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Optimal Filtering for Systems with Multiple Random Measurement Delays

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Abstract: This paper is concerned with the optimal filtering problem for discrete-time stochastic linear system with multiple random measurement delays. Without the state augmentation, the system is transferred to an equivalent system without measurement delays and with random MV (moving average) colored measurement noise. An unbiased optimal filter is developed in the least mean square sense. Its solution depends on the recursion of a Riccati equation and a Lyapunov equation. A simulation shows the effectiveness of the proposed algorithm.

1. INTRODUCTION

In recent years, the research on networked systems has gained lots of interest (Basin et al., 2004; Halevi & Ray, 1988; Nilsson et al., 1998; Yang, 2006). In networked systems, the random delays and packet dropouts are unavoidable in data transmission by unreliable communication networks from sensors to a processing center and from the processing center to end users. The data available in control and estimation may not be up-to-date due to stochastic delays or packet dropouts. So estimation, signal processing and control in the networked systems are very challenging (Zhang & Xie, 2007).

In wireless networks, the systems with stochastic sensor delays, packet dropouts and uncertain observations can be described by a stochastic parameter system (Nahi, 1969; Hadidi et al, 1979; Wang et al, 2003; Yaz et al, 1996, 1998; Ray et al, 1993). The optimal estimation problem for systems with uncertain measurements is investigated in Nahi (1969) and Hadidi et al. (1979), where sensor data that are simply the measurement noises at some samples are used for updating the estimate, resulting in undesirable estimation performance. In fact, previous measurements rather than noises should be used in the absence of valid current sensor data. Wang et al., (2003) consider the variance-constrained filtering problem for systems with measurement and parameter uncertainties.

Yaz et al. (1996, 1998) discuss the least mean square filtering problem for systems with one random sampling delay. The filters derived, however, are not optimal since a non-white noise due to augmentation is treated as a white noise. Ray et al. (1993) gives a modification of the minimum variance state estimator to accommodate the effects of random delays in sensor data arrival at the controller terminal. In Yaz et al. (1998), the state estimation problem for discrete-time linear systems with stochastic parameters is treated based on a linear matrix inequality approach and the results are applied to the problem of state estimation with random sensor delays or packet dropouts. An extended Kalman filter is given for interconnected networks with delayed measurements in Su & Lu (2001) whereas Nakamori et al. (2003, 2005) deal with the recursive least-squares linear estimation for signals with random delays by using a covariance information approach. Robust estimation problems for systems with random delays and uncertain measurements are also investigated in Wang et al. (2004, 2006). Recently, the optimal H_2 filtering for systems with random sensor delays, multiple packet dropouts and uncertain observations is presented in Sahebsara et al. (2007) where a unified stochastic parameter model is used. Franck et al. (2007) design the filter for system with multiple sensor having different delays. Furthermore, multiple random sampling delays models are also discussed. We also note that the mean square stochastic stability for some kind of systems with stochastic delays has been investigated (Koning, 1984; Kolmanovskii et al., 2003; Sinopoli et al., 2004).

So far, to the best of the author's knowledge, the research of estimation problem for systems with random measurement delays is mainly focused on one sampling delay; however, the estimation problem for systems with the bounded multiple random delays is seldom studied. In this paper, we investigate the filtering problem for systems with multiple random measurement delays, where the largest random delay is limited within a bound. It is transferred to an equivalent system without delays and with random MV (moving average) coloured measurement noise. An unbiased optimal filter is designed in the least mean square sense. Its solution depends on a Riccati equation and a Lyaponov equation.

2. PROBLEM FORMULATION

Consider the discrete time-invariant linear stochastic system with multiple packet dropouts

$$x(t+1) = \Phi x(t) + \Gamma w(t) \tag{1a}$$

$$z(t) = Hx(t) + v(t) \tag{1b}$$

$$y(t) = \xi_0(t)z(t) + (1-\xi_0(t))\xi_1(t)z(t-1) + \cdots + (1-\xi_0(t))(1-\xi_1(t))\cdots(1-\xi_{N-1}(t))z(t-N),\ N \ge 1 \tag{1c}$$

where $x(t) \in R^n$ is the state, $z(t) \in R^m$ is the output, $y(t) \in R^m$ is the received measurement, $w(t) \in R^r$ and $v(t) \in R^m$ are white noises, and Φ, Γ, H are constant matrices with suitable dimensions, and $\xi_i(t)$, $0 \le i \le N-1$ are mutually independent scalar binary distributed random variables, i.e., $\mathrm{P}\{\xi_i(t)=1\}=\alpha_i$ and $\mathrm{P}\{\xi_i(t)=0\}=1-\alpha_i$ and is uncorrelated with other random variables. Furthermore, we only know the probability α_i but don't know the values of $\xi_i(t)$ at each time instant.

