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Brockett problem for systems with feedback delay

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Abstract: The Brockett problem is posed for systems with feedback delay in the form: what kind of time-varying controllers should be used to obtain asymptotic stability? Stabilization of delayed systems is a challenging task in control theory, since these systems usually have infinitely many poles. In this paper, the act-and-wait control concept is investigated as a possible technique to reduce the number of poles of systems with feedback delay. The Brockett problem is rephrased for the act-and-wait control system.

Keywords: feedback delay; periodic control; stability.

1. INTRODUCTION

Consider the linear system

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{1}$$

$$y(t) = Cx(t) \tag{2}$$

with $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^l$. Consider the delayed feedback controller

$$u(t) = Dy(t-\tau), \tag{3}$$

where τ is the time delay of the feedback loop. We assume that the delay is a fixed parameter of the control system and cannot be eliminated or tuned during the control design. There are several sources of such time delays, e.g., acquisition of response and excitation data, information transmission, on-line data processing, computation and application of control forces.

System (1)-(2) with controller (3) implies the delay differential equation (DDE)

$$\dot{x}(t) = Ax(t) + BDCx(t-\tau). \tag{4}$$

Due to the time delay, system (4) has infinite number of poles (called also characteristic roots or characteristic exponents) determined by the transcendental characteristic equation

$$\det\left(\lambda I - A - BDC\,\mathrm{e}^{-\tau\lambda}\right) = 0. \tag{5}$$

The system is asymptotically stable if all the poles are located in the left half of the complex plane. Stability conditions for the system's parameters can be given by monitoring the number of unstable poles (see, e.g., Stépán, 1989, Atay, 1999, Olgac and Sipahi, 2002, Michiels and Roose, 2003). The difficulty of this problem is that infinitely many poles should be controlled by finite number of control parameters, i.e., by the elements of matrix D.

An effective way of managing pole placement problem is the use of periodic controllers. The problem of stabilization by means of time-periodic feedback gains in non-delayed systems has been presented by Brockett (1998) as one of the challenging open problems in control theory. For non-delayed systems, the Brockett problem can be posed as:

Problem 1. For given matrices A, B and C, under what circumstances does there exist a time-varying controller

$$u(t) = G(t)y(t), \tag{6}$$

such that the system is asymptotically stable?

Together with some papers on discrete-time systems (Aeyels and Willems, 1992, Leonov, 2002a, Artstein and Weiss, 2005), partial results have been presented by Leonov (2002b) and Allright et al. (2005) for piecewise constant control gains and by Moreau and Aeyels (2004) for sinusoidal control gains. The solution to the problem for a wide class of systems – without delay – was presented by Boikov (2005).

For systems with feedback delay, the Brockett problem can be composed as:

Problem 2. For given matrices A, B and C and for given feedback delay τ, under what circumstances does there exist a time-varying controller

$$u(t) = G(t)y(t-\tau), \tag{7}$$

such that the system is asymptotically stable?

Since the system has infinitely many poles due to the time delay, this stabilization problem is quite complicated. One possible approach to the problem is the application of the act-and-wait control technique.

2. THE ACT-AND-WAIT CONTROL TECHNIQUE

The act-and-wait controller is a special case of periodic controllers, where the feedback term is switched off and on periodically. The technique was introduced by Insperger (2006) and Stépán and Insperger (2006) for continuous-time systems, and by Insperger and Stépán (2007) for discrete-time systems.

Consider the time-varying controller (7) with the T-periodic matrix

$$G(t) = \begin{cases} 0 & \text{if} \quad 0 \le t \bmod T < t_\mathrm{w} \\ \mathit{\Gamma}(t) & \text{if} \quad t_\mathrm{w} \le t \bmod T < t_\mathrm{w} + t_\mathrm{a} = T \end{cases}, \tag{8}$$

