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A factorization approach for the ℓ_∞–gain of discrete–time linear systems ⋆

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Abstract: A method is presented for the computation of an upper bound for the ℓ_∞–gain of discrete–time BIBO–stable linear systems. The bound is proved to be tight for single–input positive systems. The approach is suitable to deal with the problem of the synthesis of a static output feedback ensuring that the ℓ_∞–gain of the closed loop dynamics is below a desired threshold: a sufficient criterion is provided which consists of the solution of a system of linear inequalities. Numerical examples are reported.

Keywords: L–infinity gain, Static output feedback, Factorization, Positive systems.

1. INTRODUCTION

The problem consisting of designing a controller so that the ℓ_∞–gain of the closed loop dynamics is below a desired threshold, or minimized, is called the ℓ_1–control problem and was introduced in (Vidyasagar, 1986). The ℓ_1 control is a natural approach to the synthesis in the presence of persistent noise disturbance, for this reason the problem has been the subject of a certain amount of literature (Dahleh–Pearson, 1987; Bobillo–Dahleh, 1993; Shamma, 1996; Khammash, 1996; Elia–Dahleh, 1998; Bamieh–Dahleh, 1998). The main proposed approaches take advantage of the convex structure of the set of all stabilizing controllers and, either the problem is transformed into an infinite dimensional linear optimization, or a linear (or quadratic) programming formulation is presented. Hence, algorithmic procedures are carried out for numerical approximation of the solution. Differently from the case of dynamic controllers, the problem of minimizing the ℓ_∞–gain by means of static output feedback has been less investigated. In this case, the main difficulties rise from the fact that the set of stabilizing control gains is not convex.

Assuming that a good synthesis methodology may be found if suitable analysis tools are available, we have first turned our attention to the problem of evaluating the ℓ_∞–gain of a BIBO–stable linear system. In this respect, the main results (Balakrishnan–Boyd, 1992; Hurak et al., 2002) are still based on algorithmic procedures that do not appear to be practical for extension to control synthesis problems.

The main contribution of this paper consists in providing an easy method for the computation of an upper bound for the ℓ_∞–gain of a BIBO–stable linear system. Although the proposed bound is not always feasible (i.e., it can be computed only for some particular systems) and often quite conservative, yet it turns out to be useful in some interesting cases. In particular, the bound is proved to be tight for single–input *positive* systems. Furthermore, the proposed method can be extended to deal with control synthesis: a sufficient criterion is provided that allows one to find a static output feedback $u = Ky$ so that the ℓ_∞–gain of the closed loop dynamics is below a desired threshold. This can be done by solving a system of linear inequalities.

The paper is organized as follows: in Section 2 we formulate the problem; Section 3 is concerned with the analysis of the ℓ_∞–gain while the control synthesis problem is faced in Section 4.

Notation and terminology: A square matrix $A \in \mathbb{R}^{n\times n}$ is said to be *Schur* iff all its eigenvalues have magnitude strictly less than 1. By e_i we denote the i–th vector of the canonical basis. Let $x \in \mathbb{R}^n$: x' is the transpose of x, $x_i := e_i' x$ is the i–th component of the vector. For $M \in \mathbb{R}^{h\times k}$, $M_{i,j} := e_i' M e_j$ is the (i,j)–th entry of the matrix. I_k is the identity matrix in $\mathbb{R}^{k\times k}$. $\mathbb{R}^+ := \{x \in \mathbb{R} \,|\, x \geq 0\}$.

By a *signal* we mean a function $\boldsymbol{v} : \mathbb{N} \to \mathbb{R}^{h\times k}$, whereas $v(t)$ denotes its value at time t. The signal $\boldsymbol{v}$ is said to be *positive* iff $\forall\, i = 1,\ldots,h$, $\forall\, j = 1,\ldots,k$ and $\forall\, t \in \mathbb{N}$, $v_{i,j}(t) \geq 0$. The *null* signal $\boldsymbol{v} \equiv 0$ is denoted by $\mathbf{0}$.

2. PRELIMINARIES AND PROBLEM FORMULATION

Let us introduce the normed space of signals $\ell_\infty(\mathbb{R}^p)$. First, recall that the infinity norm of a matrix $M \in \mathbb{R}^{h\times k}$ is given by:

$$\|M\|_\infty = \max_{i=1,\ldots,h} \sum_{j=1}^{k} |M_{i,j}|\,. \tag{1}$$

Consider

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$$\ell_\infty(\mathbb{R}^p) := \left\{ \boldsymbol{v} : \mathbb{N} \to \mathbb{R}^p \mid \sup_{t\in\mathbb{N}} \|v(t)\|_\infty < +\infty \right\}$$

endowed with the norm

$$\|\boldsymbol{v}\|_\infty := \sup_{t\in\mathbb{N}} \|v(t)\|_\infty .$$

Consider now a discrete–time linear system

$$\Sigma(A,B,C) := \begin{cases} x(t+1) = Ax(t) + B\big(u(t)+e(t)\big) \\ y(t) = Cx(t) \\ x \in \mathbb{R}^n, \quad u, e \in \mathbb{R}^m, \quad y \in \mathbb{R}^q, \end{cases} \tag{2}$$

where $u(t)$ is the control variable, $e(t)$ is an input–matched disturbance and $y(t)$ is the measured output. Let $\boldsymbol{g}$ be the impulse response of the system, namely

$$g(t) = \begin{cases} 0 & \text{if } t = 0 \\ CA^{t-1}B & \text{if } t \geq 1, \end{cases}$$

and $G(z) := \sum_{t=0}^{+\infty} g(t)z^{-t} = C(zI - A)^{-1}B$ be the corresponding transfer matrix.

