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# Fixed-Order Controller Design for Polytopic Systems Using Rank Deficiency in a Sylvester Matrix

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**Abstract:** Fixed-order controller design for LTI-SISO polytopic systems is investigated using rank deficiency constraint on the controller Sylvester resultant matrix. It is shown that the non-convexity of fixed-order controller design problem can be contracted in a rank deficiency constraint on Sylvester resultant matrix of the controller. Then, an improved convex approximation of the rank deficiency constraint is used that leads to a convex fixed-order controller design problem represented via LMIs. The effectiveness of the proposed method is shown by applying it to an experimental system.

## 1. INTRODUCTION

Fixed-order controller design has been always a challenging problem for control engineers and has attracted many recent researchers. The research is motivated by the real-time implementation of systems with a very high sampling rate, where the fast computation of the command is crucial and also by many other practical applications, such as embedded control systems for the space and aeronautics industries, where the simplicity of the code and the hardware are of great importance.

Several analytical solutions are available in the literature (see e.g. Levine and Athans (1970)). However, the main difficulty in these results is that they are not computationally efficient. It means that there do not exist fast and reliable methods to compute optimal fixed order controllers. The main problem stems from the fundamental algebraic property that the stability domain in the space of polynomial's parameters is non-convex for polynomials with order higher than two (Ackermann, 1993). This problem can be formulated as Bilinear Matrix Inequality (BMI) (Safonov et al., 1994) that has been shown to be generally NP-hard (Toker and Ozbay, 1995). Several works have been accomplished to solve the non-convex problem in special cases. In Hol et al. (2003), a non-linear algorithm is adopted which converges, under mild conditions, to a point that satisfies only the first order necessary conditions of optimality. Another way to solve the non-convex problem is to gather all the non-convexity in a rank constraint (Henrion et al., 1999; Gahinet and Apkarian, 1994). However, instead of solving the non-convex problem, many designers prefer to solve a suboptimal convex problem such as Wang and Chow (2000); Henrion et al. (2003); Shiau and Tseng (2004); Khatibi et al. (2007), where Strictly Positive Realness (SPRness) is used as a key point to develop an inner convex approximation of the non-convex set of the parameters of all stabilizing controllers.

The problem of fixed-order controller design becomes more complicated when a fixed-order controller should stabilize a model with the structured parametric uncertainty. In Henrion et al. (2003), a fixed-order stabilizing controller for a polytopic system is parameterized via LMIs that originate from the positivity in polynomials, based on fixing a so-called central polynomial. The same approach is followed by Khatibi et al. (2007), where robust regional pole-placement is performed by a proper choice of the central polynomial. However, the mentioned approaches suffer from the conservatism imposed by fixing the central polynomial and thus if the proposed optimization problem becomes infeasible, it is not possible to conclude that there does not exist a stabilizing controller of the desired order for the uncertain system. From another point of view, in these approaches, a convex inner approximation of the non-convex set of all fixed-order stabilizing controllers is developed, which have some conservatism. In Karimi et al. (2007), a convex set of all stabilizing controllers of a polytopic system is given over an infinite-dimensional space. A finite-dimensional approximation of this set is obtained using the orthonormal basis functions and represented by a set of LMIs thanks to the KYP lemma. Moreover, an LMI based convex optimization problem for robust pole placement with sensitivity function shaping in two- and infinity-norm is proposed. In this approach the conservatism is reduced by increasing the controller order.

In this paper a new constraint is added to this optimization problem in order to obtain a low-order controller. This constraint is a rank deficiency constraint on Sylvester resultant matrix of the numerator and denominator of the controller. Since this constraint is not convex, some convex approximations based on trace minimization are introduced and compared.