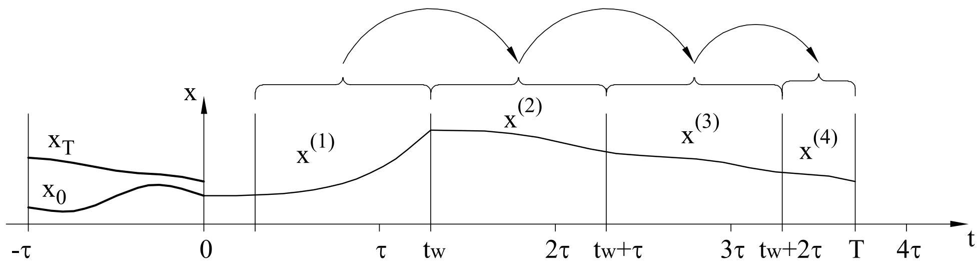

Fig. 1. Piecewise solution segments of equation (9) with (8) for $t_\mathrm{w} \geq \tau$ and $2\tau < t_\mathrm{a} \leq 3\tau$ ($k = 2$)

where $\varGamma(t) : [t_\mathrm{w}, T] \to \mathbb{R}^{m\times l}$ is an integrable matrix function. Here, t_a and t_w are the length of the acting and the waiting periods, respectively, and $t_\mathrm{a} + t_\mathrm{w} = T$ is the length of one act-and-wait period.

Clearly, in this case, the poles of the time-varying system

$$\dot{x}(t) = Ax(t) + BG(t)Cx(t-\tau) \qquad (9)$$

should be monitored. The difficulty of this stabilization problem lies in the fact that the system has infinitely many poles due to the time delay similarly to the time-independent system (4).

The general solution of DDE (9) for the initial function x_0 can be formulated as

$$x_t = \mathcal{U}(t)x_0, \qquad (10)$$

where $\mathcal{U}(t)$ is the solution operator of the system, and the function x_t is defined by the shift

$$x_t(s) = x(t+s), \qquad s \in [-\tau, 0]. \qquad (11)$$

Stability properties are determined by the monodromy operator $\mathcal{U}(T)$. The nonzero elements of the spectrum of $\mathcal{U}(T)$ are called characteristic multipliers (or poles), also defined by

$$\mathrm{Ker}(\mu\mathcal{I} - \mathcal{U}(T)) \neq \{0\}. \qquad (12)$$

The system is asymptotically stable if all the infinitely many characteristic multipliers lie in the open unit disc of the complex plane.

In Insperger (2006), it was shown that if $t_\mathrm{w} \geq \tau$, then the system can be described by an $n \times n$ monodromy matrix, consequently, only n poles determine the stability instead of infinitely many ones. In this paper, it is shown that the dimension of the monodromy operator is finite for certain parameter combinations even if $t_\mathrm{w} < \tau$. The main results are formalized as follows

Theorem 3. The number of nonzero poles of system (9) with (8) is equal to

Case 1 : n if $t_\mathrm{w} \geq \tau$,
Case 2 : $(k+1)n$ if $t_\mathrm{w} < \tau$ and $(k+1)\tau - t_\mathrm{w} \leq T \leq k\tau + t_\mathrm{w}$, $k \in \mathbb{Z}^+$.

In the next sections, the proof of this theorem is provided by the construction of the solution over the act-and-wait period T for both cases.

3. CASE 1: $t_\mathrm{w} \geq \tau$

In this section, it is shown that the dimension of system (9) with (8) is equal to n if $t_\mathrm{w} \geq \tau$. Consider the case $k\tau < t_\mathrm{a} \leq (k+1)\tau$ where k is arbitrary non-negative integer. Then, the solution can be constructed piecewise over the succeeding intervals $[0, t_\mathrm{w}]$, $[t_\mathrm{w}, t_\mathrm{w} + \tau]$, ..., $[t_\mathrm{w} + k\tau, T]$ as follows (see Fig. 1 for $k = 2$).

Since the delayed term is switched off during the waiting period, the first section of the solution can be given as

$$x^{(1)}(t) = \varPhi^{(1)}(t)x(0), \qquad 0 \leq t \leq t_\mathrm{w}, \qquad (13)$$

with $\varPhi^{(1)}(t) = \mathrm{e}^{At}$. Here, the index (1) refers to the number of the segment of the solution.

