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A Carleman approximation scheme for a stochastic optimal control problem in the continuous-time framework ⋆

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Abstract: The paper investigates the optimal control problem for a stochastic linear differential system, driven by a persistent disturbance generated by a nonlinear stochastic exogenous system. The assumption of *incomplete information* has been assumed, that is neither the state of the system, nor the state of the exosystem are directly measurable. The standard quadratic cost functional has been considered. The approach followed consists of applying the ν-degree Carleman approximation scheme to the exosystem, which provides a stochastic bilinear system. Then, the optimal regulator is obtained (i.e. the solution to the minimum control problem among all the affine transformations of the measurements). Better performances of the regulator are expected using higher order system approximations.

Keywords: Stochastic control; Optimal control; Carleman approximation; Filtering theory; Bilinear systems.

1. INTRODUCTION

Given the probability triple $(\Omega, \mathcal{F}, \mathcal{P})$, consider the optimal control problem for the following linear stochastic differential system described by the Itô equations:

$$\begin{aligned} dx(t) &= Ax(t)dt + Hu(t)dt + Mz(t)dt + NdW^x(t), \\ & \qquad\qquad\qquad\qquad\qquad\quad x(0) = x_0, \\ dy(t) &= Cx(t)dt + GdW^y(t), \qquad\quad y(0) = 0 \quad \text{a.s.,} \end{aligned} \tag{1}$$

where $x(t) \in I\!R^n$ is the state of the system, $u(t) \in I\!R^p$ is the control input, $y(t) \in I\!R^q$ is the measured output, $W^x(t) \in I\!R^b$, $W^y(t) \in I\!R^d$ are independent standard Wiener processes with respect to a family of increasing σ-algebras $\{\mathcal{F}_t,\ t \geq 0\}$. The standard assumption $\text{rank}(GG^T) = q$ is made. The initial state x_0 is an $\mathcal{F}_0$-measurable random vector, independent of $W^x(t)$ and $W^y(t)$. $z(t) \in I\!R^m$ is a persistent disturbance generated by the following nonlinear stochastic exogenous differential system (the *exosystem*):

$$dz(t) = \phi\big(z(t)\big)dt + FdW^z(t), \qquad z(0) = z_0, \tag{2}$$

where $\phi : I\!R^m \mapsto I\!R^m$ is a smooth nonlinear map and $W^z(t) \in I\!R^h$ is a standard Wiener processes with respect to $\{\mathcal{F}_t,\ t \geq 0\}$, independent of the state and output noise processes $W^x(t)$ and $W^y(t)$.

In the case of output regulation, coping with the task of tracking an assigned trajectory and/or rejecting persistent disturbances, the control problem is standardly stated in a deterministic framework and important results in the designing of a suitable control law have been reached, even in the case of uncertainties affecting the model and/or the exosystem. If the dynamics of the exosystem is not known, but it is known that it belongs to a prescribed family of functions, the so called *internal-model principle* allows to reconstruct in some way this lack of information (see Francis et al. (1976) and Isidori et al. (1990) for linear and nonlinear systems, respectively). For instance, an internal-model based control is able to cope with uncertainties affecting the amplitude and phase of an exogenous sinusoid, but it requires the knowledge of the frequency; in order to overcome this limitation, in Serrani et al. (2001) an adaptation mechanism has been used so that the natural frequencies of the internal model are tuned to match those of the unknown exosystem.

In this paper, a stochastic framework has been considered, according to which the optimal control problem of minimizing the following quadratic cost functional on the finite horizon $[0, t_f]$ has been stated:

$$J(u) = \frac{1}{2}\ I\!E\bigg\{x^T(t_f)Sx(t_f) + \int_0^{t_f} \Big(x^T(t)Qx(t) + u^T(t)Ru(t)\Big)dt\bigg\}, \tag{3}$$

with S, Q symmetric positive-semidefinite matrices, and R a symmetric positive-definite matrix. As a matter of fact, the coupling of the index (3) to the differential constraints (1-2) gives rise to a nonlinear stochastic optimal control problem. According to the *incomplete information* case here adopted, we do not assume to have direct measurements neither of the state x of the system, nor of the

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state z of the exosystem. A general setting of the problem consists of looking for the optimal solution u among all the Borel transformations of the measured output y. In this case, the optimal solution requires the knowledge of the conditional probability density, whose computation leads, in general, to an infinite-dimensional problem. For instance, in Charalambous et al. (1998) the authors overcome the drawback of incomplete information for a class of nonlinear optimal control problems by means of a set of partial differential equations providing the conditional probability density. A different research line consists of looking for *suboptimal* controllers providing the optimal solutions in a restricted family of measurable functions of the observations. Such an approach has been recently applied by Carravetta et al. (2007) to the optimal control of linear systems with state- and control-dependent noise. The control law provided achieves the optimal solution among all the affine transformations of the measurements.

On the other hand, even neglecting the stochastic disturbances, according to the *complete information* case, the application of the maximum principle to nonlinear systems does not ensure an analytical solution to the resulting nonlinear *Two-Point Boundary-Value* (TPBV) problem (see Bryson et al. (1995)). In the last decades a great deal of literature has been developed in order to obtain implementable control schemes from the maximum principle optimality conditions, see Betts (1999) and references therein. In a recent paper of Tang (2005), a *successive-approximation* approach has been adopted obtaining approximate solutions to the nonlinear TPBV problem. However, these techniques do not provide a real time algorithm, approaching the problem from a numerical point of view.