System (2) is said to be *BIBO–stable* iff $\forall\, \boldsymbol{u} \in \ell_\infty(\mathbb{R}^m)$ it holds that $\boldsymbol{g}*\boldsymbol{u} \in \ell_\infty(\mathbb{R}^q)$, where $(\boldsymbol{g}*\boldsymbol{u})(t) := \sum_{\tau=0}^{t-1} g(t-\tau)u(\tau)$. In this case, it is well known that the linear operator

$$\begin{aligned} \mathcal{G} : \ell_\infty(\mathbb{R}^m) &\to \ell_\infty(\mathbb{R}^q) \\ \boldsymbol{u} &\mapsto \boldsymbol{g} * \boldsymbol{u} \end{aligned}$$

is bounded and its induced operator norm

$$\|\mathcal{G}\|_\infty := \sup_{\boldsymbol{u}\in\ell_\infty(\mathbb{R}^m)\setminus\{\boldsymbol{0}\}} \frac{\|\boldsymbol{g}*\boldsymbol{u}\|_\infty}{\|\boldsymbol{u}\|_\infty}$$

is such that

$$\|\mathcal{G}\|_\infty = \max_{i=1,\dots,q} \sum_{\tau=0}^{+\infty} \sum_{j=1}^{m} |g_{i,j}(\tau)| = \tag{3a}$$

$$= \max_{i=1,\dots,q} \sum_{j=1}^{m} \sum_{\tau=0}^{+\infty} |g_{i,j}(\tau)| . \tag{3b}$$

More details can be found in (Desoer–Vidyasagar, 1975). According to equation (1), the expression in (3b) can be rewritten as

$$\|\mathcal{G}\|_\infty = \left\| \begin{pmatrix} \|\mathcal{G}_{1,1}\|_\infty & \cdots & \|\mathcal{G}_{1,m}\|_\infty \\ \vdots & \ddots & \vdots \\ \|\mathcal{G}_{q,1}\|_\infty & \cdots & \|\mathcal{G}_{q,m}\|_\infty \end{pmatrix} \right\|_\infty , \tag{4}$$

where $\|\mathcal{G}_{i,j}\|_\infty = \sum_{\tau=0}^{+\infty} |g_{i,j}(\tau)|$ is the induced operator norm of $\ell_\infty(\mathbb{R}) \ni \boldsymbol{v} \overset{\mathcal{G}_{i,j}}{\longmapsto} \boldsymbol{g}_{i,j} * \boldsymbol{v} \in \ell_\infty(\mathbb{R})$. Thus, the study of $\|\mathcal{G}\|_\infty$ for a MIMO system can be reduced to that for SISO systems.

The operator $\mathcal{G}$ is referred to as the *input/output operator* associated to system (2) and $\|\mathcal{G}\|_\infty$ is called the *ℓ_∞–gain* of the system. We use script symbols to denote input/output operators.

Remark 1. The norm $\|\mathcal{G}\|_\infty$ should not be confused with the norm $\|G\|_\infty$: the former is concerned with the input/output operator defined on $\ell_\infty(\mathbb{R}^m)$ (hence, in the time domain); the latter is the H_∞–norm of the transfer matrix $G(z)$ and, actually, is the ℓ_2–gain of the system. This paper deals with $\|\mathcal{G}\|_\infty$ only.

System (2) is BIBO–stable if and only if the poles of $G(z)$ have magnitude strictly less than 1. Therefore, if system (2) is reachable and observable, then it is BIBO–stable if and only if A is a Schur matrix.

A particularly interesting class of systems (2) is:

Definition 1. (Farina–Rinaldi, 2000) System (2) is said to be *externally positive* iff the impulse response $\boldsymbol{g}$ is positive.

The apparent drawback with the expressions for $\|\mathcal{G}\|_\infty$ given in equations (3) is that, in general, they require the computation of an infinite series. We are hence interested in the following two problems:

Problem 1 (Analysis of the ℓ_∞–gain) For a given BIBO–stable system (2), find $\gamma > 0$ such that $\|\mathcal{G}\|_\infty \leq \gamma$.

Problem 2 (Control synthesis: static output feedback) For a given system (2) and $\gamma > 0$, find $K \in \mathbb{R}^{m\times q}$ such that, under the static output feedback $u(t) = Ky(t)$, the closed loop system

$$\begin{cases} x(t+1) = (A + BKC)x(t) + Be(t) \\ y(t) = Cx(t) \end{cases} \tag{5}$$

is BIBO–stable and, denoted by $\mathcal{G}_K$ its input/output operator, it holds that $\|\mathcal{G}_K\|_\infty \leq \gamma$.