Rank minimization is a challenging issue in control engineering. Although it has been shown that many control problems can be reduced to a rank minimization problem (see Mesbahi and Papavassilopoulos (1997) and the references therein), the existing convexified rank minimization methods are more heuristic than rigorous and hence, not so

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efficient. However, in this paper, the problem is formulated such that a matrix should be just rank-deficient and there is no need to minimize the rank of a matrix. To form a convex problem, the approximation proposed by Fazel et al. (2003) is used and its efficiency is improved by using a weighted trace function instead of the regular one.

The paper is organized as follows : The preliminaries and problem formulation are introduced in Section 2. Main results of the paper are found in Section 3. Section 4 shows the efficiency of the proposed methods by applying it to an experimental system. Concluding remarks are given in Section 5.

## 2. PRELIMINARIES

The goal is to convexly parameterize fixed-order stabilizing controllers for a system with polytopic uncertainty. In Karimi et al. (2007) the authors give a convex parameterization of all stabilizing controllers for a polytopic system. The proposed parameterization can be used for fixed-order controller design. However, it needs to fix a polynomial to preserve the convexity, which brings conservatism (see Khatibi et al. (2007) and Henrion et al. (2003) for further details). In this paper, the same approach is followed and thus, the result of Karimi et al. (2007) is briefly recalled.

Consider a SISO LTI plant represented by a finite order rational transfer function $G$ in discrete- or in continuous-time. Assume that $N$ and $M$ are the coprime factors of $G$ such that

$$G = NM^{-1}, \qquad N, M \in \mathcal{RH}_\infty \tag{1}$$

where $\mathcal{RH}_\infty$ is the set of proper stable rational transfer functions with bounded infinity norm. It is shown in Karimi et al. (2007) that the set of all stabilizing controllers is given by :

$$\mathcal{K} : \{K = XY^{-1} \mid MY + NX \in \mathcal{S}\} \tag{2}$$

where $X, Y \in \mathcal{RH}_\infty$ and $\mathcal{S}$ is the convex set of all Strictly Positive Real (SPR) transfer functions. As mentioned before, to design a fixed-order controller, the denominator of $MY + NX$ is fixed and called central polynomial. It is clear that by choosing a central polynomial, the convex feasibility set of (2) would be an interior approximation of the non-convex set of all stabilizing controllers of the desired order. An unsuitable choice of the central polynomial may cause a null feasibility set for a polytopic system. This conservatism is removed in Karimi et al. (2007) by letting the order of $X$ and $Y$ be increased. By increasing the order of $X$ and $Y$, not only some stabilizing controllers of this new order are included in the feasible set of the problem, but also more controllers of lower orders enter in the feasible set. This can be explained as follows. Suppose there exists an $m$-th order stabilizing controller $K = X_0Y_0^{-1}$, where $MY_0 + NX_0 \notin \mathcal{S}$. This means that such a controller is not a feasible point of (2) if the order of $X$ and $Y$ is fixed to $m$. However, it can be proved that there always exists a biproper $m_0$-th order multiplier $F \in \mathcal{RH}_\infty$ such that $MY + NX \in \mathcal{S}$, where $X = X_0F$ and $Y = Y_0F$. This means that the $m$-th order controller is inside the feasible set of (2) with a $(m + m_0)$-th order controller that has $m_0$ zero-pole cancellation. This way, by increasing the order of $X$ and $Y$ the feasible set of (2) covers all stabilizing controllers. The advantage of this method is that it can be easily applied to polytopic and multimodel systems because by contrast to Youla parameterization, the controller parameterization does not depend on the system parameters.

