem 1: Assume that the system Σ of (2) is unstable and has a non-zero initial state, and let $T^*(M,u)$ be given by (9). Then, the following are valid.

(i) There is a maximal time $t_f^* := \sup_{u\in U} T^*(M,u) < \infty$, and

(ii) There is an input function $u^* \in U$ satisfying $t_f^* = T^*(M,u^*)$.

Proof: The set U is weakly compact by Lemma 1 and, by Lemma 4, the functional $T^*(M,u)$ is weakly upper semi-continuous in u over U. Consequently, we can apply the generalized Weierstrass Theorem (e.g., Zeidler 1985, pg. 152), which states the following in our current terminology: A weakly upper semicontinuous function attains a maximum on a weakly compact set. Hence, $T^*(M,u)$ attains a maximum over the set of inputs U.♦

In conclusion, we have shown that, after a feedback failure occurs, there is an optimal input function $u^*(t)$ that keeps the open loop response below a specified error bound for a duration of at least t_f^*, irrespective of the perturbation matrices. While driven by the optimal input function $u^*(t)$, the actual duration of time t_f during which the system's response remains below the specified error bound depends, of course, on the particular perturbation matrix D present in the system. However, for all permissible perturbation matrices, $t_f \geq t_f^*$, and there is a perturbation matrix for which $t_f = t_f^*$.

We turn now to the issue of calculating and implementing an optimal input function $u^*(t)$. In the next section, we show that performance close to optimal performance can be achieved by using a bang-bang input function. Bang-bang functions are relatively easy to calculate and implement, since everything is determined by their switching times.

4. BANG-BANG APPROXIMATIONS

To obtain a bang-bang approximation of the optimal input function $u^*(t)$, we have to soften slightly our optimization requirements. Recall that our objective is to control the system Σ of (2) in the presence of feedback failure, subject to the perturbations described by (3). The optimal input function $u^*(t)$ keeps the state trajectory of Σ below the bound M for the longest possible time t_f^* that is compatible with all perturbations $D \in \Delta$. To obtain a convenient approximation of the optimal input function, we allow the state trajectory of Σ to slightly exceed the bound M. Specifically, let $x^*(t,D)$ be the state trajectory of Σ generated by the optimal input function $u^*(t)$ for a particular uncertainty $D \in \Delta$. We are looking for a bang-bang input function $u^\pm(t)$ for Σ that generates a state trajectory $x^\pm(t,D)$ that deviates only slightly from $x^*(t,D)$ for all $t \in [0, t_f^*]$ for all $D \in \Delta$. The next statement indicates that such an input function can be found.

Theorem 2. Let Σ be the system of Theorem 1 and let t_f^* be the optimal time of Theorem 1(i). Then, for every $\varepsilon > 0$, there is a bang-bang input function $u^\pm \in U$ for which the following are true.

(i) $u^\pm$ has a only finite number of switches, and

(ii) The state trajectory $x^\pm(t,D)$ of Σ created by $u^\pm$ satisfies $||x^*(t,D) - x^\pm(t,D)|| < \varepsilon$ for all $t \in [0, t_f^*]$ and all $D \in \Delta$.

Proof. Fix a real number $\varepsilon > 0$. Recall that all input functions $u(t)$ of Σ are bounded by K, that $t_f^* < \infty$ by Theorem 1, and that all perturbation matrices $D \in \Delta$ have entries of magnitude not exceeding $d > 0$. Let $\eta > 0$ be a real number (to be chosen later), and recall that $A' = A + D_A$ and $B' = B + D_B$, where $D_A \in \Delta_A$ and $D_B \in \Delta_B$. Due to the uniform continuity of the function $e^{A't}$, there is a real number $\delta(\eta) > 0$ such that the function $\mu(t',t) := e^{-A't'} - e^{-A't}$ satisfies $||\mu(t',t)|| \leq \eta$ for all $t', t \in [0, t_f^*]$ satisfying $|t' - t| < \delta(\eta)$. Also, let $\beta := \sup \{||B + D_B|| : D_B \in \Delta_B\}$ and let $N := \sup \{e^{A't} : D_A \in \Delta_A, t \in [0, t_f^*]\}$; here, N exists due the fact that all involved quantities are bounded. Let $0 < \gamma \leq \delta(\eta)$ be any number for which t_f^*/γ is an integer. We build a partition of the interval $[0, t_f^*]$ into segments of length γ, namely, the partition determined by the points $0, \gamma, 2\gamma, \ldots$

Recalling that the input function $u(t)$ of Σ is an m-dimensional vector with each component bounded by K, we define a bang-bang input function $u^{\pm}(t)$ through its components $u_1^{\pm}(t), u_2^{\pm}(t), ..., u_m^{\pm}(t)$ as follows: for each component $i = 1, 2, ..., m$, we select in each interval $[q\gamma, (q+1)\gamma]$ a switching time θ_{qi}, $q = 0, 1, 2, ..., i = 1, 2, ..., m$, and set

$$u_i^{\pm}(t) := \begin{cases} +K & \text{for } t \in [q\gamma, \theta_{qi}), \\ -K & \text{for } t \in [\theta_{qi}, (q+1)\gamma), \end{cases}$$

where the value of θ_{qi} is selected to satisfy the equality

$$\int_{q\gamma}^{(q+1)\gamma} u_i^*(\tau)d\tau = K\int_{q\gamma}^{\theta_{qi}} d\tau - K\int_{\theta_{qi}}^{(q+1)\gamma} d\tau = K[2(\theta_{qi} - q\gamma) - \gamma].$$