In this paper the solution to the optimal control problem is achieved by means of the Carleman approximation of a chosen degree ν of the nonlinear stochastic exosystem (2) in the form of a bilinear system (linear drift and state-dependent noise) with respect to a suitable defined extended state made of the Kronecker powers of the original state z up to degree ν. Such a methodology has been recently successfully applied in order to improve the performances of the Extended Kalman Filter both in the discrete-time (see Germani et al. (2005a), Germani et al. (2005b)) and in the continuous-time frameworks (see Germani et al. (2007)). Then, the Carleman bilinear approximation of the exosystem is coupled to the state equations (1) and the optimization problem is restated in a bilinear setting. Unfortunately, the optimal solution in the general case of incomplete information requires the building of the optimal filter, which is still not implementable according to a finite-dimensional algorithm. It is worthwhile, then, to look for suboptimal estimates. Here, we propose the *optimal linear regulator*, by extending the results of Carravetta et al. (2007), consisting of the optimal solution among of all the $\mathbb{R}^p$-valued square-integrable affine transformations of the observations.

The same approach, based on the Carleman approximation scheme, has been applied in a recent paper by the authors for the equivalent optimal control problem in the discrete-time framework (see Mavelli et al. (2007)).

In the sequel I_m will denote the identity matrix of order m, $O_{r\times c}$ a matrix of zeros in $\mathbb{R}^{r\times c}$ and $\mathbb{E}\{\cdot\}$ the expectation value operator.

2. CARLEMAN APPROXIMATION OF THE STOCHASTIC EXOSYSTEM

Under standard analyticity hypotheses, the exosystem equations can be written by using the Taylor polynomial expansion around a given point $\tilde{z}$. Denote with F_j the j-th column of matrix F, and with $W_j^z(t)$ the j-th element of vector $W^z(t)$. According to the Kronecker formalism, the differential system in (2) becomes:

$$dz(t) = \sum_{i=0}^{\infty} \Phi_i(\tilde{z})\big(z(t)-\tilde{z}\big)^{[i]}dt + \sum_{j=1}^{h} F_j dW_j^z(t), \tag{4}$$

with the square brackets denoting the Kronecker power (see Carravetta et al. (1997) for a quick survey on the Kronecker algebra and its main properties) and:

$$\Phi_i(z) = \frac{1}{i!}\left(\nabla_z^{[i]}\otimes\phi\right). \tag{5}$$

The differential operator $\nabla_z^{[i]}\otimes$ applied to a generic function $\psi=\psi(z):\mathbb{R}^m\mapsto\mathbb{R}^p$ is defined as follows

$$\nabla_z^{[0]}\otimes\psi=\psi,\quad \nabla_z^{[i+1]}\otimes\psi=\nabla_x\otimes\big(\nabla_z^{[i]}\otimes\psi\big),\quad i\geq 1, \tag{6}$$

with $\nabla_z=[\partial/\partial z_1\ \cdots\ \partial/\partial z_m]$ and $\nabla_z\otimes\psi$ the Jacobian of the vector function ψ.

Based on the building of an extended vector made up of the powers of the state of the original nonlinear system, in the derivation of the Carleman approximation the Itô formula for the computation of stochastic differentials, written using the Kronecker formalism, is required (see Liptser et al. (1977)): consider the stochastic process

$$d\chi_t = f(\chi_t)dt + g(\chi_t)dW_t = f(\chi_t)dt + \sum_{k=1}^{p} g_k(\chi_t)dW_{k,t}, \tag{7}$$

where $\chi_t\in\mathbb{R}^n$, $f:\mathbb{R}^n\mapsto\mathbb{R}^n$, $g:\mathbb{R}^n\mapsto\mathbb{R}^{n\times p}$, $g_k(\cdot)$, $k=1,\dots,p$ are the columns of $g(\cdot)$, $W_t=\begin{bmatrix}W_{1,t} & \cdots & W_{p,t}\end{bmatrix}^T$ is a standard Wiener processes in $\mathbb{R}^p$; given a transformation $\zeta_t=\varphi(t,\chi_t)$, with $\varphi:\mathbb{R}\times\mathbb{R}^n\mapsto\mathbb{R}^q$ twice differentiable, the Itô differential $d\zeta_t$ can be written as:

$$d\zeta_t = \frac{\partial\varphi}{\partial t}dt + (\nabla_\chi\otimes\varphi)d\chi_t + \frac{1}{2}(\nabla_\chi^{[2]}\otimes\varphi)\bar{g}(\chi_t)dt \tag{8}$$

with $\bar{g}(\chi_t)=\sum_{k=1}^{p} g_k^{[2]}(\chi_t)$. According to (8), the differential of the Kronecker power $z^{[k]}$, $k\geq 2$, can be written as:

$$\begin{aligned} dz^{[k]}(t) = &\left(\big(\nabla_z\otimes z^{[k]}\big)\phi\big(z(t)\big) + \frac{1}{2}\big(\nabla_z^{[2]}\otimes z^{[k]}\big)\bar{F}\right)dt \\ &+\big(\nabla_z\otimes z^{[k]}\big)F dW^z(t), \quad \text{with} \quad \bar{F}=\sum_{j=1}^{h} F_j^{[2]}; \end{aligned} \tag{9}$$

Finally, by exploiting some properties of the Kronecker algebra:

$$\begin{aligned} dz^{[k]}(t) = &\, U_m^k\big(I_m\otimes z^{[k-1]}(t)\big)\phi\big(z(t)\big) \\ &+\frac{1}{2}O_m^k\big(I_{m^2}\otimes z^{[k-2]}(t)\big)\bar{F}dt \\ &+U_m^k\big(I_m\otimes z^{[k-1]}(t)\big)F dW^z(t), \qquad k\geq 2 \end{aligned} \tag{10}$$

with U_m^k, O_m^k suitably defined matrices (see Germani et al. (2007) for more details).