Consider a polytopic system with $q$ vertices such that the $i$-th vertex constitutes the parameters of the model $G_i = N_iM_i^{-1}$ where $N_i$ and $M_i \in \mathcal{RH}_\infty$ are the coprime factors of $G_i$. The aim is to parameterize all stabilizing controllers for this polytopic system. The controller parameterization is given by :

$$\mathcal{K} : \{K = XY^{-1} \mid M_iY + N_iX \in \mathcal{S}, \quad i = 1, \ldots, q\} \tag{3}$$

where $X, Y \in \mathcal{RH}_\infty$. Furthermore, a complete design problem for a polytopic system with sensitivity loop-shaping as proposed in Karimi et al. (2007), can be stated as follows :

$$\begin{array}{ll} & \text{Minimize } \max_i \gamma_i \\ \text{Subject to:} & \\ M_iY + N_iX \in \mathcal{S} & \text{for } i = 1, \ldots, q \\ \|M_iY + N_iX - 1\|_\infty < \gamma_i & \text{for } i = 1, \ldots, q \\ \|W_1M_iY\| < 1 - \gamma_i & \text{for } i = 1, \ldots, q \end{array} \tag{4}$$

where $W_1$ is a weighting filter. A solution to the above problem guarantees the closed-loop stability with a weighted infinity-norm less than 1 for the sensitivity functions of all models in the polytopic system. The disadvantage of this method is that it may result in a high-order controller. As a result, the objective of this paper is to introduce a new constraint to force the solver to find non-coprime $X$ and $Y$ to decrease the order of the resulting controller.

## 3. REDUCED-ORDER CONTROLLER DESIGN

As mentioned in the above section, to cope with the non-convexity of the set of parameters of all stabilizing controllers, the order of the controller is relaxed. Thus, a low-order controller can be obtained only if a set of convex constraint and convex cost function forces the solver to find a controller that has a desired number of zero-pole cancellation.

### *3.1 Sylvester resultant*

It is shown that the number of common roots between two $m$-th order polynomials $x$ and $y$ is the same as the number of rank deficiency of their first Sylvester resultant matrix (Chen, 1999). Let the $k$-th Sylvester resultant matrix $S_k$ be defined as follows :

$$S_k = \underbrace{\begin{bmatrix} 1 & & \\ y_{m-1} & \ddots & \\ \vdots & \ddots & 1 \\ y_1 & \ddots & y_{m-1} \\ y_0 & \ddots & \vdots \\ & \ddots & y_1 \\ & & y_0 \end{bmatrix}}_{m-(k-1)} \underbrace{\begin{bmatrix} x_m & & \\ x_{m-1} & \ddots & \\ \vdots & \ddots & x_m \\ x_1 & \ddots & x_{m-1} \\ x_0 & \ddots & \vdots \\ & \ddots & x_1 \\ & & x_0 \end{bmatrix}}_{m-(k-1)} \tag{5}$$

where $1 \leq k \leq m$. Suppose that $x$ and $y$ are the polynomials of the numerator and the denominator of the controller such that $K = XY^{-1} = x/y$. Taking into account the structure of $S_k$, it is very obvious that the rank deficiency of $S_1$ is strongly related to that of $S_k$, such that $Rank(S_1) \leq 2m - k$ if and only if $S_k$ is not full rank (see Rupperecht (1999) and Kaltofen et al. (2005) (Th. 2.3)). Therefore, to force a controller of order $m$, to have $k$ zero-pole cancellation, the rank of matrix $S_1$ should be $2m - k$, which in turn means that $S_k$ should be rank-deficient. Rank constraints are non-convex and an optimization containing such constraints has been shown to be NP-hard (Vandenberghe and Boyd, 1996; Recht et al., 2007). However, instead of rank minimization, we need only the $S_k$ to be rank-deficient. The rank deficiency of $S_k$ can be easily represented via a bilinear matrix equality (BME). Defining $s_k^1$ as the first column of $S_k$ and $\hat{S}_k$ such that $S_k = [s_k^1 \; \hat{S}_k]$, the $k$-th Sylvester resultant matrix $S_k$ is rank-deficient if and only if there exists a vector $u$, such that $\hat{S}_k u = s_k^1$. Therefore, the following feasibility problem parameterizes all fixed-order stabilizing controllers of order $(m - k)$ :

$$\begin{cases} M_i Y + N_i X \in \mathcal{S} \text{ for } i = 1, \ldots, q \\ \hat{S}_k u = s_k^1 \end{cases} \tag{6}$$

where, $x, y$ and $u$ are the variables. The complete design problem by appending 6 to 4 will be referred to as BME problem. Obviously, the equality constraint is not convex, due to the multiplication of the variables in the left hand side. Thus, we need to find a convex approximation of this constraint to be able to solve it with efficient convex solvers.