Model (1c) means that the random measurement delays can be encountered in data transmission of network control systems. We assume that the largest random delay is limited by N, N>0 is a certain integer.

Assumption 1. $w(t)$ and $v(t)$ are independent white noises with zeros mean and variances Q_w and Q_v.

Assumption 2. The initial state $x(0)$ with mean zero and covariance P_0 is independent of $w(t)$, $v(t)$ and $\xi(t)$.

Assumption 3. The matrix Φ is non-singular.

The assumptions 1 and 2 are general for time-delay systems (1a)-(1b). Assumption 3 is acceptable since system (1a)-(1b) can be a direct consequence of discretization of a continuous-time system.

Our aim is to design the following filter based on the measurements $(y(t), y(t-1), \cdots, y(0))$:

$$\hat{x}(t+1)=F(t)\hat{x}(t)+K(t)y(t+1) \tag{2}$$

with the initial value $\hat{x}(0)=\mu_0$. We will determine the gain matrices $F(t)$ and $K(t)$ such that it satisfies the unbiasedness and the least mean square criterion.

Remark 1. From distribution of $\xi_i(t)$, we have the properties $\mathrm{E}[\xi_i(t)]=\alpha_i$, $\mathrm{Cov}[\xi_i(t)]=\alpha_i(1-\alpha_i)$, $\mathrm{E}[\xi_i^2(t)]=\alpha_i$, $\mathrm{E}[(1\text{-}\xi_i(t))^2]=1-\alpha_i$, $\mathrm{E}[\xi_i(t)(1\text{-}\xi_i(t))]=0$, $\mathrm{E}[\xi_i(k)(1\text{-}\xi_i(t))]=\alpha_i(1-\alpha_i)$, $k\neq t$, and $\mathrm{E}[\xi_i(k)\xi_j(t)]=\alpha_i\alpha_j$, $k\neq t$ or $i\neq j$.

3. OPTIMAL FILTER DESIGN

Before designing the optimal filter of system (1), we first transfer (1) to an equivalent system and introduce some lemmas.

Substituting (1b) into (1c), we have

$$y(t)=\sum_{i=0}^{N}a_i(t)Hx(t-i)+\sum_{i=0}^{N}a_i(t)v(t-i),\ N\geq 1 \tag{3}$$

where $a_0(t)=\xi_0(t)$, $a_i(t)=\prod_{k=0}^{i-1}(1-\xi_k(t))\xi_i(t)$, $0<i<N$, and $a_N(t)=\prod_{k=0}^{N-1}(1-\xi_k(t))$.

Further, using (1a) by iteration and Assumption 3, we have relation

$$x(t-i)=\Phi^{-i}x(t)-\sum_{k=1}^{i}\Phi^{k-i-1}\Gamma w(t-k) \tag{4}$$

Substituting (4) into (3), then (3) can be rewritten as

$$y(t)=\sum_{i=0}^{N}a_i(t)H\Phi^{-i}x(t)-\sum_{i=1}^{N}\sum_{k=1}^{i}a_i(t)H\Phi^{k-i-1}\Gamma w(t-k)+\sum_{i=0}^{N}a_i(t)v(t-i) \tag{5}$$

Now, we obtain the equivalent model (1a) and (5) that describe the system with N-order random MA (moving average) colored measurement noise.

The following lemmas will be used in the later sections.

Lemma 1. Random variable $a_i(t)$ has the following properties

$$\bar{a}_0=\mathrm{E}[a_0(t)]=\alpha_0;\ \bar{a}_i=\mathrm{E}[a_i(t)]=\prod_{k=0}^{i-1}(1-\alpha_k)\alpha_i,\ 0<i<N;$$

$$\bar{a}_N=\mathrm{E}[a_N(t)]=\prod_{k=0}^{N-1}(1-\alpha_k);\ \mathrm{E}[a_i(t)a_j(k)]=\bar{a}_i\bar{a}_j,\ k\neq t \tag{6}$$

$$A_{ij}=\mathrm{E}[(a_i(t)-\bar{a}_i)(a_j(t)-\bar{a}_j)]=\begin{cases}\bar{a}_i(1-\bar{a}_i), & i=j\\ -\bar{a}_i\bar{a}_j, & i\neq j\end{cases} \tag{7}$$

Proof. We can readily obtain (6) and (7) from Remark 1. The detailed is omitted. □

Lemma 2. For system (1a), we have the correlation function $q(t)=\mathrm{E}[x(t)x^{\mathrm{T}}(t)]$ satisfying

$$q(t+1)=\Phi q(t)\Phi^{\mathrm{T}}+\Gamma Q_w\Gamma^{\mathrm{T}} \tag{8}$$

with the initial value $q(0)=P_0+\mu_0\mu_0^{\mathrm{T}}$.