Now, we utilize the fact that the waiting period is larger than (or equal to) the time delay, and that the solution over $0 \leq t \leq t_\mathrm{w}$ is given by equation (13). Thus, in the interval $t_\mathrm{w} < t \leq t_\mathrm{w} + \tau$, equation (9) can be written as

$$\dot{x}(t) = Ax(t) + B\varGamma(t)C\varPhi^{(1)}(t-\tau)x(0), \qquad t_\mathrm{w} < t \leq t_\mathrm{w} + \tau. \qquad (14)$$

The solution for the initial condition $x(t_\mathrm{w}) = x^{(1)}(t_\mathrm{w}) = \varPhi^{(1)}(t_\mathrm{w})x(0)$ can be given in the form

$$x^{(2)}(t) = \varPhi^{(2)}(t)x(0), \qquad t_\mathrm{w} < t \leq t_\mathrm{w} + \tau \qquad (15)$$

with

$$\varPhi^{(2)}(t) = \mathrm{e}^{At} + \int_{t_\mathrm{w}}^{t} \mathrm{e}^{A(t-s)}\, B\varGamma(s)C\varPhi^{(1)}(s-\tau)\,\mathrm{d}s. \qquad (16)$$

If the solution in the h^{th} interval is given as

$$x^{(h)}(t) = \varPhi^{(h)}(t)x(0), \qquad t_\mathrm{w} + (h-2)\tau < t \leq t_\mathrm{w} + (h-1)\tau, \qquad (17)$$

then the solution in the next interval can be given by the recursive form

$$x^{(h+1)}(t) = \varPhi^{(h+1)}(t)x(0), \qquad t_\mathrm{w} + (h-1)\tau < t \leq t_\mathrm{w} + h\tau, \qquad (18)$$

with

$$\varPhi^{(h+1)}(t) = \mathrm{e}^{At} + \int_{t_\mathrm{w}}^{t} \mathrm{e}^{A(t-s)}\, B\varGamma(s)C\varPhi^{(h)}(s-\tau)\,\mathrm{d}s. \qquad (19)$$

Finally, the solution at $t = T$ can be given as

$$x(T) = x^{(k+2)}(T) = \varPhi^{(k+2)}(T)x(0) \qquad (20)$$

This way, the monodromy mapping

$$x_T = \mathcal{U}(T)x_0 \qquad (21)$$

is reduced to

$$\begin{pmatrix} x(T) \\ \tilde{x}_T \end{pmatrix} = \begin{pmatrix} \varPhi^{(k+2)}(T) & O \\ \tilde{f}_{k+2} & \mathcal{O} \end{pmatrix} \begin{pmatrix} x(0) \\ \tilde{x}_0 \end{pmatrix}, \qquad (22)$$