### 3.2 Convex approximation of rank deficiency

There are very few results in convexification of the rank constraints in the literature. A well-known cost function to minimize the rank of a square positive semi-definite matrix is its *trace*, which is linear w.r.t. the matrix elements and is equal to the $\ell_1$-norm of the vector of its eigenvalues. It is quite obvious that to force a vector to become sparse, minimizing its $\ell_1$-norm is the best, compared to the other $\ell_p$-norms. Fazel et al. (2003) have used this cost function to minimize the rank of a matrix. Fortunately, we need just to make a matrix rank-deficient and there is no need to really minimize its rank. However, the related matrix, $S_k$ is not a symmetric positive semi-definite matrix. Thus, $S_k^T S_k$ is an appropriate candidate, which is a symmetric positive semi-definite matrix and its rank is the same as the rank of $S_k$ (Haddad and Bernstein, 1990). The *trace* of $S_k^T S_k$, which is the $\ell_1$-norm of its eigenvalues, is proportional to the $\ell_2$-norm of the controller parameters, i.e.

$$trace(S_k^T S_k) = (m - k + 1) \left[ 1 + \sum_{i=0}^{m} x_i^2 + \sum_{j=1}^{m} y_j^2 \right]$$

where $x_i$ and $y_i$ are the parameters of the denominator and the numerator of the controller, respectively. This quadratic cost function is a convex function and can be minimized to force zero-pole cancellation in the controller. To append this cost function to (4), for a rather high order controller, we run the optimization problem (4) and find the optimal $\gamma_{opt}$ and then, fix a $\gamma_0$ such that $\gamma_{opt} \leq \gamma_0 < 1$, to have a larger feasible set. Then, the following optimization problem is used :

$$\begin{array}{ll} \multicolumn{2}{c}{\text{Minimize } \left( \sum_{i=0}^{m} x_i^2 + \sum_{j=1}^{m} y_j^2 \right)} \\ \text{Subject to:} & \\ M_i Y + N_i X \in \mathcal{S} & \text{for } i = 1, \ldots, q \\ \|M_i Y + N_i X - 1\|_\infty < \gamma_0 & \text{for } i = 1, \ldots, q \\ \|W_1 M_i Y\| < 1 - \gamma_0 & \text{for } i = 1, \ldots, q \end{array} \tag{7}$$

This problem will be referred to as Direct problem, because it minimizes the trace of $S_k^T S_k$ directly.

**Remark :** In Direct method (7), $k$ does not have any role, because the trace of $S_k^T S_k$ and $S_1^T S_1$ are the same up to a fixed multiplier. Therefore, there is no control on the number of zero-pole cancellation in the controller.

The quadratic cost function can be converted to a linear cost function using the following lemma (Fazel et al., 2003) :

*Lemma 1. Rank $S_k \leq l$ if and only if there exists symmetric matrices $U$ and $V$ such that*

$$Rank\ U + Rank\ V \leq 2l, \quad \begin{bmatrix} U & S_k \\ S_k^T & V \end{bmatrix} \geq 0 \tag{8}$$