Proof. These directly follow from (1a). □

For system (1a)-(3), we can design the optimal estimator by augmented approach. However, the augmented approach will bring expensive computational cost and large memory space due to the high-dimension state. In the following, we will design the optimal filter (2).

Theorem 1. For system (1a)-(5) satisfying Assumptions 1-3, the gain matrices of the optimal filter (2) are computed by

$$F(t)=\Phi-K(t)M \tag{9}$$

$$M=\sum_{i=0}^{N}\bar{a}_iH\Phi^{-(i-1)} \tag{10}$$

$$K(t)=[\Phi P(t)M^{\mathrm{T}}+\alpha_0\Gamma Q_w\Gamma^{\mathrm{T}}H^{\mathrm{T}}-\Phi P_{\eta\tilde{x}}^{\mathrm{T}}(t)]\Lambda^{-1}(t) \tag{11}$$

$$\Lambda(t)=MP(t)M^{\mathrm{T}}+P_\eta(t)+\alpha_0H\Gamma Q_w\Gamma^{\mathrm{T}}H^{\mathrm{T}}-MP_{\eta\tilde{x}}^{\mathrm{T}}(t)-P_{\eta\tilde{x}}(t)M^{\mathrm{T}} \tag{12}$$

$$\begin{aligned}P_\eta(t)=&\sum_{i=0}^{N}\sum_{j=0}^{N}A_{ij}H\Phi^{-(i-1)}q(t)(\Phi^{-(j-1)})^{\mathrm{T}}H^{\mathrm{T}}+\\&\sum_{i=2}^{N}\sum_{k=1}^{i-1}\bar{a}_iH\Phi^{k-i}\Gamma Q_w\Gamma^{\mathrm{T}}(\Phi^{k-i})^{\mathrm{T}}H^{\mathrm{T}}+\sum_{i=0}^{N}\bar{a}_iQ_v-\\&\sum_{i=0}^{N}\sum_{j=2}^{N}\sum_{k=1}^{j-1}A_{ij}H\Phi^{k-i}\Gamma Q_w\Gamma^{\mathrm{T}}(\Phi^{k-j})^{\mathrm{T}}H^{\mathrm{T}}-\\&\sum_{i=2}^{N}\sum_{j=0}^{N}\sum_{k=1}^{i-1}A_{ij}H\Phi^{k-i}\Gamma Q_w\Gamma^{\mathrm{T}}(\Phi^{k-j})^{\mathrm{T}}H^{\mathrm{T}}\end{aligned} \tag{13}$$

$$P_{\eta\tilde{x}}(t)=\sum_{l=1}^{N}P_\eta^lK^{\mathrm{T}}(t-l)F^{\mathrm{T}}(t,t-l+1)+\sum_{l=1}^{N}P_{\eta\zeta}^l(t)F^{\mathrm{T}}(t,t-l+1) \tag{14}$$

$$P_\eta^l = \sum_{k=l+1}^{N-1}\sum_{i=k+1}^{N}\sum_{j=k-l+1}^{N} \bar{a}_i\bar{a}_j H\Phi^{k-i}\Gamma Q_w\Gamma^{\mathrm{T}}(\Phi^{k-j-l})^{\mathrm{T}}H^{\mathrm{T}} + \sum_{i=l}^{N}\bar{a}_i\bar{a}_{i-l}Q_v \tag{15}$$

$$P_{\eta\zeta}^l(t) = \sum_{i=l+1}^{N}\bar{a}_i H\Phi^{l-i}\Gamma Q_w\Gamma^{\mathrm{T}} - \sum_{i=l+1}^{N}\alpha_0\bar{a}_i H\Phi^{l-i}\Gamma Q_w\Gamma^{\mathrm{T}}H^{\mathrm{T}}K^{\mathrm{T}}(t-l) \tag{16}$$

The filtering error variance is computed by

$$P(t+1) = \Phi P(t)\Phi^{\mathrm{T}} + \Gamma Q_w\Gamma^{\mathrm{T}} - K(t)\Lambda(t)K^{\mathrm{T}}(t) \tag{17}$$

where we define $F(t,t-l) = \prod_{i=1}^{l}F(t-i)$ with $F(i,i) = I_n$. A_{ij} is computed by Lemma 1. $q(t)$ is computed by Lemma 2. The initial values are $P(0) = P_0$ and $q(0) = P_0 + \mu_0\mu_0^{\mathrm{T}}$.