where function $\tilde{x}_t$ is defined by the shift

$$\tilde{x}_t(s) = x(t+s), \qquad s \in [-\tau, 0). \qquad (23)$$

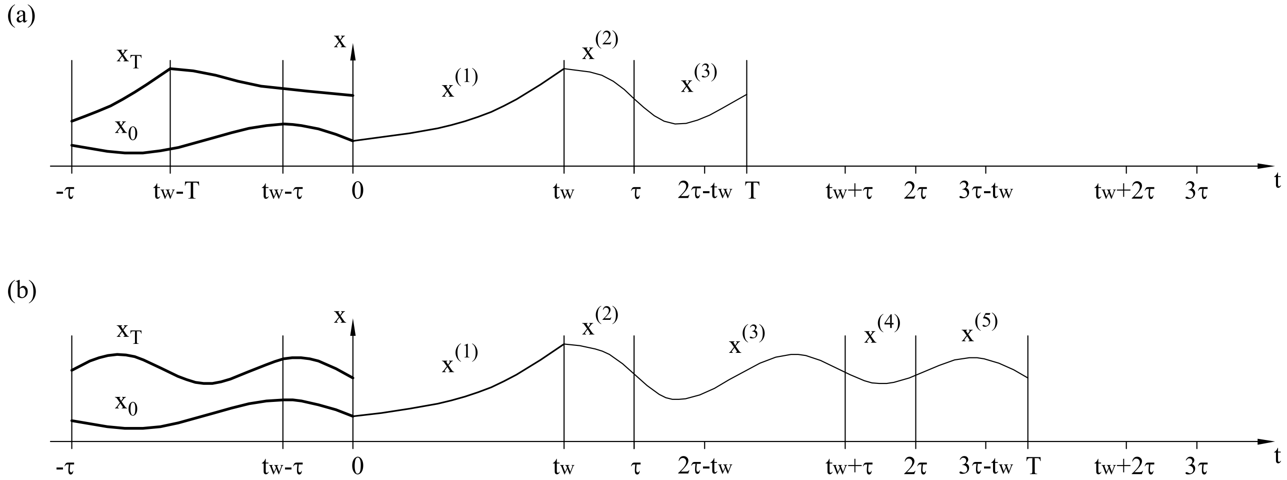

Fig. 2. Graphs of the solution of equation (9) with (8) for different act-and-wait periods T. Panel (a): $t_\mathrm{w} < \tau$ with $2\tau - t_\mathrm{w} < T \le t_\mathrm{w} + \tau$. Panel (b): $t_\mathrm{w} < \tau$ with $3\tau - t_\mathrm{w} < T \le t_\mathrm{w} + 2\tau$.

Note that $s = 0$ is excluded here as opposed to equation (11). In equation (22), O denotes the zero functional, $\mathcal{O}$ denotes the zero operator and $\tilde{f}_{k+2}$ is the function

$$\tilde{f}_{k+2}(s) = \begin{cases} \Phi^{(k+1)}(s) & \text{if} \quad -\tau \le s < k\tau - t_\mathrm{a} \\ \Phi^{(k+2)}(s) & \text{if} \quad k\tau - t_\mathrm{a} \le s < 0 \end{cases} . \quad (24)$$

Equation (22) shows that function x_T can be determined using only the initial state $x(0)$ and does not depend on the initial function $\tilde{x}_0$. Thus, the monodromy operator has only n nonzero eigenvalue that are just equal to the eigenvalues of $\Phi^{(k+2)}(T)$, and all the further infinitely many eigenvalues are zero. Clearly, the system is asymptotically stable if the eigenvalues of $\Phi^{(k+2)}(T)$ are in modulus less then 1. In this sense, $\Phi^{(k+2)}(T)$ serves as an $n \times n$ monodromy matrix.

This way, we have constructed a finite dimensional monodromy matrix for Case 1 of Theorem 3.

For example, if $k = 0$, i.e., $0 < t_\mathrm{a} \le \tau$ then

$$\Phi^{(2)}(T) = \mathrm{e}^{AT} + \int_{t_\mathrm{w}}^{T} \mathrm{e}^{A(T-s)} \, B\Gamma(s)C \, \mathrm{e}^{A(s-\tau)} \, \mathrm{d}s. \quad (25)$$

If $k = 1$, i.e., $\tau < t_\mathrm{a} \le 2\tau$, then

$$\begin{aligned} \Phi^{(3)}(T) = {} & \mathrm{e}^{AT} + \int_{t_w}^{T} \mathrm{e}^{A(T-s)} \, B\Gamma(s)C \, \mathrm{e}^{A(s-\tau)} \, \mathrm{d}s \\ & + \int_{t_w+\tau}^{T} \mathrm{e}^{A(T-s_1)} \, B\Gamma(s_1)C \\ & \times \int_{t_w}^{s_1-\tau} \mathrm{e}^{A(s_1-s_2-\tau)} \, B\Gamma(s_2)C \, \mathrm{e}^{A(s_2-\tau)} \, \mathrm{d}s_2 \, \mathrm{d}s_1. \quad (26) \end{aligned}$$

4. CASE 2: $t_\mathrm{w} < \tau$ AND $(k+1)\tau - t_\mathrm{w} \le$T$\le t_\mathrm{w} + k\tau$

The constructive step-by-step solution presented in the previous subsection is not applicable if the waiting period is shorter than the time delay ($t_\mathrm{w} < \tau$). It can be shown that under certain conditions, the system can still be transformed into a finite dimensional map. In this case, the solution is constructed piecewise over the succeeding intervals $[0, t_\mathrm{w}]$, $[t_\mathrm{w}, \tau]$, $[\tau, t_\mathrm{w} + \tau]$, $[t_\mathrm{w} + \tau, 2\tau]$, ... as it is shown in Fig. 2. In the next subsection, the cases $k = 1$, $k = 2$, $k > 2$ are considered.