In (Boyd–Doyle, 1987), an upper bound for the ℓ_∞–gain of a linear system is given in terms of the singular values of the Hankel operator (Glover, 1984). This result has been the basic tool to carry out numerical algorithms that solve Problem 1. E.g., in (Balakrishnan–Boyd, 1992; Hurak et al., 2002), the series defining $\|\mathcal{G}\|_\infty$ is truncated and the result of (Boyd–Doyle, 1987) is used to provide a bound on the norm of the tail of the series. Although these methods allow one to find good estimates of the ℓ_∞–gain of a system, on the other hand do not appear to be practical to deal with control synthesis problems. In this paper instead, we propose a method to bound the ℓ_∞–gain of a system that, even if quite conservative in general, it is suitable to deal also with Problem 2.

3. A FACTORIZATION APPROACH TO THE ANALYSIS OF THE ℓ_∞–GAIN

In this section we consider Problem 1. A solution is proposed which is based on the factorization of the overall dynamics in terms of subsystems whose computation of the ℓ_∞–gain is simple. The proposed method cannot be applied to any system and, in general, may be quite conservative. On the other hand it is shown to be tight for an interesting class of systems, namely single–input externally positive systems.

FIR systems and *externally positive* systems are two cases where the computation of the ℓ_∞–gain is simple.

Definition 2. Consider system (2) and let $\boldsymbol{g}$ be its impulse response. The system is said to be *finite impulse response (FIR)* iff $\exists\, r \in \mathbb{N}$, such that $\forall\, t > r$, $g(t) = 0$.

Therefore, system (2) is FIR if and only if $G(z) = \frac{1}{z^r}\sum_{t=1}^{r} g(t)z^{r-t}$ for some $r \in \mathbb{N}$. For a FIR system, the computation of $\|\mathcal{G}\|_\infty$ is trivial as the series in equation (3a) is a finite sum. Hence, $\|\mathcal{G}\|_\infty = \max_{i=1,\dots,q} \sum_{\tau=1}^{r}\sum_{j=1}^{m} |g_{i,j}(\tau)|$. Let us associate to a FIR system the matrix

$$\mathtt{G} := \left[\, g(1) \mid \cdots \mid g(r) \,\right] \in \mathbb{R}^{q\times mr}, \tag{6}$$

then, according to equation (1),

$$\|\mathcal{G}\|_\infty = \|\mathtt{G}\|_\infty . \tag{7}$$

As far as externally positive systems are concerned:

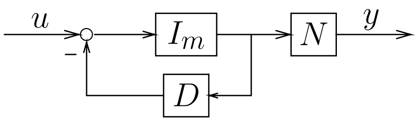

Fig. 1. Block diagram representation of the factorization of $G(z)$ considered in Theorem 1.

Proposition 1. (**ℓ_∞–gain of externally positive systems)** If system (2) is BIBO–stable and externally positive, then

$$\|\mathcal{G}\|_\infty = \|G(1)\|_\infty .$$

Proof. Because $\boldsymbol{g}$ is positive, then $\forall\, i = 1, \dots, q$ and $\forall\, j = 1, \dots, m$,

$$G_{i,j}(1) = \sum_{t=0}^{+\infty} g_{i,j}(t) = \|\mathcal{G}_{i,j}\|_\infty .$$

The thesis follows by equation (4). ■

We are ready to introduce the main result of this section. Consider system (2), assume without loss of generality that $q \geq m$ and let $G(z)$ be the transfer matrix of the system. It is always possible to factorize $G(z)$ in the form

$$G(z) = N(z)\big(I_m + D(z)\big)^{-1}, \tag{8}$$

where $N(z)$ and $D(z)$ are the transfer matrices of FIR systems[1] (see Fig. 1). Three methods to obtain this factorization are described in next Remark 2.

Theorem 1. **(Bound for $\|\mathcal{G}\|_\infty$)** Consider system (2), assume without loss of generality that $q \geq m$ and let the transfer matrix of the system be factorized as in equation (8). If $\|\mathcal{D}\|_\infty < 1$, then the system is BIBO–stable and

$$\|\mathcal{G}\|_\infty \leq \frac{\|\mathcal{N}\|_\infty}{1 - \|\mathcal{D}\|_\infty} .$$

Proof. Denote by $\mathcal{I}_m : \ell_\infty(\mathbb{R}^m) \to \ell_\infty(\mathbb{R}^m)$ the identity operator. By a well known result in functional analysis, since $\|\mathcal{D}\|_\infty < 1$, then the operator $(\mathcal{I}_m + \mathcal{D})^{-1} : \ell_\infty(\mathbb{R}^m) \to \ell_\infty(\mathbb{R}^m)$ is well–defined and $\|(\mathcal{I}_m+\mathcal{D})^{-1}\|_\infty \leq \frac{1}{1-\|\mathcal{D}\|_\infty}$. Also, $\mathcal{N} : \ell_\infty(\mathbb{R}^m) \to \ell_\infty(\mathbb{R}^q)$ because $N(z)$ is a FIR system. From the factorization (8) of $G(z)$, it follows that $\mathcal{G} = \mathcal{N} \circ (\mathcal{I}_m + \mathcal{D})^{-1} : \ell_\infty(\mathbb{R}^m) \to \ell_\infty(\mathbb{R}^q)$ and hence,

$$\|\mathcal{G}\|_\infty \leq \|\mathcal{N}\|_\infty \|(\mathcal{I}_m + \mathcal{D})^{-1}\|_\infty \leq \frac{\|\mathcal{N}\|_\infty}{1 - \|\mathcal{D}\|_\infty} .$$

■

Remark 2. **(Computation of the factorization** (8)**)** Let us present some methods that, for any strictly proper transfer matrix $G(z)$ ($q \geq m$), allow one to obtain a factorization as in equation (8).