The ν-degree Carleman approximation of the stochastic nonlinear exosystem (2) is, then, achieved according to the following steps:

1. consider the Taylor expansion around a given point $\tilde{z}$ of the nonlinear terms of the differentials $dz^{[k]}$, $k = 1, \dots, \nu$, by using (4) for $k = 1$ and suitably exploiting the Kronecker product properties in (10) for $2 \le k \le \nu$:

$$dz^{[k]}(t) = \sum_{i=0}^{\infty} A_{ki}(\tilde{z})\big(z(t) - \tilde{z}\big)^{[i]} dt + \sum_{j=1}^{h} \big(\mathbf{B}_{kj} z^{[k-1]}(t) + \mathbf{F}_{kj}\big) dW_j^z(t); \tag{11}$$

2. neglect in the summations in (11) the higher order terms greater than ν and expand the binomials $(z(t)-\tilde{z})^{[i]}$ according to the Kronecker binomial power formula (see Germani et al. (2007)):

$$dz^{[k]}(t) \simeq \sum_{i=0}^{\nu} \mathbf{A}_{ki}(\tilde{z}) z^{[i]}(t) dt + \sum_{j=1}^{h} \big(\mathbf{B}_{kj} z^{[k-1]}(t) + \mathbf{F}_{kj}\big) dW_j^z(t); \tag{12}$$

3. substitute in (12) the powers $z^{[k]}$ with a vector Z_k^ν of the same dimensions:

$$dZ_k^\nu(t) = \sum_{i=0}^{\nu} \mathbf{A}_{ki}(\tilde{z}) Z_i^\nu(t) dt + \sum_{j=1}^{h} \big(\mathbf{B}_{kj} Z_{k-1}^\nu(t) + \mathbf{F}_{kj}\big) dW_j^z(t). \tag{13}$$

Comparing the equations (12) and (13), it is clear that $Z_k^\nu(t)$ is aimed to approximate $z^{[k]}(t)$, $k = 1, \dots, \nu$.

Finally, the bilinear approximation may be written in a more compact form, defining the extended state $Z^\nu(t) = \big[Z_1^{\nu T}(t) \ \cdots \ Z_\nu^{\nu T}(t)\big]^T \in I\!R^{m_\nu}$, $m_\nu = m + m^2 + \cdots m^\nu$:

$$dZ^\nu(t) = \mathcal{A}_z^\nu(\tilde{z}) Z^\nu(t) dt + \mathcal{D}_z^\nu(\tilde{z}) dt + \sum_{j=1}^{h} \big(\mathcal{B}_{z,j}^\nu Z^\nu(t) + \mathcal{F}_{z,j}^\nu\big) dW_j^z(t), \tag{14}$$

with $Z^\nu(0) = Z_0^\nu = \big[z_0^T \ \cdots \ z_0^{[\nu]T}\big]^T$. Explicit computations of the matrices involved in the Carleman approximations in (11-14) can be found in Germani et al. (2007).

In order to couple the ν-degree Carleman approximation (14) to the differential system (1), the original state and output vectors $x(t)$, $y(t)$ are substituted by vectors of the same dimension $x^\nu(t)$, $y^\nu(t)$, so that the following equations are obtained:

$$\begin{aligned} dx^\nu(t) &= Ax^\nu(t)dt + Hu(t)dt + \mathcal{M}_z^\nu Z^\nu(t) + NdW^x(t), \\ dZ^\nu(t) &= \mathcal{A}_z^\nu(\tilde{z}) Z^\nu(t) dt + \mathcal{D}_z^\nu(\tilde{z}) dt + \sum_{j=1}^{h} \big(\mathcal{B}_{z,j}^\nu Z^\nu(t) + \mathcal{F}_{z,j}^\nu\big) dW_j^z(t), \\ dy^\nu(t) &= Cx^\nu(t)dt + GdW^y(t), \end{aligned} \tag{15}$$

with $\mathcal{M}_z^\nu = \big[M \quad O_{n\times(m_\nu - m)}\big]$ and $x^\nu(0) = x_0$, $Z^\nu(0) = Z_0^\nu$. By defining the following vectors and matrices (in (18) N_j denotes the j-th column of matrix N):

$$\mathcal{X}^\nu(t) = \begin{bmatrix} x^\nu(t) \\ Z^\nu(t) \end{bmatrix}, \quad \mathcal{A}^\nu(\tilde{z}) = \begin{bmatrix} A & \mathcal{M}_z^\nu \\ O_{m_\nu\times n} & \mathcal{A}_z^\nu(\tilde{z}) \end{bmatrix}, \quad \mathcal{H}^\nu = \begin{bmatrix} H \\ O_{m_\nu\times p} \end{bmatrix}, \quad \mathcal{D}^\nu(\tilde{z}) = \begin{bmatrix} O_{n\times 1} \\ \mathcal{D}_z^\nu(\tilde{z}) \end{bmatrix}, \tag{16}$$