4.1 *Case $k = 1$: $t_\mathrm{w} < \tau$ with $2\tau - t_\mathrm{w} < T \le t_\mathrm{w} + \tau$*

The sketch of the piecewise solution of the system is shown in Fig. 2, panel (a). Since the delayed term is switched off during the waiting period, the first section of the solution can be given as

$$x^{(1)}(t) = \mathrm{e}^{At} \, x(0), \qquad 0 < t \le t_\mathrm{w}. \quad (27)$$

In the interval $t_\mathrm{w} < t \le \tau$, equation (9) with (8) reads

$$\dot{x}(t) = Ax(t) + B\Gamma(t)Cx_0(t-\tau), \quad t_\mathrm{w} < t \le \tau \quad (28)$$

with the initial condition $x(t_\mathrm{w}) = x^{(1)}(t_\mathrm{w}) = \mathrm{e}^{At_\mathrm{w}} \, x(0)$. The corresponding solution segment is

$$x^{(2)}(t) = \mathrm{e}^{At} \, x(0) + \int_{t_\mathrm{w}-\tau}^{t-\tau} \mathrm{e}^{A(t-s-\tau)} \, B\Gamma(s+\tau)Cx_0(s) \, \mathrm{d}s, \qquad t_\mathrm{w} < t \le \tau. \quad (29)$$

The state at $t = \tau$ is given as

$$\begin{aligned} x(\tau) &= x^{(2)}(\tau) \\ &= \mathrm{e}^{A\tau} \, x(0) + \underbrace{\int_{t_\mathrm{w}-\tau}^{0} \mathrm{e}^{-As} \, B\Gamma(s+\tau)Cx_0(s) \, \mathrm{d}s}_{=:\, I_1(0)} . \quad (30) \end{aligned}$$

Here, $I_1(0)$ is a special weighted integral of the initial function x_0. In general, $I_1(t)$ can be defined as

$$I_1(t) := \int_{t_\mathrm{w}-\tau}^{0} \mathrm{e}^{-As} \, B\Gamma(s+\tau)Cx_t(s) \, \mathrm{d}s. \quad (31)$$

In the interval $\tau < t \le T$, equation (9) with (8) reads

$$\dot{x}(t) = Ax(t) + B\Gamma(t)Cx^{(1)}(s-\tau), \qquad \tau < t \le T \quad (32)$$

with the initial condition $x(\tau) = x^{(2)}(\tau) = \mathrm{e}^{A\tau} \, x(0) + I_1(0)$. The third solution segment is

$$x^{(3)}(t) = \Psi_{11}^{(1)}(t)x(0) + \Psi_{12}^{(1)}(t)I_1(0), \qquad \tau < t \le T, \quad (33)$$

where

$$\Psi_{11}^{(1)}(t) = \mathrm{e}^{At} + \int_{\tau}^{t} \mathrm{e}^{A(t-s)}\, B\Gamma(s)C\, \mathrm{e}^{A(s-\tau)}\, \mathrm{d}s, \tag{34}$$

$$\Psi_{12}^{(1)}(t) = \mathrm{e}^{A(t-\tau)}. \tag{35}$$

Note that function $x^{(2)}$ depends on the initial function x_0, while $x^{(1)}$ and $x^{(3)}$ do not depend on x_0.

