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An Approach for the MRP Parameterization Under Lead Time Uncertainty: Branch and Cut Algorithm

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Abstract: Inventory control in a Supply chain is crucial for companies who wish to satisfy their customer demands on time as well as controlling costs. A common approach is to use the MRP techniques. However, these techniques are based on the supposition that lead times are known. In a Supply chain the lead times are often random variables. Therefore, an efficient exact approach to aid in MRP parameterization under lead time uncertainties was developed; more precisely the approach has as objective to calculate planned lead times when the component procurement times are random. The aim is to find the values of planned lead times which minimize the sum of the average component holding cost and the average backlogging cost. The developed approach is based on a mathematical model of this problem with discrete decision variables and on a Branch and Cut algorithm.

1. INTRODUCTION

Efficient replenishment planning is a very important problem for industry. A poor inventory control policy leads to overstocking or stockout situations. In the former, the generated inventories are expensive and in the later there are shortages and penalties due to unsatisfied customer demands.

Material Requirements Planning (MRP) is a commonly accepted approach for replenishment planning in major companies and for Supply Chain Management (Axsäter, 2006). However, MRP is based on the supposition that the demand and lead times are known. Replenishment order dates (release dates) are calculated for a series of discrete time intervals (time buckets) based on the demand and taking into account the fixed lead time: the release date is equal to the due date (demand) minus the lead time.

This premise of deterministic environment seems somewhat off base since most production occurs stochastically and product lead times and finished product demands rarely are forecasted reliably (due to machine breakdown, transport delay, customer demand variations…). Therefore, in real life, the deterministic assumptions embedded in MRP are often too limited.

Fortunately, the MRP approach can be adapted for replenishment planning under uncertainties by searching the optimal values of its parameters. For the case of random lead times, the planned lead time will be equal to the forecasted lead time plus the safety lead time (safety stock). These planned lead time values are a trade-off between overstocking and stockout while minimizing the total cost. This problem is called MRP parameterization.

In literature, the majority of publications are devoted to the MRP parameterization under customer demand uncertainties. As to random lead times, the number of publications is modest in spite of their significant importance. The motivation of this paper is contributing the development of new efficient methods for MRP parameterization under lead time uncertainties.

The supply planning problems under lead time uncertainties are not sufficiently studied especially in the context of assembly systems (Porteus, 1990).

In this paper, for this task of MRP parameterization for assembly systems, an abstract inventory/ production model with several types of components and one type of finished product is proposed. In this mathematical model the finished product demand is constant and assembly capacity is considered infinite. The lead times for orders made at different periods (time buckets) for the same type of component are independent and identically distributed discrete random variables. This abstraction allows us to concentrate on the problem at hand and obtain relevant results for planned lead time optimization.

The body of the paper is organized as follows. Section 2 presents related works. Section 3 deals with the problem description. Section 4 leads to the mathematical model of the problem. Section 5 develops an optimization algorithm. Experimental results are reported in Section 6, and finally, some concluding remarks are given in Section 7.

2. RELATED WORKS

The problem of MRP parameterization under lead time uncertainties has been often studied via simulation. Gupta and Brennan (1995) show that lead time uncertainty has a large influence on the total inventory management cost. Ho and Ireland (1998) illustrate that lead time uncertainty affects stability of a MRP system no matter what lot-sizing method used or demand forecast error obtained. Molinder (1997) study the problem of planned lead time (safety lead time/safety stock) calculation via simulation and proposes a simulated annealing algorithm to find appropriate safety stock and/or safety lead time. The simulations show that the overestimated planned lead times conduct to excessive inventory, and underestimated planned lead times introduce shortages and delays.

In assembly systems there are several suppliers at each stage, and so, there is dependence among the different component inventories at the same stage. Yano (1987) considers a particular problem for two-level assembly systems with only two types of components at stage 2 and one type of components at stage 1. The delivery times for the three components are stochastic continuous variables. The problem is to find the planned lead times for MRP minimizing the sum of holding and tardiness costs.