• **Method 1:** factorization (8) can be obtained in the form of a right coprime rational matrix factorization of $G(z)$ (see (Kucera, 1991)). The standard state space approach to obtain such a factorization is the following: let $\Sigma(A, B, C)$ be a reachable and observable linear system whose transfer matrix is $G(z)$ and K be a matrix such that all the eigenvalues of $A + BK$ are in 0. Then

$$\begin{cases} N(z) = C\big(zI - (A + BK)\big)^{-1} B \\ D(z) = K\big(zI - (A + BK)\big)^{-1} B \end{cases}$$

are such that equation (8) holds[2].

Next methods 2 and 3 are purely algebraic approaches.

• **Method 2:** let $d(z)$ be the monic least common multiple of the denominators of $G(z)$ and $r := \deg(d)$. Let

$$\bar{D}(z) := \frac{d(z)}{z^r} I_m ,$$

then equation (8) holds with

$$\begin{cases} N(z) = G(z)\bar{D}(z) \\ D(z) = \bar{D}(z) - I_m . \end{cases}$$

• **Method 3:** for $j = 1, \dots, m$, let $d_j(z)$ be the monic least common multiple of the denominators appearing in the j–th column of $G(z)$ and $r_j := \deg(d_j)$. Let[3]

$$\bar{D}(z) := \operatorname{diag}\left\{ \frac{d_1(z)}{z^{r_1}}, \dots, \frac{d_m(z)}{z^{r_m}} \right\} ,$$

then equation (8) holds with

$$\begin{cases} N(z) = G(z)\bar{D}(z) \\ D(z) = \bar{D}(z) - I_m . \end{cases}$$

In general, $N(z)$ and $\bar{D}(z)$ resulting from method 3 are not right coprime rational matrices. In case they are, the factorizations resulting from methods 1 and 3 coincide.

Next Proposition 2 is just a particularization of Theorem 1 to single–input systems. This particular case allows us to point out that, for a special class of externally positive single–input systems, the bound resulting from Theorem 1 is indeed an equality (see Corollary 1 below).

Proposition 2. **(Single–input systems)** Consider the transfer matrix $G(z)$ of a strictly proper linear system with $u \in \mathbb{R}$ and $y \in \mathbb{R}^q$. Let $d(z) = z^n - \sum_{k=1}^n f_k z^{k-1}$ be the polynomial of the poles of the system and $G^{(I)}(z)$ be defined by $G_k^{(I)}(z) := z^{k-1}/d(z)$, $k = 1, \dots, n$. Consider $C \in \mathbb{R}^{q \times n}$ such that $G(z) = CG^{(I)}(z)$. If $f := \sum_{k=1}^n |f_k| < 1$, then the system is BIBO–stable and

$$\|\mathcal{G}\|_\infty \leq \frac{\|C\|_\infty}{1 - f} .$$

Proof. Since $\sum_{k=1}^n |f_k| < 1$, then the poles of $G(z)$ have magnitude strictly less than 1 and the system is BIBO–stable. As $G(z) = CG^{(I)}(z)$, then $G(z)$ can be factorized in the form

$$\begin{cases} G(z) = N(z)\big(1 + D(z)\big)^{-1}, \qquad \text{with} \\ N(z) = \dfrac{1}{z^n} C \begin{pmatrix} 1 \\ z \\ \vdots \\ z^{n-1} \end{pmatrix} = \dfrac{1}{z^n} \displaystyle\sum_{t=1}^{n} C e_t z^{t-1} \\ D(z) = \dfrac{-\sum_{t=1}^n f_t z^{t-1}}{z^n} . \end{cases} \tag{9}$$

Thus, $\|\mathcal{D}\|_\infty = \sum_{t=1}^n |f_t| < 1$ and $\|\mathcal{N}\|_\infty = \|C\|_\infty$ (see equation (7)). We can hence apply Theorem 1 which yields

[1] If $q \leq m$, consider a factorization $G(z) = \big(I_q + D(z)\big)^{-1} N(z)$.

[2] If $q \leq m$, just consider a left coprime rational matrix factorization: let L be such that all the eigenvalues of $A + LC$ are in 0 and let $N(z) = C\big(zI - (A + LC)\big)^{-1} B$ and $D(z) = C\big(zI - (A + LC)\big)^{-1} L$.

[3] Where $\operatorname{diag}\left\{ \frac{d_1(z)}{z^{r_1}}, \dots, \frac{d_m(z)}{z^{r_m}} \right\} := \sum_{j=1}^m \frac{d_j(z)}{z^{r_j}} e_j e_j'$, $e_j \in \mathbb{R}^m$.

$$\|\mathcal{G}\|_\infty \le \frac{\|\mathcal{N}\|_\infty}{1-\|\mathcal{D}\|_\infty} = \frac{\|C\|_\infty}{1-f}\,.$$
■

Corollary 1. Under the assumptions of Proposition 2, if $\forall k = 1,\dots,n,\ f_k \ge 0$ and, $\forall i = 1,\dots,q$ and $\forall j = 1,\dots,n,\ C_{i,j} \ge 0$, then

$$\|\mathcal{G}\|_\infty = \frac{\|C\|_\infty}{1-f}\,.$$

Proof. The impulse response is positive, in fact: system $\Sigma(F,B,C)$, where

$$F = \begin{pmatrix} 0 & 1 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \\ f_1 & f_2 & \dots & f_n \end{pmatrix} \quad \text{and} \quad B = e_n\,,$$

is a realization of $G(z)$ and the entries of the matrices F, B and C are non–negative. Therefore, by Proposition 1, $\|\mathcal{G}\|_\infty = \|G(1)\|_\infty$ and, by equation (9),

$$\|G(1)\|_\infty = \left\| \frac{1}{1-f}\, C \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \right\|_\infty = \frac{\|C\|_\infty}{1-f}\,.$$
■

Let us illustrate, through numerical examples, how to apply Theorem 1.