The state after one act-and-wait period can be given by setting $t = T$:

$$x(T) = \Psi_{11}^{(1)}(T)x(0) + \Psi_{12}^{(1)}(T)I_1(0). \tag{36}$$

Here, $x(T)$ is determined as a linear combination of the initial state $x(0)$ and the weighted integral $I_1(0)$. In order to obtain a discrete map, the integral

$$I_1(T) = \int_{t_{\mathrm{w}}-\tau}^{0} \mathrm{e}^{-As}\, B\Gamma(s+\tau)Cx_T(s)\,\mathrm{d}s \tag{37}$$

should also be expressed as a linear combination of $x(0)$ and $I_1(0)$. Here, x_T can be given as

$$x_T(s) = \begin{cases} x^{(1)}(s+T) & \text{if} \quad -\tau \le s \le t_{\mathrm{w}} - T \\ x^{(2)}(s+T) & \text{if} \quad t_{\mathrm{w}} - T < s \le \tau - T \\ x^{(3)}(s+T) & \text{if} \quad \tau - T < s \le 0 \end{cases}. \tag{38}$$

Using the condition $2\tau - t_{\mathrm{w}} \le T$, it can be seen that $\tau - T \le t_{\mathrm{w}} - \tau$, thus the integral (37) depends only on $x^{(3)}$:

$$I_1(T) = \int_{t_{\mathrm{w}}-\tau}^{0} \mathrm{e}^{-As}\, B\Gamma(s+\tau)Cx^{(3)}(s+T)\,\mathrm{d}s. \tag{39}$$

Utilizing that $x^{(3)}$ depends linearly on $x(0)$ and $I_1(0)$, but does not depend on the initial function x_0, the integral (39) results in

$$I_1(T) = \Psi_{21}^{(1)}(T)x(0) + \Psi_{22}^{(1)}(T)I_1(0), \tag{40}$$

where

$$\Psi_{21}^{(1)}(T) = \int_{t_{\mathrm{w}}-\tau}^{0} \mathrm{e}^{-As}\, B\Gamma(s+\tau)C\Psi_{11}^{(1)}(s+T)\,\mathrm{d}s, \tag{41}$$

$$\Psi_{22}^{(1)}(T) = \int_{t_{\mathrm{w}}-\tau}^{0} \mathrm{e}^{-As}\, B\Gamma(s+\tau)C\Psi_{12}^{(1)}(s+T)\,\mathrm{d}s. \tag{42}$$

Using (33) and (40), the monodromy mapping can be written in the form

$$\begin{pmatrix} x(T) \\ I_1(T) \\ w_T \end{pmatrix} = \begin{pmatrix} \Psi_{11}^{(1)}(T) & \Psi_{12}^{(1)}(T) & O \\ \Psi_{21}^{(1)}(T) & \Psi_{22}^{(1)}(T) & O \\ f_x & f_{I1} & \mathcal{O} \end{pmatrix} \begin{pmatrix} x(0) \\ I_1(0) \\ w_0 \end{pmatrix}. \tag{43}$$

Here, the function w_t is defined as

$$w_t = x_t - x(t)s_x - I_1(t)s_{I1}, \tag{44}$$

where s_x and s_{I1} are the right eigenvectors of the monodromy operator $\mathcal{U}(T)$ corresponding to $x(t)$ and $I_1(t)$, respectively. Functions f_x and f_{I1} describe the dependence of w_T on $x(0)$ and $I_1(0)$. Equation (43) shows that x_T can be determined as a linear combination of $x(0)$ and $I_1(0)$. In this sense, matrix

$$\Psi^{(1)}(T) = \begin{pmatrix} \Psi_{11}^{(1)}(T) & \Psi_{12}^{(1)}(T) \\ \Psi_{21}^{(1)}(T) & \Psi_{22}^{(1)}(T) \end{pmatrix} \tag{45}$$

serves as an $2n \times 2n$ mondoromy matrix. Thus, the stability is determined by the $2n$ eigenvalues of $\Psi^{(1)}(T)$. All the remaining infinitely many eigenvalues are set to zero.