A single period model and an optimization algorithm were developed. Tang and Grubbström (2003) consider a two component assembly system with stochastic lead times (for components) and fixed finished product demand. This study is similar to (Yano, 1987). However, here, the process time at level 1 is also assumed to be stochastic, the due date is known and the optimal planned lead times are smaller than the due date. The objective is to minimize the total stockout and inventory holding costs. The Laplace transform procedure is used to capture the stochastic properties of lead times. The optimal safety lead times, which are the difference between planned and expected lead time are derived. Another interesting single period model was proposed in (Chu et al., 1993) which deals with a punctual fixed demand for one finished product. The model gives optimal values of the component planned lead times for one-level assembly systems with random component procurement times.

Wilhelm and Som (1998) studied a two-component assembly system using queuening models and showed that a renewal process can be used to describe the end-item inventory level evolution. The optimization of several component stocks is replaced by the optimization of finished product stock. To perform this replacement, a simplified supply policy for component ordering was introduced. Another multi-period model is proposed in (Gurnani et al., 1996) for assembly systems with two types of components and the lead time probability distributions are limited to two periods.

For comprehensive reviews of the literature on the other models which can be used for MRP parameterization under uncertainties see (Yeung et al., 1998; Dolgui et al., 2005; Mula et al., 2006).

In this paper, the definition of planned lead times in an MRP environment for assembly systems under component lead times uncertainties is considered. A solution of this problem gives for each component type used an optimal value of planned lead time. To solve this multi-component and multi-period (for all time buckets) inventory control problem a specially developed mathematical model is proposed. To our knowledge, there isn't any other multi-period model in literature that gives an optimal solution in the case of MRP controlled assembly systems with several types of components and random lead times.

This article is the sequel to our earlier paper (Louly and Dolgui, 2002). In this paper, a more universal case is presented, when the unit holding costs aren't the same for all components and the component lead times are not iid random variables. No restrictive hypothesis is made on such random variables; we only suppose that the distribution probabilities are known in advance.

3. OPTIMISATION MODEL

In the MRP approach, replenishment order dates (release date) for each component are calculated for a series of discrete time intervals (time buckets) based on the demand and taking into account a fixed lead time: the release date is equal to the due date (demand) minus the lead time. For the case of random lead times, in industry, a supply reliability coefficient (□ 1) is assigned to each supplier. The planned lead times for MRP are calculated by multiplying the contractual lead time by the corresponding supplier reliability coefficient. The choice of these coefficients (which give safety lead times) is based on past experience. However, this approach is subjective and can be non optimal if we need to minimize the total cost for MRP systems. The supplier reliability coefficients (safety lead times and so planned lead times) can be calculated more precisely taking into account inventory holding and backlogging costs, if it is possible to formulate a corresponding inventory control model and to solve it optimally. Such an inventory control model must be simple (to be solvable) but representative, integrating all major factors influencing the planned lead time calculation.

For component planned lead time calculation in an MRP environment for assembly systems with several types of components and random component lead times, in this paper a model is introduced. This model will help us to solve the considered problem of MRP parameterization, i.e. to find optimal planned lead times for components when the actual lead times are random variables.

For this model, the finished product demand per period (multi-period model) is assumed to be known and constant and the assembly capacity is infinite. Several types of components are needed to assembly one finished product. The unit holding cost per period for each type of component (hi) and the unit backlogging cost (b) for the finished product are known. The lead times (L_i) for orders made at different periods for the same type of component i, are independent and identically distributed discrete random variables. The distribution probabilities for the different types of components can be not identical. These distributions are known, and their upper values are finite.

The finished products are delivered at the end of each period and unsatisfied demands are backordered and have to be treated later (when sufficient numbers of components of each type are in stock). The supply policy for components is *lot for lot*: one lot of each type of component is ordered at the beginning of each period.