Proof. From (5), we have

$$y(t+1) = \sum_{i=0}^{N}a_i(t+1)H\Phi^{-(i-1)}x(t) - \sum_{i=2}^{N}\sum_{k=1}^{i-1}a_i(t+1)H\Phi^{k-i}\Gamma w(t-k) + \sum_{i=0}^{N}a_i(t+1)v(t+1-i) + a_0(t+1)H\Gamma w(t) \tag{18}$$

From (1a), (9) and (18), we have the filtering error equation

$$\tilde{x}(t+1) = [\Phi - F(t) - K(t)\sum_{i=0}^{N}a_i(t+1)H\Phi^{-(i-1)}]x(t) + F(t)\tilde{x}(t) + [\Gamma - a_0(t+1)K(t)H\Gamma]w(t) + K(t)\sum_{i=2}^{N}\sum_{k=1}^{i-1}a_i(t+1)H\Phi^{k-i}\Gamma w(t-k) - K(t)\sum_{i=0}^{N}a_i(t+1)v(t+1-i) \tag{19}$$

where $\tilde{x}(t) = x(t) - \hat{x}(t)$. To guarantee the unbiasedness, we require $\tilde{x}(0) = 0$ and

$$\mathrm{E}[\Phi - F(t) - K(t)\sum_{i=0}^{N}a_i(t+1)H\Phi^{-(i-1)}] = 0 \tag{20}$$

which yields (9) and (10) by using Lemma 1. Then, (19) can be rewritten as

$$\tilde{x}(t+1) = F(t)\tilde{x}(t) + K(t)\eta(t) + \zeta(t) \tag{21}$$

where $\zeta(t)$ and $\eta(t)$ are defined as

$$\zeta(t) = [\Gamma - a_0(t+1)K(t)H\Gamma]w(t),\quad \eta(t) = Z(t) + W(t) - V(t) \tag{22}$$

with

$$Z(t) = \sum_{i=0}^{N}(\bar{a}_i - a_i(t+1))H\Phi^{-(i-1)}x(t),\quad W(t) = \sum_{i=2}^{N}\sum_{k=1}^{i-1}a_i(t+1)H\Phi^{k-i}\Gamma w(t-k),\quad V(t) = \sum_{i=0}^{N}a_i(t+1)v(t+1-i) \tag{23}$$

So, the filtering error covariance matrix is derived as follows

$$P(t+1) = F(t)P(t)F^{\mathrm{T}}(t) + K(t)P_\eta(t)K^{\mathrm{T}}(t) + Q_\zeta(t) + K(t)P_{\eta\tilde{x}}(t)F^{\mathrm{T}}(t) + F(t)P_{\tilde{x}\eta}(t)K^{\mathrm{T}}(t) \tag{24}$$

where we define $P(t) = \mathrm{E}[\tilde{x}(t)\tilde{x}^{\mathrm{T}}(t)]$, $P_\eta(t) = \mathrm{E}[\eta(t)\eta^{\mathrm{T}}(t)]$ and $P_{\eta\tilde{x}}(t) = \mathrm{E}[\eta(t)\tilde{x}^{\mathrm{T}}(t)]$ with $P_{\eta\tilde{x}}(t) = P_{\tilde{x}\eta}^{\mathrm{T}}(t)$. $Q_\zeta(t) = \mathrm{E}[\zeta(t)\zeta^{\mathrm{T}}(t)]$ is computed by

$$Q_\zeta(t) = \Gamma Q_w\Gamma^{\mathrm{T}} - \alpha_0\Gamma Q_w\Gamma^{\mathrm{T}}H^{\mathrm{T}}K^{\mathrm{T}}(t) - \alpha_0 K(t)H\Gamma Q_w\Gamma^{\mathrm{T}} + \alpha_0 K(t)H\Gamma Q_w\Gamma^{\mathrm{T}}H^{\mathrm{T}}K^{\mathrm{T}}(t) \tag{25}$$