4.2 *Case $k = 2$: $t_{\mathrm{w}} < \tau$ with $3\tau - t_{\mathrm{w}} < T \le 2\tau + t_{\mathrm{w}}$*

Using similar algorithm as for the case $k = 1$, it can be shown that for $k = 2$, 3-dimensional discrete map can be constructed:

$$\begin{pmatrix} x(T) \\ I_1(T) \\ I_2(T) \end{pmatrix} = \Psi^{(2)} \begin{pmatrix} x(0) \\ I_1(0) \\ I_2(0) \end{pmatrix}, \tag{46}$$

where $I_1(t)$ is defined in (31) and $I_2(t)$ is defined as

$$I_2(t) := \int_{t_{\mathrm{w}}-\tau}^{0} \mathrm{e}^{-As_1}\, B\Gamma(s_1+2\tau)C \times \int_{t_{\mathrm{w}}-\tau}^{s_1} \mathrm{e}^{A(s_1-s_2)}\, B\Gamma(s_2+\tau)Cx_t(s_2)\,\mathrm{d}s_2\,\mathrm{d}s_1. \tag{47}$$

Here, $\Psi^{(2)}$ is a $3n \times 3n$ mondoromy matrix not detailed here. Thus, in this case, the system has $3n$ eigenvalues.

4.3 *Case $k > 2$: $t_{\mathrm{w}} < \tau$ with $(k+1)\tau - t_{\mathrm{w}} < T \le k\tau + t_{\mathrm{w}}$*

Similarly to cases $k = 1$ and $k = 2$, it can be shown that for the general case $(k+1)\tau - t_{\mathrm{w}} < T \le t_{\mathrm{w}} + k\tau$, the system can always be described by an $(k+1)n \times (k+1)n$ monodromy matrix, denoted by $\Psi^{(k)}$. Thus, the number of nonzero poles in this case is $(k+1)n$.

This way, we constructed the finite dimensional monodromy matrix for Case 2 of Theorem 3.

5. CASE $t_{\mathrm{w}} < \tau$ AND $t_{\mathrm{w}} + (k-1)\tau <$T $< (k+1)\tau - t_{\mathrm{w}}$

The case when $t_{\mathrm{w}} < \tau$ and $t_{\mathrm{w}} + (k-1)\tau < T < (k+1)\tau - t_{\mathrm{w}}$ were excluded in the above analysis. In these cases, the algorithm of constructing finite dimensional discrete maps over the act-and-wait period does not work. Still, it is not sure if such discrete maps does not exists for certain acting and waiting period lengths. Discovering the properties of these parameter regions requires further analysis.

6. CONCLUSION

The number of nonzero poles was analyzed for systems with feedback delay under act-and-wait control for different acting (t_{a}) and waiting period lengths (t_{w}). The results for the two cases regarding the relation between t_{a} and t_{w} are summarized in Table 1. The geometric representation in the plane $(t_{\mathrm{a}}, t_{\mathrm{w}})$ are presented in Figure 3.

Clearly, the smallest number of nonzero poles is obtained if the waiting period is chosen to be larger than the feedback delay. In this case, the resulting time-periodic and time-delayed system can be described by an $n \times n$ monodromy matrix, and the stability depends only on n poles.

The Brockett problem can now be rephrased as

Problem 4. Consider system (1)-(2) with the act-and-wait controller (7). Assume that matrices A, B and C and the feedback delay τ are given. Assume that $G(t)$ is given as in (8) and $t_{\mathrm{w}} \ge \tau$, thus an $n \times n$ monodromy matrix can be constructed. Under what circumstances does there exist a time-dependent function $\Gamma(t) : [t_{\mathrm{w}}, T] \to \mathbb{R}^{m \times l}$ such that the system is asymptotically stable, i.e., all the n poles are in modulus less than one?

Table 1. Summary of the dimension of the monodromy operator for equation (9) with (8).

condition for t_{w} and t_{a}	monodromy operator	number of nonzero poles
$t_{\mathrm{w}} \geq \tau$ $t_{\mathrm{a}} > k\tau$ $t_{\mathrm{a}} \leq (k+1)\tau$	$\begin{pmatrix} \Phi^{(k+2)}(T) & O \\ \tilde{f}_{k+2} & \mathcal{O} \end{pmatrix}$	n
$t_{\mathrm{w}} < \tau$ $t_{\mathrm{a}} \leq k\tau$ $t_{\mathrm{w}} \geq \dfrac{(k+1)\tau - t_{\mathrm{a}}}{2}$	$\begin{pmatrix} \Psi^{(k)}(T) & O \\ f_{x,I} & \mathcal{O} \end{pmatrix}$	$(k+1)n$