$$\mathcal{B}_j^\nu = \begin{cases} O_{(n+m_\nu)\times(n+m_\nu)}, & j = 1, \dots, b \\ \begin{bmatrix} O_{n\times n} & O_{n\times m_\nu} \\ O_{m_\nu\times n} & \mathcal{B}_{z,j-b}^\nu \end{bmatrix}, & j = b+1, \dots, b+h \end{cases} \tag{17}$$

$$\mathcal{F}_j^\nu = \begin{cases} \begin{bmatrix} N_j \\ O_{m_\nu\times 1} \end{bmatrix}, & j = 1, \dots, b \\ \begin{bmatrix} O_{n\times 1} \\ \mathcal{F}_{z,j-b}^\nu \end{bmatrix}, & j = b+1, \dots, b+h \end{cases} \tag{18}$$

$$\mathcal{C}^\nu = \big[C \quad O_{q\times m_\nu}\big], \quad \mathcal{W}(t) = \begin{bmatrix} W^x(t) \\ W^z(t) \end{bmatrix}, \tag{19}$$

system (15) is put in a more compact form:

$$\begin{aligned} d\mathcal{X}^\nu(t) &= \mathcal{A}^\nu(\tilde{z})\mathcal{X}^\nu(t)dt + \mathcal{H}^\nu u(t)dt + \mathcal{D}^\nu(\tilde{z})dt + \sum_{j=1}^{b+h} \big(\mathcal{B}_j^\nu \mathcal{X}^\nu(t) + \mathcal{F}_j^\nu\big) d\mathcal{W}_j(t), \\ \mathcal{X}^\nu(0) &= \big[x_0^T \quad Z_0^{\nu T}\big]^T, \\ dy^\nu(t) &= \mathcal{C}^\nu \mathcal{X}^\nu(t)dt + GdW^y(t). \end{aligned} \tag{20}$$

According to (16), the cost functional (3) becomes:

$$J_\nu(u) = \frac{1}{2} I\!E\Big\{\mathcal{X}^{\nu T}(t_f)\mathcal{S}_\nu\mathcal{X}^\nu(t_f) + \int_0^{t_f} \Big(\mathcal{X}^{\nu T}(t)\mathcal{Q}_\nu\mathcal{X}^\nu(t) + u^T(t)Ru(t)\Big)dt\Big\}, \tag{21}$$

with:

$$\mathcal{S}_\nu = \begin{bmatrix} S & O_{n\times m_\nu} \\ O_{m_\nu\times n} & O_{m_\nu\times m_\nu} \end{bmatrix}, \quad \mathcal{Q}_\nu = \begin{bmatrix} Q & O_{n\times m_\nu} \\ O_{m_\nu\times n} & O_{m_\nu\times m_\nu} \end{bmatrix}. \tag{22}$$

3. OPTIMAL LINEAR REGULATOR

The optimal control problem here proposed is solved by applying the *optimal linear regulator* to the bilinear differential system (20), obtained by means of the Carleman approximation of the nonlinear stochastic exosystem. More precisely we seek a solution $u(t)$ to the minimum of the index (21), which belongs to the space $\mathcal{L}_t^p(y^\nu)$ of all the $I\!R^p$-valued square-integrable affine transformations of the random variables $\{y^\nu(\tau),\ 0 \le \tau \le t \le t_f\}$:

$$\min_{u(t)\in\mathcal{L}_t^p(y^\nu)} J_\nu(u), \quad \text{with} \quad u, \mathcal{X}^\nu, y^\nu \quad \text{subject to (20)}.$$

Such a problem has been properly formalized (quadratic functional cost and bilinear differential system) and solved in Carravetta et al. (2007); however, the solution proposed in Carravetta et al. (2007) cannot be directly applied, because in addition to the case investigated in that paper, here we have also a pure additive noise and a given deterministic drift in the state-differential equation (20). Therefore, this Section is devoted to extend the results of Carravetta et al. (2007) to such a more general framework.

Let us state a preliminary lemma before giving the main result.

Lemma 1. Let $V(t)$, $t \in I\!R$, a twice differentiable matrix function taking values in $I\!R^{n\times n}$, and $M(t)$ the martingale defined by:

$$M(t) = \sum_{k=1}^{q} \int_0^t \big(B_k X(\tau) + F_k\big) dW_k(\tau), \tag{23}$$

where $W = \begin{bmatrix} W_1 & \cdots & W_q \end{bmatrix}^T$ is a Wiener process in $I\!R^q$, and $X(t)$ is a process in $I\!R^n$, such that $X(t)$ is independent of $\{W(s),\ s \le t\}$; $B_k, F_k,\ k = 1,\dots,q$ are suitably dimensioned given matrices. Then, the following equality holds:

$$\begin{aligned}
&\frac{1}{2}\sum_{i,j=1}^{n}\left(\frac{\partial^2 (x^T V(t) x)}{\partial x_i \partial x_j}\right)_{x=X(t)} d\langle M_i, M_j\rangle_t \\
&= X^T(t)\Big(\sum_{k=1}^{q} {B_k}^T V(t) B_k\Big) X(t) dt \\
&\quad + 2X(t)^T \Big(\sum_{k=1}^{q} {B_k}^T V(t) F_k\Big) dt + \sum_{k=1}^{q} {F_k}^T V(t) F_k dt,
\end{aligned} \tag{24}$$

where $d\langle M_i, M_j\rangle_t$ is the mutual quadratic variation process of the i-th and j-th entries of the process $M(t)$.