From the definition of $\eta(t)$ in (22), we have

$$P_\eta(t) = \mathrm{E}[\eta(t)\eta^{\mathrm{T}}(t)] = \mathrm{E}[Z(t)Z^{\mathrm{T}}(t)] + \mathrm{E}[W(t)W^{\mathrm{T}}(t)] + \mathrm{E}[V(t)V^{\mathrm{T}}(t)] + \mathrm{E}[Z(t)W^{\mathrm{T}}(t)] + \mathrm{E}[W(t)Z^{\mathrm{T}}(t)] \tag{26}$$

where

$$\mathrm{E}[Z(t)Z^{\mathrm{T}}(t)] = \mathrm{E}\left\{\left[\sum_{i=0}^{N}(\bar{a}_i - a_i(t+1))H\Phi^{-(i-1)}x(t)\right]\times\left[\sum_{j=0}^{N}(\bar{a}_j - a_j(t+1))H\Phi^{-(j-1)}x(t)\right]^{\mathrm{T}}\right\} = \sum_{i=0}^{N}\sum_{j=0}^{N}A_{ij}H\Phi^{-(i-1)}q(t)(\Phi^{-(j-1)})^{\mathrm{T}}H^{\mathrm{T}} \tag{27}$$

$$\mathrm{E}[W(t)W^{\mathrm{T}}(t)] = \mathrm{E}\left\{\left[\sum_{i=2}^{N}\sum_{k=1}^{i-1}a_i(t+1)H\Phi^{k-i}\Gamma w(t-k)\right]\times\left[\sum_{j=2}^{N}\sum_{k=1}^{j-1}a_j(t+1)H\Phi^{k-j}\Gamma w(t-k)\right]^{\mathrm{T}}\right\} = \sum_{i=2}^{N}\sum_{k=1}^{i-1}\bar{a}_i H\Phi^{k-i}\Gamma Q_w\Gamma^{\mathrm{T}}(\Phi^{k-i})^{\mathrm{T}}H^{\mathrm{T}} \tag{28}$$

$$\mathrm{E}[V(t)V^{\mathrm{T}}(t)] = \mathrm{E}\left\{\left[\sum_{i=0}^{N}a_i(t+1)v(t+1-i)\right]\times\left[\sum_{j=0}^{N}a_j(t+1)v(t+1-j)\right]^{\mathrm{T}}\right\} = \sum_{i=0}^{N}\bar{a}_i Q_v \tag{29}$$

$$\mathrm{E}[Z(t)W^{\mathrm{T}}(t)] = \mathrm{E}\left\{\left[\sum_{i=0}^{N}(\bar{a}_i - a_i(t+1))H\Phi^{-(i-1)}(\Phi^{j-1}x(t-j+1) + \sum_{k=1}^{j-1}\Phi^{k-1}\Gamma w(t-k))\right]\times\left[\sum_{j=2}^{N}\sum_{k=1}^{j-1}a_j(t+1)H\Phi^{k-j}\Gamma w(t-k)\right]^{\mathrm{T}}\right\} = -\sum_{i=0}^{N}\sum_{j=2}^{N}\sum_{k=1}^{j-1}A_{ij}H\Phi^{k-i}\Gamma Q_w\Gamma^{\mathrm{T}}(\Phi^{k-j})^{\mathrm{T}}H^{\mathrm{T}} \tag{30}$$

Substituting (27)-(30) into (26) yields (13).

From (21)-(22), $P_{\eta\tilde{x}}(t)$ is derived as follows

$$P_{\eta\tilde{x}}(t) = \mathrm{E}[\eta(t)\tilde{x}^{\mathrm{T}}(t-1)]F^{\mathrm{T}}(t-1) + \mathrm{E}[\eta(t)\eta^{\mathrm{T}}(t-1)]K^{\mathrm{T}}(t-1) + \mathrm{E}[\eta(t)\zeta^{\mathrm{T}}(t-1)] \tag{31}$$

(31) by iteration yields

$$P_{\eta\tilde{x}}(t) = \mathrm{E}[\eta(t)\tilde{x}^{\mathrm{T}}(t-N)]F^{\mathrm{T}}(t,t-N) + \sum_{l=1}^{N}\mathrm{E}[\eta(t)\eta^{\mathrm{T}}(t-l)]K^{\mathrm{T}}(t-l)F^{\mathrm{T}}(t,t-l+1) + \sum_{l=1}^{N}\mathrm{E}[\eta(t)\zeta^{\mathrm{T}}(t-l)]F^{\mathrm{T}}(t,t-l+1) \tag{32}$$