Note that a solution θ_{qi} exists for all $q = 1, 2, ...$ and all $i = 1, 2, ..., m$ due to the fact that $|u_i^*(t)| \le K$ for all $t \ge 0$. Then, we obtain the equality

$$\int_{q\gamma}^{(q+1)\gamma} [u_i^*(\tau) - u_i^{\pm}(\tau)]d\tau = 0,\ q = 1, 2, ... \tag{16}$$

Finally, let $x^{\pm}(t)$ be the state function generated by the system Σ when driven by the input function $u_i^{\pm}(t)$, and let $x^*(t)$ be the trajectory induced by the optimal input function $u^*(t)$. Noting that the perturbation matrix D is the same in both cases (we are activating the same system sample), one obtains (using (16))

$$||x^*(t) - x^{\pm}(t)|| =$$

$$= ||e^{A't}[x_0 + \int_0^t e^{-A'\tau}B'u^*(\tau)d\tau] - e^{A't}[x_0 + \int_0^t e^{-A'\tau}B'u^{\pm}(\tau)d\tau]|| =$$

$$= ||e^{A't}\int_0^t e^{-A'\tau}B'[u^*(\tau) - u^{\pm}(\tau)]d\tau|| \le N||\int_0^t e^{-A'\tau}B'[u^*(\tau) - u^{\pm}(\tau)]d\tau||$$

$$= N||[\sum_{r=0}^{q-1}\int_{r\gamma}^{(r+1)\gamma} e^{-A'\tau}B'[u^*(\tau) - u^{\pm}(\tau)]d\tau + \int_{q\gamma}^{t} e^{-A'\tau}B'[u^*(\tau) - u^{\pm}(\tau)]d\tau||$$

$$\le N\{||\sum_{r=0}^{q-1}[e^{-A'r\gamma}B'\int_{r\gamma}^{(r+1)\gamma}[u^*(\tau) - u^{\pm}(\tau)]d\tau +$$

$$\int_{r\gamma}^{(r+1)\gamma}\mu(\tau, r\gamma)B'[u^*(\tau) - u^{\pm}(\tau)]d\tau]|| + ||\int_{q\gamma}^{t} e^{-A'\tau}B'[u^*(\tau) - u^{\pm}(\tau)]d\tau||\} \le$$

$$\le N\{\sum_{r=0}^{q-1}\int_{r\gamma}^{(r+1)\gamma}||\mu(\tau, r\gamma)||\ ||B'||\ [||u^*(\tau)|| + ||u^{\pm}(\tau)||]d\tau +$$

$$+ \int_{q\gamma}^{t}||e^{-A'\tau}||\ ||B'||\ [||u^*(\tau)|| + ||u^{\pm}(\tau)||]d\tau\} \le 2KN\beta[\eta t_f^* + N\gamma].$$

We choose now the value of η so that $2KN\beta\eta t_f^* < \varepsilon/2$. Then, we choose $0 < \gamma \le \min\{\delta(\eta), \varepsilon/(4KN^2\beta)\}$ so that t_f^*/γ is an integer. For these selections, we obtain $||x^*(t) - x^{\pm}(t)|| < \varepsilon$ for all $t \in [0, t_f^*]$, and our proof concludes. ♦

Note that the approximation holds for all permissible perturbation matrices, and it is independent of the perturbation. The cost of making ε smaller is an increase in the number of switches of the bang-bang function $u^{\pm}(t)$. A more detailed discussion of the results presented in this note, as well as a characterization of the conditions under which the optimal input function $u^*(t)$ is itself a bang-bang function, is provided in Chakraborty and Hammer (2008).

5. EXAMPLE

Consider the one-dimensional system $\dot{x}(t) = ax(t) + u(t)$, where the time constant a is subject to the uncertainty $1.2 \le a \le 1.4$. The system has the input bound $|u(t)| \le 2$ for all t, and the initial condition $x(0) = 1$. The objective is to find an input function $u^*(t)$ that keeps the state amplitude below the bound $x^2(t) \le 1.96$ for the longest period of time, irrespective of the value a adopts within its uncertainty range. The optimal input is shown in the left plot, and the corresponding state trajectories for different values of a are plotted on the right. We can see that the optimal input is not a bang-bang function (the upper value of $u(t)$ in the graph is not 2).

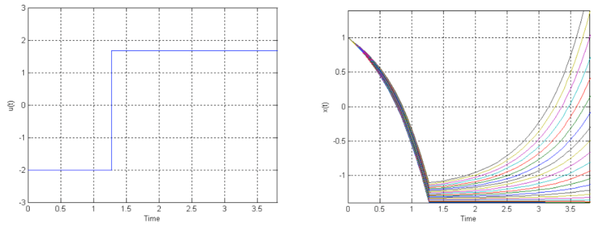

Figure 1: Optimal input has one switch: $M = 1.96$, $t_f = 3.7$

A bang-bang input approximation with 16 switches is shown in the left plot below. As we can see, the state trajectories plotted on the right are close to the optimal ones.

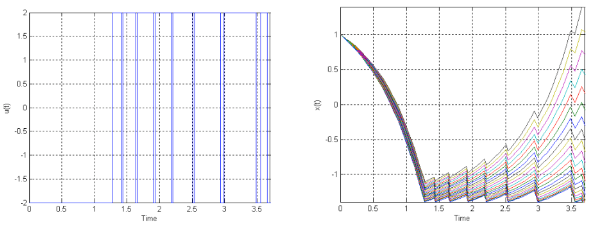

Figure 2: Approximate bang-bang input: 16 switches

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