The proof (here omitted for brevity) follows the same steps of the corresponding Lemma 3.1 in Carravetta et al. (2007).

Theorem 2. Suppose a solution exists for the following backward generalized Riccati equations:

$$\begin{aligned}
\dot V(t) &= -\mathcal{A}^{\nu T}(\tilde z) V(t) - V(t)\mathcal{A}^{\nu}(\tilde z) - \mathcal{Q}_\nu \\
&\quad - \sum_{k=b+1}^{b+h} \big(\mathcal{B}_k^{\nu T} V(t) \mathcal{B}_k^{\nu}\big) + V(t)\mathcal{H}^{\nu} R^{-1} \mathcal{H}^{\nu T} V(t),
\end{aligned} \tag{25}$$

$$V(t_f) = \mathcal{S}_\nu,$$

$$\begin{aligned}
\dot g(t) &= -2V(t)\mathcal{D}^{\nu}(\tilde z) - \mathcal{A}^{\nu T}(\tilde z) g(t) \\
&\quad + V(t)\mathcal{H}^{\nu} R^{-1}\mathcal{H}^{\nu T} g(t) - 2\sum_{k=b+1}^{b+h} \mathcal{B}_k^{\nu T} V(t) \mathcal{F}_k^{\nu},
\end{aligned} \tag{26}$$

$$g(t_f) = 0,$$

with $V(t) = V^T(t) \ge 0$. Then the solution to the optimal control problem of minimizing the cost criterion (21), under the differential constraints (20), with $u(t) \in \mathcal{L}_t^p(y^\nu)$ is given by:

$$u^o(t) = L^o(t)\widehat{\mathcal{X}^{\nu}}(t) + \alpha(t), \tag{27}$$

with

$$L^o(t) = -R^{-1}\mathcal{H}^{\nu T} V(t), \quad \alpha(t) = -\frac{1}{2} R^{-1}\mathcal{H}^{\nu T} g(t), \tag{28}$$

where $\widehat{\mathcal{X}^{\nu}}(t)$ is the optimal (in the sense of the minimum error variance) estimate of $\mathcal{X}^{\nu}(t)$ among all the $I\!R^{n+m_\nu}$-valued square-integrable affine transformations of $\{y^\nu(\tau),\ 0 \le \tau \le t \le t_f\}$.

Proof. Define the process:

$$\xi_t = \mathcal{X}^{\nu T}(t) V(t) \mathcal{X}^{\nu}(t) + g^T(t)\mathcal{X}^{\nu}(t). \tag{29}$$

By exploiting the final conditions in (25-26), we have:

$$\begin{aligned}
\int_0^{t_f} d\xi_t &= \mathcal{X}^{\nu T}(t_f)\mathcal{S}_\nu \mathcal{X}^{\nu}(t_f) - \mathcal{X}^{\nu T}(0) V(0) \mathcal{X}^{\nu}(0) \\
&\quad - g^T(0)\mathcal{X}^{\nu}(0),
\end{aligned} \tag{30}$$

so that the index (21) may be rewritten as follows:

$$\begin{aligned}
J_\nu(u) = \frac{1}{2} I\!E \Bigg\{ &\int_0^{t_f} \Big(\mathcal{X}^{\nu T}(t)\mathcal{Q}_\nu \mathcal{X}^{\nu}(t) + u^T(t) R u(t)\Big) dt \\
&+ \int_0^{t_f} d\xi_t + \mathcal{X}^{\nu T}(0) V(0) \mathcal{X}^{\nu}(0) + g^T(0)\mathcal{X}^{\nu}(0) \Bigg\}.
\end{aligned} \tag{31}$$

Recall that, in case of a scalar function $\varphi(\cdot,\cdot)$, the Itô formula in (8) particularizes as:

$$d\zeta_t = \frac{\partial \varphi}{\partial t} dt + \big(\nabla_\chi \otimes \varphi\big) d\chi_t + \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2 \varphi}{\partial \chi_i \partial \chi_j} d\langle M_i, M_j\rangle_t, \tag{32}$$

with $M(t)$ as in (**??**). Then, applying (32) to the process ξ_t:

$$\begin{aligned}
d\xi_t &= \Big(\mathcal{X}^{\nu T}(t)\dot V(t)\mathcal{X}^{\nu}(t) + \dot g^T(t)\mathcal{X}^{\nu}(t)\Big) dt + g^T(t) d\mathcal{X}^{\nu}(t) \\
&\quad + d\mathcal{X}^{\nu T}(t) V(t)\mathcal{X}^{\nu}(t) + \mathcal{X}^{\nu T}(t) V(t) d\mathcal{X}^{\nu}(t) \\
&\quad + \frac{1}{2}\sum_{i,j=1}^{n+m_\nu}\left(\frac{\partial^2 (x^T V(t) x)}{\partial x_i \partial x_j}\right)_{x=\mathcal{X}^{\nu}(t)} d\langle M_i, M_j\rangle_t,
\end{aligned} \tag{33}$$

where

$$dM(t) = \sum_{k=1}^{b+h} \Big(\mathcal{B}_k^{\nu}\mathcal{X}^{\nu}(t) + \mathcal{F}_k^{\nu}\Big) d\mathcal{W}_k(t). \tag{34}$$