Noting (21)-(23), we have $\mathrm{E}[\eta(t)\tilde{x}^{\mathrm{T}}(t-N)] = 0$. (14) can be obtained from (32) by defining $P_{\eta}^{l}(t) = \mathrm{E}[\eta(t)\eta^{\mathrm{T}}(t-l)]$ and $P_{\eta\zeta}^{l}(t) = \mathrm{E}[\eta(t)\zeta^{\mathrm{T}}(t-l)]$. Next, we will derive (15)-(16).
From the definition of $\eta(t)$ in (22), we also have

$$P_{\eta}^{l}(t) = \mathrm{E}[\eta(t)\eta^{\mathrm{T}}(t-l)] = \mathrm{E}[Z(t)Z^{\mathrm{T}}(t-l)] + \mathrm{E}[W(t)W^{\mathrm{T}}(t-l)] + \mathrm{E}[V(t)V^{\mathrm{T}}(t-l)] + \mathrm{E}[Z(t)W^{\mathrm{T}}(t-l)] + \mathrm{E}[W(t)Z^{\mathrm{T}}(t-l)] \tag{33}$$

From (23) and Lemma 1, we have $\mathrm{E}[Z(t)Z^{\mathrm{T}}(t-l)] = 0$, $\mathrm{E}[Z(t)W^{\mathrm{T}}(t-l)] = 0$ and $\mathrm{E}[W(t)Z^{\mathrm{T}}(t-l)] = 0$ for $l \ge 1$.

$$\mathrm{E}[W(t)W^{\mathrm{T}}(t-l)] = \mathrm{E}\left\{\left[\sum_{i=2}^{N}\sum_{k=1}^{i-1}a_i(t+1)H\Phi^{k-i}\Gamma w(t-k)\right]\times\left[\sum_{j=2}^{N}\sum_{k=1}^{j-1}a_j(t+1-l)H\Phi^{k-j}\Gamma w(t-k-l)\right]^{\mathrm{T}}\right\}$$
$$= \mathrm{E}\left\{\left[\sum_{k=1}^{N-1}\sum_{i=k+1}^{N}a_i(t+1)H\Phi^{k-i}\Gamma w(t-k)\right]\times\left[\sum_{k=l+1}^{N+l-1}\sum_{j=k-l+1}^{N}a_j(t+1-l)H\Phi^{k-j-l}\Gamma w(t-k)\right]^{\mathrm{T}}\right\}$$
$$= \sum_{k=l+1}^{N-1}\sum_{i=k+1}^{N}\sum_{j=k-l+1}^{N}\bar{a}_i\bar{a}_jH\Phi^{k-i}\Gamma Q_w\Gamma^{\mathrm{T}}(\Phi^{k-j-l})^{\mathrm{T}}H^{\mathrm{T}} \tag{34}$$

$$\mathrm{E}[V(t)V^{\mathrm{T}}(t-l)] = \mathrm{E}\left\{\left[\sum_{i=0}^{N}a_i(t+1)v(t+1-i)\right]\times\left[\sum_{j=0}^{N}a_j(t-l+1)v(t-l+1-j)\right]^{\mathrm{T}}\right\} = \sum_{i=l}^{N}\bar{a}_i\bar{a}_{i-l}Q_v \tag{35}$$

Substituting (34) and (35) into (33) and noting that $P_{\eta}^{l}(t)$ is not relative to time t, we have (15) by letting $P_{\eta}^{l}(t) = P_{\eta}^{l}$.
From (22), we have

$$P_{\eta\zeta}^{l}(t) = \mathrm{E}[\eta(t)\zeta^{\mathrm{T}}(t-l)] = \mathrm{E}[Z(t)\zeta^{\mathrm{T}}(t-l)] + \mathrm{E}[W(t)\zeta^{\mathrm{T}}(t-l)] \tag{36}$$

From Lemma 1, we have $\mathrm{E}[Z(t)\zeta^{\mathrm{T}}(t-l)] = 0$ for $l \ge 1$, and

$$\mathrm{E}[W(t)\zeta^{\mathrm{T}}(t-l)] = \mathrm{E}\left\{\left[\sum_{i=2}^{N}\sum_{k=1}^{i-1}a_i(t+1)H\Phi^{k-i}\Gamma w(t-k)\right]\times\left[(\Gamma - a_0(t+1-l)K(t-l)H\Gamma)w(t-l)\right]^{\mathrm{T}}\right\}$$
$$= \sum_{i=l+1}^{N}\bar{a}_iH\Phi^{l-i}\Gamma Q_w\Gamma^{\mathrm{T}} - \sum_{i=l+1}^{N}\alpha_0\bar{a}_iH\Phi^{l-i}\Gamma Q_w\Gamma^{\mathrm{T}}H^{\mathrm{T}}K^{\mathrm{T}}(t-l) \tag{37}$$

Substituting (37) into (36) yields (16).