Because the supply policy is the *Lot for Lot* and the demand is constant, the same quantities of components are ordered at the beginning of each period. Thus, only planned lead times are unknown parameters for this model. They are the decision variables in our optimization approach. The model considers random component lead times and also the dependence among inventories of the different components suitable for assembly systems (when there is a stockout of only one component, consequently, there is no possibility to assemble the finished product).

To simplify the equations of this paper, without lost of generality, it is assumed that the finished product demand is equal to one unit per period, and that one finished product is assembled from on unit of each type of component.

Let's use the following model notations:

1_f function equal to 1 if f is true, and 0 otherwise,
n the number of types of components used for the assembly of the product,
$E[.]$ the mathematical expectation operator,
h_i unit holding cost of the component i per period,
b unit backlogging cost of the finished product per period,
k reference of a period (period index),
L_i lead time of the components i (discrete random variable),
L_i^k lead time of the components i ordered at period k (discrete random variable),
u_i upper value of the lead time for components i ($1 \le L_i \le u_i$; i=1,2,...,n); u=max(u_i),
N_i^k number of orders for the component i that have not yet arrived at the end of the period k,
N_i steady state number of orders for the component i that have not yet arrived at the end of a period,
y_i planned lead time of the components i (integer decision variable, $1 \le y_i \le u_i$),
X the vector of the decision variables for the multi-period case $(x_1, ..., x_n)$, where x_i=y_i-1
Z^+ function equal to the maximum of Z and 0: max (Z,0).
N_i steady state number of orders for the component i that have not yet arrived at the end of a period,
y_i planned lead time of the components i (integer decision variable,),
X the vector of the decision variables for the multi-period case, where x_i=y_i-1
Z^+ function equal to the maximum of Z and 0: max (Z,0).

4. OPTIMIZATION MODEL

For the multi-period problem, the cost of a period k is given by the following expression (Louly and Dolgui, 2002):

$$C_k(X, N^k) = \sum_{i=1}^{n} h_i(x_i - N_i^k) + H \max_{i=1,\cdots,n} (N_i^k - x_i)^+ , \qquad (1)$$

where
$X = (x_1, x_2, \ldots, x_n)$ are decision variables,

$$H = b + \sum_{i=1}^{n} h_i ,$$

$N_i^k = \sum_{j=1}^{u-1} 1_{L_i^{k-j+1} > j}$ is a random variable giving the number of orders for the component i that have not yet arrived at the end of the period k (discrete random variable),
$N^k = (N_1^k, \cdots, N_n^k)$.

To define N_i^k, i = 1,2, …, let's consider a period $k \ge u$-1. Let L_i^{k+1-j}, j = 1, 2, …, u_i-1, be the lead times (random variables) for orders made respectively at the beginning of the periods k, k-1,…, k- u + 2, for the components i. Thus, $1_{L_i^{k+1-j} > j}$ is a random variable which is equal to 1 if the order for components i made at the period $k - j + 1$ is delivered after the end of the period k, otherwise this variable is equal to 0. By definition $1_{L_i^{k-j+1} > j}$=0, if $k - j + 1 \le 0$, because these orders are already in the initial inventory (this represents the orders made before the initial date).

So, the random variables N_i^k are defined as follows:

$$N_i^k = \sum_{j=1}^{u-1} 1_{L_i^{k-j+1} > j} \qquad i = 1, \cdots, n.$$

The maximal value of component i lead time is equal to u_i, so only the previous u_i-1 orders may not yet be received. The earlier orders have already arrived, therefore:
$0 \le N_i^k \le u_i - 1$.

The expected steady state cost per period is given as follows.