Thus, by moving $U$ and $V$ towards rank deficiency, i.e. by decreasing their *trace*, $S_k$ also will become rank-deficient. The advantage of minimizing the $trace(U) + trace(V)$ rather than $trace(S_k^T S_k)$ is that the resulting cost-function is linear w.r.t. the variables, which means that the optimization problem becomes an SDP and can be solved more reliably via existing solvers such as SeDuMi (Sturm, 1999). Furthermore, in contrast to Direct method, the desired number of zero-pole cancellation in the controller, i.e. $k$, enters in the problem. The complete optimization problem, using this result will be referred to as Embedded problem :

$$\begin{array}{ll} \multicolumn{2}{c}{\text{Minimize } trace(U) + trace(V)} \\ \text{Subject to:} & \\ M_i Y + N_i X \in \mathcal{S} & \text{for } i = 1, \ldots, q \\ \|M_i Y + N_i X - 1\|_\infty < \gamma_i & \text{for } i = 1, \ldots, q \\ \|W_1 M_i Y\| < 1 - \gamma_i & \text{for } i = 1, \ldots, q \\ \begin{bmatrix} U & S_k^T \\ S_k & V \end{bmatrix} \geq 0 & \end{array} \tag{9}$$

**Remark :** A positive semi-definite matrix with a small *trace* is not necessarily rank-deficient or even near to it. The condition number, which is equal to the ratio of its maximum to minimum singular value is an index of the rank deficiency of a matrix. On the other hand, the *trace* of a hermitian positive semi-definite matrix is not only equal to $\ell_1$ norm of the vector of its eigenvalues, but also is equal to $\ell_1$ norm of its diagonals. Obviously, in a hermitian positive semi-definite matrix, if a diagonal becomes zero, certainly an eigenvalue becomes zero. However, the inverse is not true, which means a zero eigenvalue does not mean that a diagonal has become zero necessarily. Thus, one way to produce zero eigenvalues, i.e. to have a rank-deficient matrix, is to try to have a sparse vector of diagonals. One way to accomplish such a task is to modify the cost function of (9) from $trace(U)+trace(V) = \sum_{i=1}^{2m-k+1} U(i,i)+\sum_{i=1}^{2m-k+1} V(i,i)$ to $\sum_{i=1}^{2m-k+1} w^i U(i,i) + \sum_{i=1}^{2m-k+1} w^i V(i,i)$, where $w$ is a positive scalar. Simulation results show that using this weighted cost function better rank deficiency can be ob-

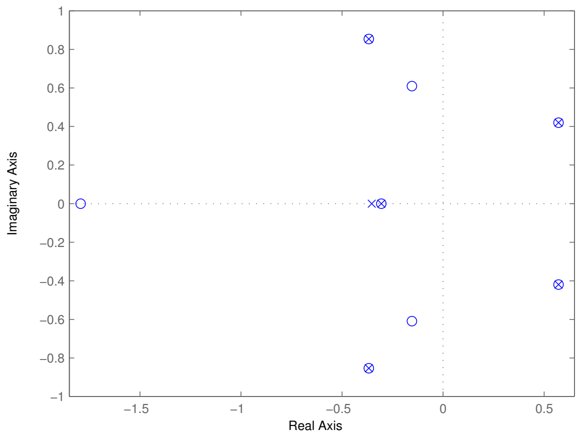


Fig. 1. Roots of polynomials $a$ and $b$ with BME method.

tained. This method will be referred to as Weighted-Embedded.

### *3.3 Example*

In the following example, the effectiveness of the rank-deficient constraint on Sylvester resultant matrix of two polynomials is examined via different convex approximations of the rank minimization problem.

Consider two ninth-order polynomials $a$ and $b$ with some unknown parameters. The goal is to see if it is possible to force them to have a desired number of common roots by means of the rank-deficient constraint on their Sylvester resultant matrix. Let $[1\ a_1\ 1.6\ a_2\ 0.5\ a_3\ a_4\ a_5\ a_6]$ and $[1\ 2\ b_1\ 1\ b_2\ 0.5\ b_3\ 0.3\ b_4]$ be the parameter vectors of the polynomials $a$ and $b$, respectively, where $a_i$ and $b_i$ are the optimization variables. The objective is to make the fourth Sylvester resultant matrix of $a$ and $b$ rank-deficient, which means that they should have four common roots. Table (1) shows the condition number of the Sylvester matrix for different methods and also the computational effort of each method. The distances between the common roots of polynomials $a$ and $b$ with BME method is of order $1e-12$ whereas for Weighted-Embedded method the distances are $8.6e-4$ and $5.0e-3$. For Embedded method there is only one pair of common roots with distance of $3.8e-3$ and in Direct method there is only one common root related to the case where $a_6 = b_4 = 0$.