Substituting (9) and (25) into (24) and completing the square, we have

$$P(t+1) = \Phi P(t)\Phi^{\mathrm{T}} + \Gamma Q_w\Gamma^{\mathrm{T}} + \left\{K(t) - [\Phi P(t)M^{\mathrm{T}} + \alpha_0 H\Gamma Q_w\Gamma^{\mathrm{T}}H^{\mathrm{T}} - \Phi P_{\eta\tilde{x}}^{\mathrm{T}}(t)]\Lambda^{-1}(t)\right\}\Lambda(t) \times \left\{K(t) - [\Phi P(t)M^{\mathrm{T}} + \alpha_0 H\Gamma Q_w\Gamma^{\mathrm{T}}H^{\mathrm{T}} - \Phi P_{\eta\tilde{x}}^{\mathrm{T}}(t)]\Lambda^{-1}(t)\right\}^{\mathrm{T}} - [\Phi P(t)M^{\mathrm{T}} + \alpha_0\Gamma Q_w\Gamma^{\mathrm{T}}H^{\mathrm{T}} - \Phi P_{\eta\tilde{x}}^{\mathrm{T}}(t)]\Lambda^{-1}(t)\times[\Phi P(t)M^{\mathrm{T}} + \alpha_0\Gamma Q_w\Gamma^{\mathrm{T}}H^{\mathrm{T}} - \Phi P_{\eta\tilde{x}}^{\mathrm{T}}(t)]^{\mathrm{T}} \tag{38}$$

where $\Lambda(t)$ is defined by (12). To minimize the filtering error variance, we only require the gain matrix $K(t)$ to satisfy (11). Further, we can obtain (17) from (38). □

Remark 2. Noting (6), the optimal filter (9)-(17) is just the standard Kalman filter when there are no packet dropouts, i.e., $\alpha_0 = 1$. Also, its multiplication and division has the quantity grade of N^2n^3 less than N^3n^3 of the augmentation approach.

4. SIMULATION EXAMPLE

Consider a numerical example

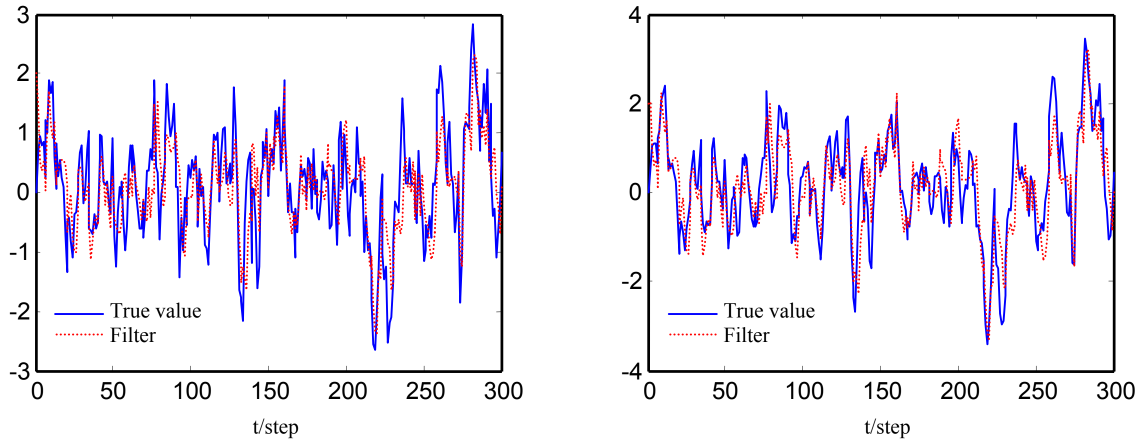
$$x(t+1) = \begin{bmatrix}0.8 & 0\\0.9 & 0.2\end{bmatrix}x(t) + \begin{bmatrix}0.6\\0.5\end{bmatrix}w(t) \tag{39}$$

$$z(t) = [1 \quad 1]x(t) + v(t) \tag{40}$$

$$y(t) = \xi_0(t)z(t) + (1-\xi_0(t))\xi_1(t)z(t-1) + (1-\xi_0(t))(1-\xi_1(t))z(t-2) \tag{41}$$