Proposition 1: The expected cost $EC(X,N)$ per period on the infinite horizon is as follows:

$$EC(X,N) = \lim_{T \to \infty} \frac{1}{T} \sum_{k=1}^{T} C_k(X, N^k) = E[C_k(X, N^k)] \qquad (2)$$

As shown in (Louly and Dolgui, 2002), using calculation of the expected value for (2), the resulting multi-period *optimization problem* can be rewritten as follows:

$$EC(X,N) = \sum_{i=1}^{n} h_i\left(x_i - E(N_i)\right) + H \sum_{j \ge 0} \left(1 - \prod_{i=1}^{n} F_{N_i}(x_i + j)\right), \qquad (3)$$

subject to: $0 \le x_i \le u_i - 1$, $i = 1, 2, \ldots, n$. (4)

where $F_{N_i}(j) = \Pr(N_i \le j)$, the value $x_i = 0$ signifies that the component i is ordered at the beginning of the target period (i.e. when assembly must be made).

5. OPTIMIZATION ALGORITHM

A Branch and Cut algorithm to solve the optimization problem (3) – (4) was developed. It is based on our previous works.

In (Louly and Dolgui, 2006), in which only holding costs with the service level constraint were studied, it was proved that for the increment functions:

a) $G_i^+(X)$ is increasing on x_i, and decreasing on x_j for all $j \ne i$,

b) $G_i^-(X)$ is decreasing on x_i and increasing on x_j for all $j \ne i$.

It is easy to see, that if the same functions $G_i^+(X)$ and $G_i^-(X)$ are introduced for the model (3)-(4), the above properties a) –b) remain valid for this paper (i.e. where the objective is the sum of the holding and backlogging costs).

The following additional property was proved:

Proposition 2: $G_i^+(X)$ and $G_i^-(X)$ verify the following inequalities:

$$-b - \sum_{j \ne i} h_j \le G_i^+(X) \le h_i, \qquad (5)$$

$$-h_i \le G_i^-(X) \le b + \sum_{j \ne i} h_j. \qquad (6)$$

Considering the above properties of the functions $G_i^+(X)$ and $G_i^-(X)$, two *Lower Bounds* for the objective function on the space [*A, B*] are.

$$LB_1 = EC(A) + \sum_{i=1}^{n} (b_i - a_i) \min\left(G_i^+(b_1, ..., b_{i-1}, a_i, ..., a_n), 0\right) \qquad (7)$$

$$LB_2 = EC(B) + \sum_{i=1}^{n} (b_i - a_i) \min\left(G_i^-(a_1, ..., a_{i-1}, b_i, ..., b_n), 0\right) \qquad (8)$$

So the lower bound is equal to:
$LB = \max(LB_1, LB_2)$ (9)

Dominance properties. First, two dominance properties are suggested:

(i) If $G_i^+(A) < 0$, then each solution X of [*A, B*] with $x_i = a_i$ is dominated.

(ii) If $G_i^-(B) < 0$, then each solution X of [*A, B*] with $x_i = b_i$ is dominated.

Then, for each i satisfying $G_i^+(A) < 0$, can be deleted all solutions with $a_i = x_i$. Then, the solutions space [*A, B*] can be reduced by replacing a_i with a_i+1. Furthermore, if $G_i^+(A) < 0$ and $a_i = b_i$, then all the solutions of [*A, B*] are then dominated. At the same time, for each i satisfying $G_i^-(B) < 0$, it is possible to delete all solutions with $b_i = x_i$. The search space [*A, B*] is then reduced by replacing b_i with b_i-1. In addition, if $G_i^-(B) < 0$ and $a_i = b_i$, then all the solutions of [*A, B*] are dominated.

These dominance properties can be used to develop efficient cut procedures for the Branch and Cut algorithm. Indeed, after the division of a node, in a Branch and Cut algorithm, two son-nodes (descendants) are created. For each son-node, some cuts are used to reduce the corresponding search spaces before the next branching.