The root map of two polynomials is shown in Fig. 1, in Fig. 2 and in Fig. 3, respectively for BME, Embedded and Weighted-Embedded methods. It is obvious that Weighted-Embedded method works better than Embedded method. The chosen weight in this example is $w = 0.5$. Moreover, the computational time in this simple problem is very short, thus the number of Linsearch steps is chosen in order to compare the computational effort. Table (1) shows that the convex approximations need much less computational effort than the non-convex BME method.

## 4. EXPERIMENTAL RESULTS

Consider the problem of controller design for the Flexible joint system made by Quanser. The module consists of a thin stainless steel link instrumented with a strain gage to measure the arm deflection. The module is designed to accentuate the effects of flexible links in robot control systems. Such flexibility is common in lightweight robots designed for space applications. To produce a system with huge uncertainty, the system is modified by adding small magnets of 24 grams to the end or to the center of the arm. The resulting system is modeled by the following transfer functions :

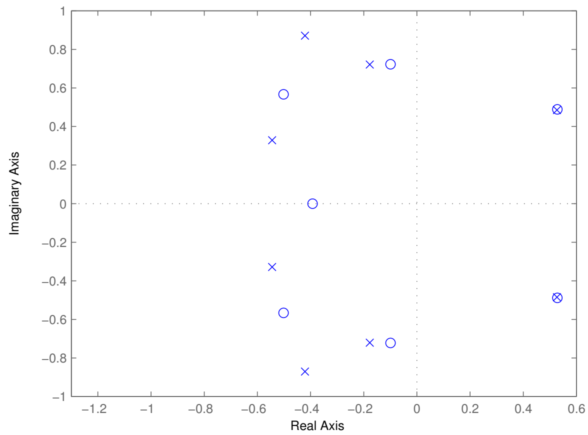


Fig. 2. Roots of polynomials $a$ and $b$ with Embedded method.

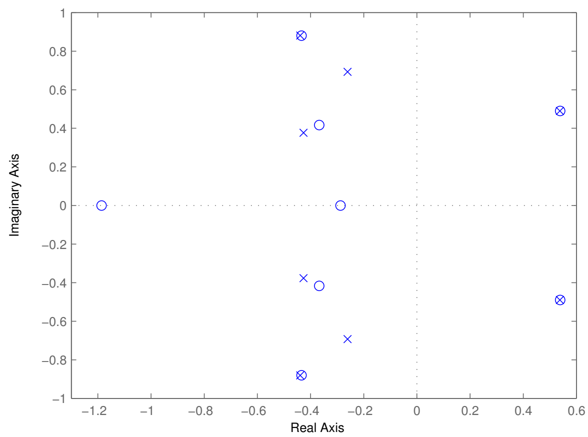


Fig. 3. Roots of polynomials $a$ and $b$ with Weighted-Embedded method.

Table 1. Comparison of the different rank minimization methods

| Method | BME | Direct | Emb | WEmb |
|---|---|---|---|---|
| Condition number | 1.3$e$6 | 5.6 | 38 | 79 |
| Linsearch steps | 471 | 46 | 61 | 99 |