Example 1. Consider a MIMO system whose transfer matrix is

$$G(z) = \begin{pmatrix} \dfrac{1}{z} & \dfrac{1}{z-1/2} \\ \dfrac{1}{z-1/4} & \dfrac{1}{z-1/5} \end{pmatrix}.$$

According to method 3 in Remark 2,

$$\bar{D}(z) = \operatorname{diag}\left\{ \frac{z(z-1/4)}{z^2}, \frac{(z-1/2)(z-1/5)}{z^2} \right\}$$

and $G(z) = N(z)\big(I_2 + D(z)\big)^{-1}$ with

$$D(z) := \frac{1}{z^2}\left[\begin{pmatrix} -1/4 & 0 \\ 0 & -7/10 \end{pmatrix} z + \begin{pmatrix} 0 & 0 \\ 0 & 1/10 \end{pmatrix} \right]$$

and

$$N(z) := \frac{1}{z^2}\left[\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} z + \begin{pmatrix} -1/4 & -1/5 \\ 0 & -1/2 \end{pmatrix} \right].$$

Thus,

$$\mathtt{D} = \big[\, d(1) \,|\, d(2) \,\big] = \begin{pmatrix} -1/4 & 0 & 0 & 0 \\ 0 & -7/10 & 0 & 1/10 \end{pmatrix},$$

and, according to equation (7), $\|\mathcal{D}\|_\infty = \|\mathtt{D}\|_\infty = \frac{4}{5} < 1$. Similarly,

$$\mathtt{N} = \big[\, n(1) \,|\, n(2) \,\big] = \begin{pmatrix} 1 & 1 & -1/4 & -1/5 \\ 1 & 1 & 0 & -1/2 \end{pmatrix},$$

and $\|\mathcal{N}\|_\infty = \|\mathtt{N}\|_\infty = \frac{5}{2}$. Hence, by Theorem 1,

$$\|\mathcal{G}\|_\infty \le \frac{5/2}{1-4/5} = \frac{25}{2}\,.$$

In this example, method 3 provides a right coprime rational matrix factorization of $G(z)$ (i.e., it coincides with method 1). Method 2, instead, cannot be applied because $d(z) = z(z-1/2)(z-1/4)(z-1/5) = z^4 - \frac{19}{20}z^3 + \frac{11}{40}z^2 - \frac{1}{40}z$ and it immediately follows that $\|\mathcal{D}\|_\infty = \frac{19}{20} + \frac{11}{40} + \frac{1}{40} = \frac{5}{4} > 1$.

Actually, it is easy to see that the corresponding impulse response $\boldsymbol{g}$ is positive, hence, according to Proposition 1, $\|\mathcal{G}\|_\infty = \|G(1)\|_\infty = 3$. This gives evidence of the fact that the proposed bound may be quite conservative. Moreover, for multi–input systems also the tightness of the bound for positive systems is lost. ♣

Example 2. Consider a MIMO system whose transfer matrix is

$$G(z) = \begin{pmatrix} \dfrac{1}{z+1/3} & \dfrac{z+1}{(z-1/2)(z+1/4)} \\ \dfrac{-2}{z-1/2} & \dfrac{z}{(z-1/4)(z+1/4)} \end{pmatrix}.$$

According to method 2 in Remark 2,

$$\bar{D}(z) = \frac{z^4 - \frac{1}{6}z^3 - \frac{11}{48}z^2 + \frac{1}{96}z + \frac{1}{96}}{z^4} I_2$$

and $G(z) = N(z)\big(I_2 + D(z)\big)^{-1}$, where

$$\begin{cases} N(z) = \dfrac{1}{z^4}\begin{pmatrix} z^3 - \frac{1}{2}z^2 - \frac{1}{16}z + \frac{1}{32} & z^3 + \frac{13}{12}z^2 - \frac{1}{12} \\ -2z^3 - \frac{2}{3}z^2 + \frac{1}{8}z + \frac{1}{24} & z^3 - \frac{1}{6}z^2 - \frac{1}{6}z \end{pmatrix} \\ D(z) = \dfrac{1}{z^4}\Big(-\frac{1}{6}I_2\, z^3 - \frac{11}{48}I_2\, z^2 + \frac{1}{96}I_2\, z + \frac{1}{96}I_2 \Big). \end{cases}$$

Thus, $\|\mathcal{N}\|_\infty = \frac{25}{6}$ and $\|\mathcal{D}\|_\infty = \frac{5}{12} < 1$. Hence, by Theorem 1,

$$\|\mathcal{G}\|_\infty \le \frac{25/6}{1-5/12} = \frac{50}{7} \simeq 7.14\,.$$

It can be seen that also in this case method 3 coincides with method 1 and provides the bound $\|\mathcal{G}\|_\infty \le \frac{400}{39} \simeq 10.26$. Thus, differently from the previous example, method 3 leads to a worse result than method 2.