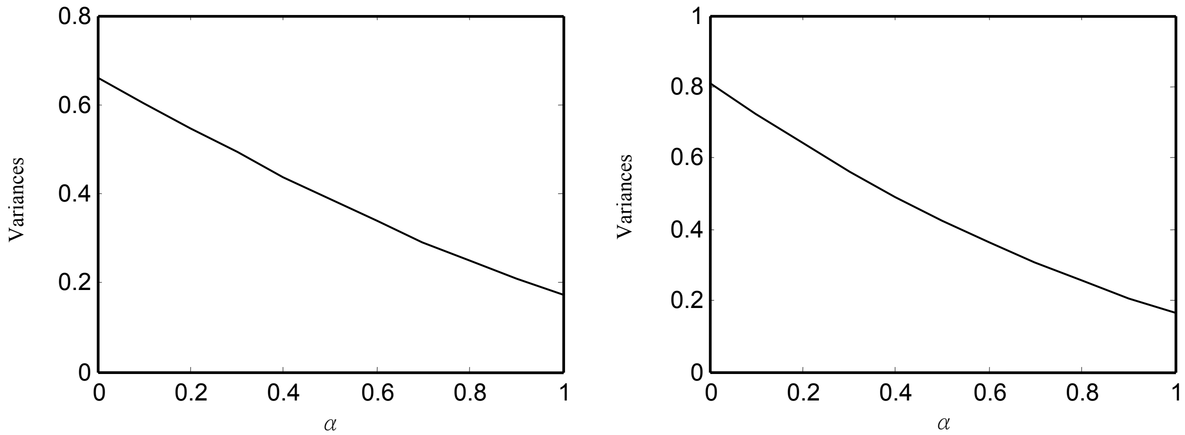
where $v(t)$ is the measurement noise with mean zero and variance σ_v^2, and is independent with Gaussian white noise $w(t)$ with mean zero and variance σ_w^2. $\xi_i(t)$, $i = 0,1$ are scalar binary distributed random variables with probability $\mathrm{P}\{\xi_i(t) = 1\} = \alpha_i$ and $\mathrm{P}\{\xi_i(t) = 0\} = 1-\alpha_i$, $i = 0,1$.

In the simulation, setting noise variances $\sigma_w^2 = 1$, $\sigma_v^2 = 1$, $\alpha_i = \alpha = 0.5$, $i = 0,1$, the initial value $x(0) = [2,2]^{\mathrm{T}}$ and $P_0 = 0.1I_2$, where I_2 is the identity matrix, and we take 300 sampling data. Applying Theorems 1, we have the optimal filter $\hat{x}(t)$ shown in Fig.1 where solid curves denote true values and dotted curves denote estimates. We can compute the steady-state filtering error variance matrix of the optimal filter as $P = \begin{bmatrix}0.3864 & 0.3840\\0.3840 & 0.4222\end{bmatrix}$. Fig.2 shows the steady-state filtering error variances of the optimal filter when $0 \le \alpha \le 1$. From Fig.2, we see that the optimal filter has better accuracy as α increases. Furthermore, we can verify that the obtained result is just the optimal Kalman filter when there is no random measurement delays, i.e., $\alpha = 1$.



(a) The first state component $x_1(t)$ and filter $\hat{x}_1(t)$ (b) The second state component $x_2(t)$ and filter $\hat{x}_2(t)$

Fig.1 The optimal filter under $\alpha = 0.5$



(a) Filtering error variances for the first state component $x_1(t)$ (b) Filtering error variances for the second state component $x_2(t)$

Fig.2 The optimal filtering error variances for different random delay rate $0 \le \alpha \le 1$

6. CONCLUSIONS

For the problem of multiple random delays in networked systems, we have derived the optimal filter in the unbiased least mean square sense. It is obtained based on an equivalent system without delays and with bounded random MV (moving average) colored measurement noise, while the state augmentation is avoided. Its solution depends on the recursion of a Riccati equation and a Lyapunov equation. Furthermore, the proposed optimal filter is reduced to the standard Kalman filter when there are no random measurement delays.

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