Table 2. Comparison of the different rank minimization methods

| Method | BME | Direct | Emb | WEmb |
|---|---|---|---|---|
| Condition number | 1.9e5 | 1.7e3 | 1.8e3 | 3.3e3 |
| Computation time (s) | 1294 | 3.31 | 3.76 | 4.26 |

$$G_1 = \frac{-0.012z^4 + 0.0166z^3 + 0.0002z^2 + 0.0003z + 0.0104}{z^5 - 1.537z^4 + 0.5z^3 - 0.503z^2 + 1.026z - 0.4856}$$
$$G_2 = \frac{-0.0115z^4 + 0.0225z^3 - 0.007z^2 - 0.0096z + 0.0117}{z^5 - 2.21z^4 + 2.042z^3 - 1.952z^2 + 1.762z - 0.6407}$$
$$G_3 = \frac{-0.0145z^4 + 0.025z^3 - 0.0076z^2 - 0.0096z + 0.0109}{z^5 - 2.286z^4 + 2.258z^3 - 2.176z^2 + 1.817z - 0.6129}$$
$$G_4 = \frac{-0.0146z^4 + 0.0203z^3 - 0.0034z^2 - 0.0089z + 0.017}{z^5 - 2.614z^4 + 3.32z^3 - 3.094z^2 + 1.926z - 0.5381}$$

which are respectively related to the system without an extra mass, a mass of 24 grams added to the end of the arm, two masses of 24 grams added to the end of the arm and two masses of 24 grams added to the center of the arm. The system is used in position control mode and thus, all systems contain an integrator. The goal is to design a 5th order controller, such that the output sensitivity function does not exceed 6 dB. The desired closed-loop dominant poles are chosen at $0.824 \pm 0.076j$, which corresponds to the natural frequency of the slowest system with a larger damping factor. The central polynomial is chosen to have the desired closed-loop characteristic poles and the rest of its roots are arbitrarily located on the origin. In order to control the maximum value of the sensitivity function, a constant weighting filter $W_1 = 0.0625$ is considered. The optimization program (4) is run for a 5th order controller. The optimal value of $\gamma$ is 0.8042 and the resulting sensitivity transfer function has an undesirable peak of 12 dB. In the next step, the controller order is increased to 10 to cover more 5th order controllers. The optimal $\gamma$ for a 10th order controller is $\gamma_{opt} = 0.7778$ and hence, $\gamma_0$ is chosen to be $\gamma_0 = 0.87$ in order to have a larger feasibility set. Now, the goal is to find a 10th order controller that can be simplified to a 5th order one. Thus, the order of the controller is selected to be 10 with $k = 5$, which means that the resulting controller is forced to have 5 zero-pole cancellation. Table (2) compares the result of different methods. It is obvious that the BME method works better than its convex approximations, while the solver time is also increased significantly. The optimizations have been done in MATLAB, using YALMIP (Löfberg, 2004) with SeDuMi (Sturm, 1999) for convex ones and PENBMI (Kocvara and Stingl, 2006) for the bilinear one. Fig. 4 shows the zero-pole map of the resulting 10th order controller of Weighted-Embedded method. Five zero-pole cancellations on the left semi-circle are clearly observed and the resulting 5th order controller is :

$$K = \frac{-2.236z^5 + 19.12z^4 - 42.39z^3 + 44.67z^2 - 24.74z + 5.955}{z^5 - 1.355z^4 + 0.9816z^3 - 0.2951z^2 - 0.02473z + 0.04834}$$

The magnitude Bode diagram of the output sensitivity transfer functions of the four models with this controller are shown in Fig. 5, both for the resulting 10th order controller and for the simplified 5th order one. It is obvious that the peak value is less than 5 dB. Furthermore, it should be mentioned that the closed-loop systems with the simplified controller are absolutely stable. The Nyquist diagrams of $M_iY + N_iX$ are shown in Fig. 6. Evidently, the simplified 5th order controller is not a feasible point of the SPRness constraint of (4). This figure illustrates the capability of the proposed method by showing that there exists a 5th order controller which is not in the feasible set of (4) because of the conservatism imposed by fixing a central polynomial, whereas it is in the feasible set of the same problem, when the order is increased to 10.