Taking advantage of equation (4) and of the external positivity of some of the components of $\boldsymbol{g}$, the exact computation of $\|\mathcal{G}\|_\infty$ can be carried out. It holds that $\|\mathcal{G}\|_\infty = \frac{76}{15} \simeq 5.067$.

More details, including the explicit computations related with method 1, can be found in (Picasso, 2008). ♣

4. CONTROL SYNTHESIS

The result of Theorem 1 is suitable to deal with Problem 2.

Theorem 2. **(Control synthesis: static output feedback)** For a discrete–time linear system $\Sigma(A,B,C)$ as in equation (2), consider Problem 2. Assume that $G(z) = C(zI-A)^{-1}B$ is factorized in the form $G(z) = N(z)\big(I_m + D(z)\big)^{-1}$ as in equation (8). Let $G_K(z)$ be the transfer matrix of the closed loop dynamics (5) and, if system (5) is BIBO–stable, denote by $\mathcal{G}_K$ its input/output operator. If $K \in \mathbb{R}^{m\times q}$ is such that $\|\mathcal{D}_K\|_\infty < 1$, where $D_K(z) := D(z) - KN(z)$, then system (5) is BIBO–stable and

$$\|\mathcal{G}_K\|_\infty \le \frac{\|\mathcal{N}\|_\infty}{1-\|\mathcal{D}_K\|_\infty}\,. \tag{10}$$

Before proving the theorem, let us derive the solution to Problem 2 in terms of linear inequalities.

Corollary 2. **(Linear inequalities formulation)** With the same notation of Theorem 2, let $\gamma \ge \|\mathcal{N}\|_\infty$. As in equation (6), let $\mathtt{N} = \big[\, n(1) \,|\, \cdots \,|\, n(r) \,\big] \in \mathbb{R}^{q\times mr}$ and $\mathtt{D} = \big[\, d(1) \,|\, \cdots \,|\, d(r) \,\big] \in \mathbb{R}^{m\times mr}$ (for suitable $r \in \mathbb{N}$) be the matrices associated to the FIR systems $N(z)$ and

$D(z)$ appearing in the factorization (8). If $\exists K \in \mathbb{R}^{m\times q}$ such that

$$\forall i = 1,\ldots,m\,, \quad \sum_{j=1}^{mr}\Big|\mathtt{D}_{i,j} - \sum_{l=1}^{q} K_{i,l}\mathtt{N}_{l,j}\Big| \le 1 - \frac{\|\mathcal{N}\|_\infty}{\gamma}\,, \tag{11}$$

then, with $u(t) = Ky(t)$, the closed loop system (5) is BIBO–stable and

$$\|\mathcal{G}_K\|_\infty \le \gamma\,.$$

Proof. By Theorem 2, a sufficient condition in order that $K \in \mathbb{R}^{m\times q}$ is such that $\|\mathcal{G}_K\|_\infty \le \gamma$ is that $\|\mathcal{D}_K\|_\infty < 1$ and $\frac{\|\mathcal{N}\|_\infty}{1-\|\mathcal{D}_K\|_\infty} \le \gamma$. This is equivalent to find $K \in \mathbb{R}^{m\times q}$ such that

$$\|\mathcal{D}_K\|_\infty \le 1 - \frac{\|\mathcal{N}\|_\infty}{\gamma}\,.$$

Because $D_K(z) = D(z) - KN(z) = \frac{1}{z^r}\sum_{t=1}^{r}\big(d(t) - Kn(t)\big)z^{r-t}$, by equation (7) it holds that $\|\mathcal{D}_K\|_\infty = \|\mathtt{D} - K\mathtt{N}\|_\infty$. Condition (11) is tantamount to requiring that $\|\mathtt{D} - K\mathtt{N}\|_\infty \le 1 - \frac{\|\mathcal{N}\|_\infty}{\gamma}$ (see equation (1)). ∎

Proof of Theorem 2. It is sufficient to show that $G_K(z) = N(z)\big(I_m + D_K(z)\big)^{-1}$, the thesis then follows by Theorem 1. The transfer matrix $G_K(z)$ can be written as $G_K(z) = \big(I_q - G(z)K\big)^{-1}G(z)$. Thus[4],

$$\begin{aligned}
&G_K(z) = \big(I_q - G(z)K\big)^{-1}G(z) = \\
&\overset{(a)}{=} G(z)\big(I_m - KG(z)\big)^{-1} = \\
&= N(z)\big(I_m + D(z)\big)^{-1}\Big(I_m - KN(z)\big(I_m + D(z)\big)^{-1}\Big)^{-1} = \\
&= N(z)\big(I_m + D(z) - KN(z)\big)^{-1} = \\
&= N(z)\big(I_m + D_K(z)\big)^{-1}.
\end{aligned}$$

∎

Theorem 2 and Corollary 2 provide a sufficient criterion for the solution of Problem 2. However, because the upper bound (10) may be in general quite conservative, in many cases inequalities (11) are not feasible even if a solution to the control synthesis problem exists. Nonetheless, the proposed technique turns out to be particularly useful in the case of state feedback for single–input systems:

Proposition 3. **(State feedback for single-input systems)** Consider the transfer matrix $G(z)$ of a strictly proper linear system with $u \in \mathbb{R}$ and $y \in \mathbb{R}^n$. Let $d(z) = z^n - \sum_{k=1}^{n} a_k z^{k-1}$ be the polynomial of the poles of the system and $G^{(I)}(z)$ be defined by $G^{(I)}_k(z) := z^{k-1}/d(z)$, $k = 1,\ldots,n$. Consider $C \in \mathbb{R}^{n\times n}$ such that $G(z) = CG^{(I)}(z)$. Then, $\forall \gamma \ge \|C\|_\infty$, a control gain $K \in \mathbb{R}^{1\times n}$ can be determined by solving a system like (11) such that the closed loop dynamics with $u = Kx$ is BIBO–stable and $\|\mathcal{G}_K\|_\infty \le \gamma$.
Moreover, if $\forall i, j = 1,\ldots,n$, $C_{i,j} \ge 0$, then a solution exists to system (11) so that the equality $\|\mathcal{G}_K\|_\infty = \gamma$ is satisfied.

Proof. First notice that, because the system is of order n and $y \in \mathbb{R}^n$, then $C \in \mathbb{R}^{n\times n}$ is invertible. It holds that $G(z) = N(z)\big(1 + D(z)\big)^{-1}$, where $N(z)$ and $D(z)$ are defined as in equation (9) (with a_t in place of f_t). In particular, $\|\mathcal{N}\|_\infty = \|C\|_\infty$. According to the notation of Corollary 2, $\mathtt{N} = C$ and $\mathtt{D} = \big(\,-a_1\ \ -a_2\ \cdots\ -a_n\big)$. For $\gamma \ge \|C\|_\infty$, there exists a solution to system (11) if and only if $\exists K \in \mathbb{R}^{1\times n}$ such that $\|\mathtt{D} - K\mathtt{N}\|_\infty \le 1 - \frac{\|C\|_\infty}{\gamma}$. Since $\mathtt{D} - K\mathtt{N} = (\mathtt{D}C^{-1} - K)C$, then $\|\mathtt{D} - K\mathtt{N}\|_\infty \le \|\mathtt{D}C^{-1} - K\|_\infty\|C\|_\infty$. Therefore, K can be chosen so as to make $\|\mathtt{D}C^{-1} - K\|_\infty$, and hence $\|\mathtt{D} - K\mathtt{N}\|_\infty$, arbitrarily small. Notice that the row vector $\mathtt{D} - K\mathtt{N}$ collects the coefficients of the polynomial of the poles of $G_K(z)$. The last statement is hence a direct consequence of Corollary 1: let $\gamma \ge \|C\|_\infty$, it is sufficient to pick $K \in \mathbb{R}^{1\times n}$ so that $\mathtt{D} - K\mathtt{N} = (-f_1\ \ -f_2\ \ \ldots\ \ -f_n)$, with $f_k \ge 0$ $\forall k = 1,\ldots,n$ and $\sum_{k=1}^{n} f_k = 1 - \frac{\|C\|_\infty}{\gamma}$. This is achieved with $K = \big(\mathtt{D} + (\,f_1\ f_2\ \ldots\ f_n\,)\big)C^{-1}$. ∎

Let us provide an example where the control synthesis technique based on Corollary 2 allows one to solve Problem 2 for a MIMO system.

Example 3. Let us consider system (2), where

$$\begin{cases} A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1/4 & 0 & 3/4 & 0 \\ 0 & 1/4 & 0 & 3/4 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \\ C = \begin{pmatrix} 0 & -1 & 2 & 0 \\ 1/2 & 0 & 0 & 1 \end{pmatrix}. \end{cases}$$

The goal is to find $K \in \mathbb{R}^{2\times 2}$ such that, with $u = Ky$, the closed loop system is BIBO–stable and $\|\mathcal{G}_K\|_\infty \le 10$.

It is a reachable and observable system whose poles are 1 and $-1/4$ (both with double multiplicity), therefore the system is not BIBO–stable. The transfer matrix of the system is

$$G(z) = \begin{pmatrix} 2z & -1 \\ 1/2 & z \end{pmatrix} \frac{1}{z^2 - \frac{3}{4}z - \frac{1}{4}}\,,$$

then $G(z) = N(z)\big(I_2 + D(z)\big)^{-1}$ with

$$D(z) := \frac{1}{z^2}\left[-\frac{3}{4}I_2 z - \frac{1}{4}I_2\right]$$

and

$$N(z) := \frac{1}{z^2}\left[\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} z + \begin{pmatrix} 0 & -1 \\ 1/2 & 0 \end{pmatrix}\right].$$

According to the notation of Corollary 2, let

$$\mathtt{D} = \begin{pmatrix} -3/4 & 0 & -1/4 & 0 \\ 0 & -3/4 & 0 & -1/4 \end{pmatrix}$$

and

$$\mathtt{N} = \begin{pmatrix} 2 & 0 & 0 & -1 \\ 0 & 1 & 1/2 & 0 \end{pmatrix}.$$