Now, by using Lemma 1, and substituting in (33) the right-hand side of the extended state equation in (20), it results:

$$\begin{aligned}
d\,\xi_t = \Bigg(&\mathcal{X}^{\nu T}(t)\dot V(t)\mathcal{X}^{\nu}(t) + \dot g^T(t)\mathcal{X}^{\nu}(t) \\
&+ \mathcal{X}^{\nu T}(t)\mathcal{A}^{\nu T}(\tilde z) V(t)\mathcal{X}^{\nu}(t) + \mathcal{X}^{\nu T}(t) V(t)\mathcal{A}^{\nu}(\tilde z)\mathcal{X}^{\nu}(t) \\
&+ 2u^T(t)\mathcal{H}^{\nu T} V(t)\mathcal{X}^{\nu}(t) + 2\mathcal{D}^{\nu T}(\tilde z) V(t)\mathcal{X}^{\nu}(t) \\
&+ g^T(t)\mathcal{A}^{\nu}(\tilde z)\mathcal{X}^{\nu}(t) + g^T(t)\mathcal{H}^{\nu} u(t) + g^T(t)\mathcal{D}^{\nu}(\tilde z) \\
&+ \mathcal{X}^{\nu T}(t)\Big(\sum_{k=b+1}^{b+h} \mathcal{B}_k^{\nu T} V(t)\mathcal{B}_k^{\nu}\Big)\mathcal{X}^{\nu}(t) \\
&+ 2\mathcal{X}^{\nu T}(t)\Big(\sum_{k=b+1}^{b+h} \mathcal{B}_k^{\nu T} V(t)\mathcal{F}_k^{\nu}\Big) + \sum_{k=1}^{b+h} \mathcal{F}_k^{\nu T} V(t)\mathcal{F}_k^{\nu}\Bigg) dt \\
&+ 2\sum_{k=b+1}^{b+h} \mathcal{X}^{\nu T}(t)\mathcal{B}_k^{\nu T} V(t)\mathcal{X}^{\nu}(t) d\mathcal{W}_k(t) \\
&+ 2\sum_{k=1}^{b+h} \mathcal{F}_k^{\nu T} V(t)\mathcal{X}^{\nu}(t) d\mathcal{W}_k(t) + \sum_{k=1}^{b+h} g^T(t)\mathcal{F}_k^{\nu} d\mathcal{W}_k(t)
\end{aligned}$$

$$+\sum_{k=b+1}^{b+h} g^T(t)\mathcal{B}_k^\nu \mathcal{X}^\nu(t)d\mathcal{W}_k(t); \tag{35}$$

(recall that the summations involving $\mathcal{B}_k^\nu$, $k=1,\dots,b$, vanish according to (17)). By substituting (35) in (31), taking into account that the expectations values of the noise-dependent terms in (35) vanish, and exploiting (25) and (26), it follows that:

$$\begin{aligned} J_\nu(u) = \frac{1}{2}I\!E\Bigg\{ & \int_0^{t_f} \mathcal{X}^{\nu T}(t)\big(V(t)\mathcal{H}^\nu R^{-1}\mathcal{H}^{\nu T}V(t)\big)\mathcal{X}^\nu(t)dt \\ & + \int_0^{t_f} \mathcal{X}^{\nu T}(t)V(t)\mathcal{H}^\nu R^{-1}\mathcal{H}^{\nu T}g(t)dt \\ & + \int_0^{t_f} 2\mathcal{X}^{\nu T}(t)V(t)\mathcal{H}^\nu u(t)dt + \int_0^{t_f} u^T(t)Ru(t)dt \\ & + \int_0^{t_f} \Big(g^T(t)\mathcal{H}^\nu u(t) + g^T(t)\mathcal{D}^\nu(\tilde z) + \sum_{k=1}^{b+h}\mathcal{F}_k^{\nu T}V(t)\mathcal{F}_k^\nu\Big)dt \\ & + \mathcal{X}^{\nu T}(0)V(0)\mathcal{X}^\nu(0) + g^T(0)\mathcal{X}^\nu(0)\Bigg\}. \end{aligned} \tag{36}$$

By considering the gain matrix $L^o(t) = -R^{-1}\mathcal{H}^{\nu T}V(t)$, and the vector $\alpha(t) = -(1/2)R^{-1}\mathcal{H}^{\nu T}g(t)$, the index J_ν is finally given by:

$$\begin{aligned} J_\nu(u) &= \frac{1}{2}I\!E\Bigg\{ \int_0^{t_f} \mathcal{X}^{\nu T}(t)L^{oT}(t)RL^o(t)\mathcal{X}^\nu(t)dt \\ &+ \int_0^{t_f} 2\mathcal{X}^{\nu T}(t)L^{oT}(t)R\alpha(t)dt - \int_0^{t_f} 2\mathcal{X}^{\nu T}(t)L^{oT}(t)Ru(t)dt \\ &+ \int_0^{t_f} u^T(t)Ru(t)dt - \int_0^{t_f} 2\alpha(t)^T Ru(t)dt + \int_0^{t_f} g^T(t)\mathcal{D}^\nu(\tilde z)dt \\ &+ \mathcal{X}^{\nu T}(0)V(0)\mathcal{X}^\nu(0) + g^T(0)\mathcal{X}^\nu(0) + \int_0^{t_f}\sum_{k=1}^{b+h}\mathcal{F}_k^{\nu T}V(t)\mathcal{F}_k^\nu dt\Bigg\} \\ &= \frac{1}{2}I\!E\Bigg\{ \int_0^{t_f} \Big(u(t) - L^o(t)\mathcal{X}^\nu(t) - \alpha(t)\Big)^T R \\ &\qquad \cdot\Big(u(t) - L^o(t)\mathcal{X}^\nu(t) - \alpha(t)\Big)dt \\ &+ \int_0^{t_f}\left(g^T(t)\mathcal{D}^\nu(\tilde z) - \alpha(t)^T R\alpha(t) + \sum_{k=1}^{b+h}\mathcal{F}_k^{\nu T}V(t)\mathcal{F}_k^\nu\right)dt \\ &+ g^T(0)\mathcal{X}^\nu(0) + \mathcal{X}^{\nu T}(0)V(0)\mathcal{X}^\nu(0)\Bigg\}. \end{aligned} \tag{37}$$

Since only the first term in the above expression of J_ν depends of the control law $u(\cdot)$, the minimum of J_ν is achieved by minimizing the quantity:

$$I\!E\left\{\|u(t) - L^o(t)\mathcal{X}^\nu(t) - \alpha(t)\|^2\right\} \tag{38}$$

for almost all $t \in [0, t_f]$. Hence, according to the constraint of $u(t) \in \mathcal{L}_t^p(y^\nu)$, the solution is given by (27-28) and the Theorem is proved.

4. THE ASSOCIATED FILTERING ALGORITHM

In the *complete information* case, that is assuming to have a direct measurement of the extended state $\mathcal{X}^\nu(t)$ (that is direct measurements of $x(t)$ and $z(t)$, actually), it is straightforward from Theorem 2 that the optimal regulator would be: $u^*(t) = L^o(t)\mathcal{X}^\nu(t) + \alpha(t)$. On the contrary, the case under investigation is that of *incomplete information*, therefore we need to solve the filtering problem of providing the estimate of $\mathcal{X}^\nu(t)$ from the available measurements y^ν. It is well known that such a problem is optimally solved (in the sense of the minimum error variance) by the conditional expectation w.r.t. all the Borel transformations of the measurements, whose computation, in general, cannot be obtained through algorithms of finite dimension. It is worthwhile, then, to look for suboptimal estimates providing implementable filtering algorithms. As previously stated in Theorem 2, the extended state estimate $\widetilde{\mathcal{X}^\nu}(t)$ here proposed is the optimal linear estimate among all the $I\!R^{n+m_\nu}$-valued square-integrable affine transformations of the measurements $\{y^\nu(\tau),\ 0 \le \tau \le t \le t_f\}$, which consists of the projection of $\mathcal{X}^\nu$ onto $\mathcal{L}_t^{n+m_\nu}(y^\nu)$:

$$\widehat{\mathcal{X}^\nu}(t) = \mathbf{\Pi}\big[\mathcal{X}^\nu(t)|\mathcal{L}_t^{n+m_\nu}(y^\nu)\big] \tag{39}$$

(formally the projection onto $\mathcal{L}_t^{n+m_\nu}(y^\nu)$ is a random variable such that the difference $\mathcal{X}^\nu(t) - \mathbf{\Pi}\big[\mathcal{X}^\nu(t)|\mathcal{L}_t^{n+m_\nu}(y^\nu)\big]$ is orthogonal to $\mathcal{L}_t^{n+m_\nu}(y^\nu)$, i.e. is uncorrelated with all random variables in $\mathcal{L}_t^{n+m_\nu}(y^\nu)$).

Theorem 3. Consider the stochastic system (20) with $u(t) = u^o(t) = L^o(t)\widehat{\mathcal{X}^\nu}(t) + \alpha(t)$, $L^o(t)$, $\alpha(t)$ as in (28), and $\widehat{\mathcal{X}^\nu}(t)$ as in (39). Then $\widehat{\mathcal{X}^\nu}(t)$ satisfies the equation:

$$\begin{aligned} d\widehat{\mathcal{X}^\nu}(t) = & \Big(\big(\mathcal{A}^\nu(\tilde z) + \mathcal{H}^\nu L^o(t)\big)\widehat{\mathcal{X}^\nu}(t) + \mathcal{H}^\nu\alpha(t) + \mathcal{D}^\nu(\tilde z)\Big)dt \\ & + K(t)\Big(dy^\nu(t) - \mathcal{C}^\nu\widehat{\mathcal{X}^\nu}(t)dt\Big), \end{aligned} \tag{40}$$

with $\widehat{\mathcal{X}^\nu}(0) = I\!E\{\mathcal{X}^\nu(0)\}$ and $K(t) = P(t)\mathcal{C}^{\nu T}(GG^T)^{-1}$, where $P(t)$ is the error covariance matrix evolving according to the equation:

$$\begin{aligned} \dot P(t) = & \mathcal{A}^\nu(\tilde z)P(t) + P(t)\mathcal{A}^{\nu T}(\tilde z) + \sum_{j=1}^{b+h}\big(\mathcal{B}_j^\nu\Psi_\nu(t)\mathcal{B}_j^{\nu T}\big) \\ & + \sum_{j=1}^{b+h}\big(\mathcal{B}_j^\nu\mu_\nu(t) + \mathcal{F}_j^\nu\big)\big(\mathcal{B}_j^\nu\mu_\nu(t) + \mathcal{F}_j^\nu\big)^T \\ & - P(t)\mathcal{C}^{\nu T}(GG^T)^{-1}\mathcal{C}^\nu P(t), \end{aligned} \tag{41}$$

with $P(0) = \mathrm{Cov}\{\mathcal{X}^\nu(0)\}$ and $\mu_\nu(t) = I\!E\{\mathcal{X}^\nu(t)\}$, $\Psi_\nu(t) = \mathrm{Cov}\{\mathcal{X}^\nu(t)\}$ obeying the following equations:

$$\begin{aligned} \dot\mu_\nu(t) &= \big(\mathcal{A}^\nu(\tilde z) + \mathcal{H}^\nu L^o(t)\big)\mu_\nu(t) + \mathcal{H}^\nu\alpha(t) + \mathcal{D}^\nu(\tilde z), \\ \dot\Psi_\nu(t) &= \big(\mathcal{A}^\nu(\tilde z) + \mathcal{H}^\nu L^o(t)\big)\Psi_\nu(t) \\ &\quad + \Psi_\nu(t)\big(\mathcal{A}^\nu(\tilde z) + \mathcal{H}^\nu L^o(t)\big)^T + \sum_{j=b+1}^{b+h}\big(\mathcal{B}_j^\nu\Psi_\nu(t)\mathcal{B}_j^{\nu T}\big) \end{aligned}$$

$$+\sum_{j=1}^{b+h}\big(\mathcal{B}_j^{\nu}\mu_{\nu}(t)+\mathcal{F}_j^{\nu}\big)\big(\mathcal{B}_j^{\nu}\mu_{\nu}(t)+\mathcal{F}_j^{\nu}\big)^T. \tag{42}$$

Proof. The proof is a straightforward consequence of Theorem 4.2 in Carravetta et al. (2007).

Note that, according to the optimal initialization of the filtering algorithm associated to the proposed control law, the second order moments of the initial extended state $\mathcal{X}^{\nu}(0)$ have to be finite and available, that means finite and available moments up to order 2 for x_0 and up to order 2ν for z_0.

Remark 4. The filter proposed in Theorem 3, provides the optimal linear estimate of $\mathcal{X}^{\nu}$ as a function of the observations y^{ν}. However, the available measurements are given by the output y (instead of y^{ν}), therefore the differential dy^{ν} in (40) should be replaced by dy.

Remark 5. The computational burden for real-time implementation can be reduced by eliminating the redundancies in the extended vector Z^{ν} (see Germani et al. (2007) for further details).

5. SIMULATION RESULTS

Numerical simulations have been performed to show the effectiveness of the proposed algorithm. The ones here reported refers to a finite horizon of [0 10]. The linear system under investigation is a third order system, whose matrices are below reported, according to the formalism of (1):

$$A=\begin{bmatrix}-1&0&1\\1&-1&0\\1&0&-2\end{bmatrix},\quad H=\begin{bmatrix}1\\0.2\\-1.5\end{bmatrix},\quad M=\begin{bmatrix}1&1\\0&-1\\2&-1\end{bmatrix},$$

$$N=\begin{bmatrix}1\\1\\1\end{bmatrix},\quad C=\begin{bmatrix}1&0&1\\2&1&-1\end{bmatrix},\quad G=\begin{bmatrix}10&0\\0&10\end{bmatrix}.$$

The state transition map $\phi(\cdot):I\!R^2\mapsto I\!R^2$ and matrix F of the nonlinear exosystem (2) are:

$$\phi\big(z(t)\big)=\begin{pmatrix}-z_1(t)+z_1(t)z_2(t)\\z_1(t)-2z_2(t)-z_1(t)z_2(t)\end{pmatrix},\quad F=\begin{bmatrix}1\\1\end{bmatrix}.$$

The initial states x_0 and z_0 are Gaussian standard random vectors (zero mean and identity covariance matrix). The origin $(0,0)$ has been chosen as the point $\tilde z$ around which the nonlinear exosystem is approximated by means of the Carleman approximation approach. The weight matrices S, Q, R of the index J in (3) are $S=Q=10\cdot I_3$, $R=0.1$.

A second order regulator (that is $\nu=2$) is implemented, whose performances have been compared to the first order regulator, consisting of the optimal linear control law applied to the original linear system (1) endowed with the standard linear approximation of the exosystem. Comparisons with the free-evolution without control have also been obtained (formally denoted by $\nu=0$).

Simulations have been performed according to the Euler-Maruyama method (see Higham (2001)) with integration step $\Delta=10^{-3}$.

Below are reported the values of the index J as the averages of 100 different simulations (with different noise realizations):

$$J_0=1.6226\cdot10^3,\quad J_1=1.5088\cdot10^3,\quad J_0=1.4052\cdot10^3.$$

Note that, by applying the second order regulator, we obtain apparent improvements.

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