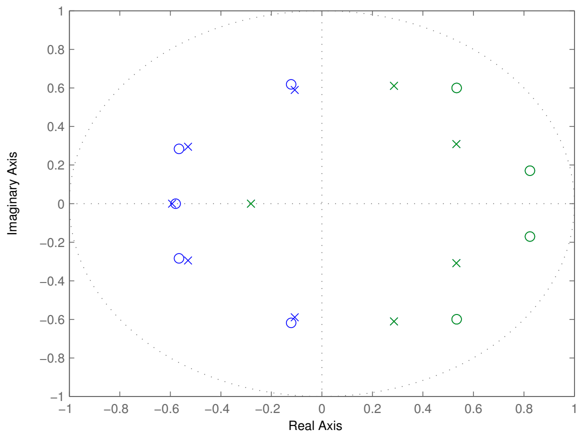


Fig. 4. Pole-zero map of the resulting 10th order controller of Weighted-Embedded method.

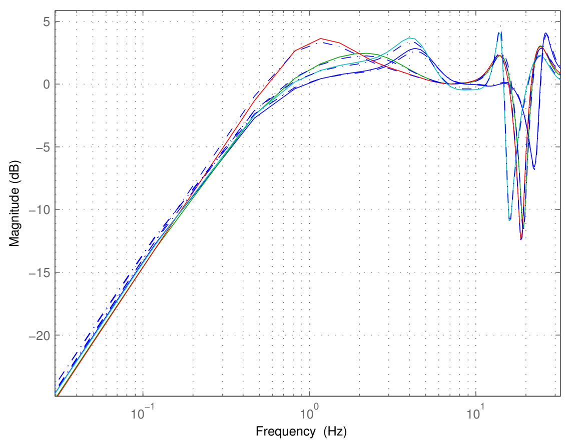


Fig. 5. Magnitude Bode diagram of the output sensitivity transfer functions of the four models with the 10th order controller (solid) and with the simplified controller of order 5 (dashed). The 10th order controller is the result of Weighted-Embedded method.

## 5. CONCLUSION

A fixed-order controller design method based on an infinite-dimensional convex parameterization of all stabilizing controllers is given. Rank deficiency constraint on the $k$-th Sylvester resultant matrix of the numerator and the denominator of the controller is the key point to obtain the fixed-order controller. Using several convex approximations of the rank constraint, in addition to a recently proposed convex method of controller design for polytopic systems, a low-order controller can be obtained. The whole problem is formulated as an LMI optimization problem with a linear cost function.

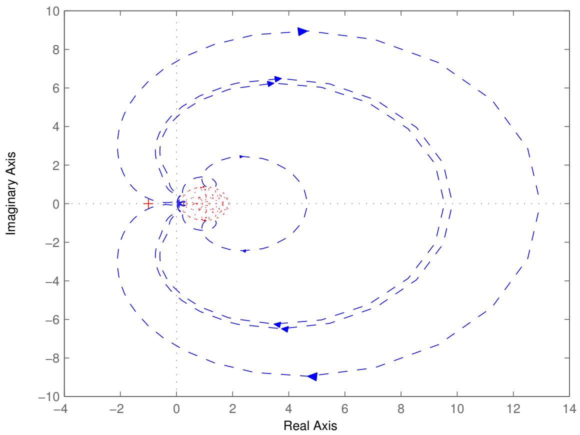


Fig. 6. Nyquist diagrams of $M_iY + N_iX$ for all four models, with the 10th order controller (dotted) and also with the 5th order simplified one (dashed). The controller is the result of Weighted-Embedded method.

The difference between the proposed method and the regular a posteriori controller order reductions is as follows. In the first step of a regular a posteriori order reduction, an optimal controller of high-order is obtained. Then, using an order reduction technique, the high-order controller is reduced to a low-order one. This new controller not only is not optimal but also, might not be even stabilizing. However, in the approach proposed in this paper, the optimization problem is forced to result in a controller with zero-pole cancellation. This method is applied to an experimental system, showing the effectiveness of the proposed approach.

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