As $\|\mathcal{N}\|_\infty = 3$, we look for $K \in \mathbb{R}^{2\times 2}$ such that $\|\mathtt{D} - K\mathtt{N}\|_\infty \le 1 - \frac{3}{10} = \frac{7}{10}$. We have

$$\mathtt{D} - K\mathtt{N} = \begin{pmatrix} -\frac{3}{4} - 2K_{1,1} & -K_{1,2} & -\frac{1}{4} - \frac{K_{1,2}}{2} & K_{1,1} \\ -2K_{2,1} & -\frac{3}{4} - K_{2,2} & -\frac{K_{2,2}}{2} & -\frac{1}{4} + K_{2,1} \end{pmatrix},$$

thus system (11) takes the form of

$$\begin{cases} \left|\frac{3}{4} + 2K_{1,1}\right| + |K_{1,2}| + \left|\frac{1}{4} + \frac{K_{1,2}}{2}\right| + |K_{1,1}| \le \frac{7}{10} \\ |2K_{2,1}| + \left|\frac{3}{4} + K_{2,2}\right| + \left|\frac{K_{2,2}}{2}\right| + \left|K_{2,1} - \frac{1}{4}\right| \le \frac{7}{10}\,. \end{cases}$$

[4] Where equality (a) holds because if $L \in \mathbb{C}^{m\times q}$, $M \in \mathbb{C}^{q\times m}$ and $\lambda \in \mathbb{C}\setminus\{0\}$ are such that both $\lambda I_m + LM \in \mathbb{C}^{m\times m}$ and $\lambda I_q + ML \in \mathbb{C}^{q\times q}$ are invertible, then $(\lambda I_q + ML)^{-1}M = M(\lambda I_m + LM)^{-1}$.

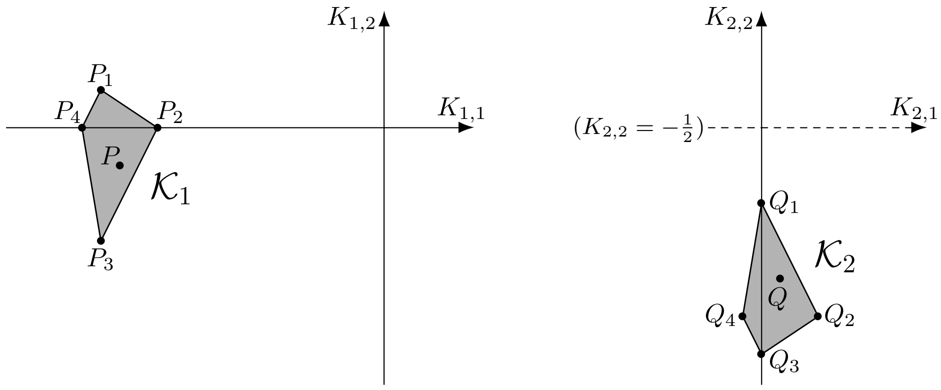

Fig. 2. The feasibility regions of system (11) in the case considered in Example 3.

The system is solved for $(K_{1,1}, K_{1,2}) \in \mathcal{K}_1 \subset \mathbb{R}^2$, where $\mathcal{K}_1$ is the quadrilateral whose vertices are

$$P_1 = \left(-\tfrac{3}{8}, \tfrac{1}{20}\right), \quad P_2 = \left(-\tfrac{3}{10}, 0\right),$$
$$P_3 = \left(-\tfrac{3}{8}, -\tfrac{3}{20}\right), \; P_4 = \left(-\tfrac{2}{5}, 0\right),$$

and for $(K_{2,1}, K_{2,2}) \in \mathcal{K}_2 \subset \mathbb{R}^2$, where $\mathcal{K}_2$ is the quadrilateral whose vertices are

$$Q_1 = \left(0, -\tfrac{3}{5}\right), \; Q_2 = \left(\tfrac{3}{40}, -\tfrac{3}{4}\right),$$
$$Q_3 = \left(0, -\tfrac{4}{5}\right), \; Q_4 = \left(-\tfrac{1}{40}, -\tfrac{3}{4}\right),$$

see Fig. 2.
One feasible choice for K is

$$K = \begin{pmatrix} -7/20 & -1/20 \\ 1/40 & -7/10 \end{pmatrix}$$

(which is identified by the points P and Q represented in Fig. 2). For such a K, we have $\|\mathtt{D} - K\mathtt{N}\|_\infty = 27/40$ and hence $\|\mathcal{G}_K\|_\infty \le \frac{3}{1-27/40} = \frac{120}{13} \simeq 9.2308$.
Notice that, because the closed loop system is reachable and observable, then $A + BKC$ is a Schur matrix. ♣

5. CONCLUSION

A method for the computation of an upper bound for the ℓ_∞–gain of discrete–time BIBO–stable linear systems is presented which is based on factorization theory. The approach is extended to deal with the static output feedback control problem.
Several issues are open for further investigations. The factorization of the transfer function allowing for the computation of the bound is not unique and results are quite different at the varying of the considered factorization. Moreover, the coprime factorization does not lead, in general, to the less conservative result. This raises the question of the search for the factorization that minimize the corresponding upper bound for $\|\mathcal{G}\|_\infty$.
While the bound is proved to be tight for a particular class of single–input positive systems, more effort is needed to identify the class of systems where the proposed approach results to be non–conservative. Further investigations will be also devoted to the applications of the proposed technique to control synthesis problems for positive systems.

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