Node extension procedure
A Branch and Bound (B&B) algorithm is based on the design of an enumeration tree. In our algorithm, each node of the enumeration tree represents a set of feasible solutions. Let [*A, B*] be a node of this tree. The descendants of this node are obtained by dividing (partitioning) the corresponding space [*A, B*] into two smaller subspaces [*A*, B^1] and [A^1, *B*] as follows: we choose i such that $i = \arg\max(b_i - a_i)$, then the descendent [*A*, B^1] (respectively [A^1, *B*]) is

the subspace given by the vectors A and B^1 (resp. A^1 and B) for whom the i-th component satisfies

$a_i \le x_i \le \frac{a_i + b_i}{2}$ (resp. $\frac{a_i + b_i}{2} + 1 \le x_i \le b_i$). After applying this node extension procedure for the node [*A, B*] we obtain two son-nodes [*A*, B^1] and [A^1, *B*], each with smaller space of the feasible solutions.

Lower Cut and Upper Cut procedures. A Branch and Cut algorithm is B&B where for each node before applying a node extension procedure, some cuts are executed. The aim is to reduce the space of feasible solutions which is associated with the node to be divided. For our algorithm, the principle is simple, as mentioned above a node corresponds to a search space [*A, B*]. The cut procedure reduces the solution space [*A, B*] replacing *A* (respectively *B*) by a larger (respectively smaller) vector. This is equivalent to cutting a part of the search space [*A, B*]. We introduce two procedures: one for cutting small values (Lower Cut procedure) and second for cutting large values (Upper Cut procedure) of the corresponding decision variables. The reduction scheme is the same for these two procedures and they return "true" when the subset [*A, B*] is entirely dominated (i.e. by applying the cuts we completely eliminated the node [*A, B*]).

6. NUMERICAL TESTS

To test the algorithm's performance, 1000 examples grouped into 100 families were generated, each family gathers 10 examples with the same number of components (n) and the same value for u=max(u_i). The maximum time of calculation for B&C algorithm was fixed at 30 seconds. The unit holding costs hi are randomly generated in the interval [1, n]. The unit backlogging cost b is generated in [100, 100n].

Table 1 gives the average computing times in seconds for 100 families of tests. Only three problems out of thousand were not solved within the limit of allocated computing time.

n / *u*	10	20	30	40	50	60	70	80	90	100
10	0.001	0.005	0.012	0.012	0.041	0.059	0.084	0.105	0.205	0.246
20	0.0031	0.017	0.048	0.0725	0.151	0.219	0.2971	0.368	0.457	0.766
30	0.01	0.039	0.1	0.1875	0.308	0.446	0.6461	0.7961	1.374	1.548
40	0.016	0.064	0.163	0.3278	0.522	0.762	1.044	1.305	2.269	2.605
50	0.024	0.1	0.245	0.547	0.807	1.153	1.591	2.02	3.397	3.808
60	0.034	0.142	0.343	0.763	1.136	1.64	2.27	2.854	4.005	5.26
70	0.045	0.187	0.462	0.996	1.464	2.177	3.005	3.768	5.047	7.102
80	0.059	0.238	0.582	1.286	1.867	2.747	3.792	4.836	6.696	8.982
90	0.075	0.3	0.731	1.516	2.309	3.405	4.684	6.03	8.612	11.175
100	0.09	0.365	0.887	1.924	2.799	4.095	5.679	7.222	10.132	13.577

Table 1. Computing time in second

7. CONCLUSIONS

An MRP parameterization problem was studied for assembly systems under component lead time uncertainties. A model was proposed. For this model two lower bounds have been obtained for the objective function. In addition, two dominance properties were suggested to construct efficient cut procedures reducing the search space at each step of the optimisation algorithm. Based on these results, a Branch and Cut (B&C) algorithm was developed and tested.

The proposed model and B&C algorithm, give the optimal value of planned lead time (safety lead times) for each component. They are calculated taking into account the distributions of probabilities of the component lead times, holding and backlogging costs. Use of the proposed algorithm minimizes the total cost for MRP controlled assembly systems under lead time uncertainties.

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