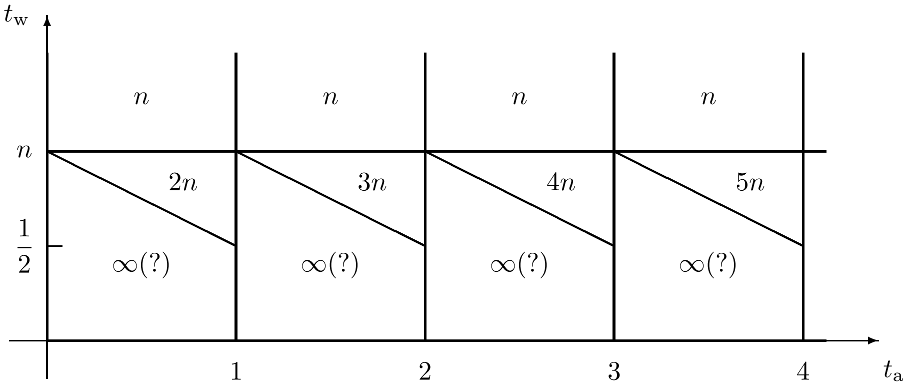

Fig. 3. Chart of the dimension of the monodromy operator for equation (9) with (8)

7. AN EXAMPLE

Consider the second-order system ($n = 2$) described by (1)-(2) with

$$A = \begin{bmatrix} 0 & 1 \\ -a & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{48}$$

Consider the time-invariant controller (3) with

$$D = \begin{bmatrix} -d_1 \\ -d_2 \end{bmatrix}^T, \quad \tau = 1. \tag{49}$$

The corresponding characteristic equation reads

$$\lambda^2 + a + d_1 \,\mathrm{e}^{-\lambda} + d_2 \lambda \,\mathrm{e}^{-\lambda} = 0. \tag{50}$$

This transcendental equation has infinitely many poles that can not arbitrarily be placed using the two control parameters d_1 and d_2. Moreover, if $a < -2$ then the system cannot even be stabilized, it is unstable for all (d_1, d_2) pairs (see, e.g., [14]). In other words, for $a < -2$, the infinitely many poles of the system cannot be placed to the left half of the complex plane.

Apply the act-and-wait controller (7) with

$$G(t) = \begin{cases} 0 & \text{if} \quad 0 \leq t \bmod T < t_{\mathrm{w}} \\ D & \text{if} \quad t_{\mathrm{w}} \leq t \bmod T < t_{\mathrm{w}} + t_{\mathrm{a}} = T \end{cases}, \tag{51}$$

where D is given in (49). This is a special case of periodic controllers: the feedback gains are switched between zero and constant values. Fix the length of the waiting and the acting periods to $t_{\mathrm{w}} = 1.2$ and $t_{\mathrm{a}} = 0.8$, thus $T = t_{\mathrm{w}} + t_{\mathrm{a}} = 2$. Since $t_{\mathrm{w}} > \tau$ and $t_{\mathrm{a}} < \tau$, the monodromy matrix can be given according to (25):

$$\Phi^{(2)}(T) = \mathrm{e}^{AT} + \int_{t_{\mathrm{w}}}^{T} \mathrm{e}^{A(T-s)} \, BDC \, \mathrm{e}^{A(s-\tau)} \, \mathrm{d}s. \tag{52}$$

Consider the case $a = -3$. In this case, the system cannot be stabilized using the time-invariant controller (3). However, it can be stabilized using the act-and-wait controller, furthermore deadbeat control can be attained as it is shown below.

Evaluation of the integral in (52) yields the 2×2 matrix given in (53) at the bottom of the page. For fixed control parameters d_1 and d_2, the eigenvalues of $\Phi^{(2)}(T)$ can be computed numerically. If both eigenvalues have magnitude less than one, than the system is asymptotically stable.

The stability chart in the (d_1, d_2) plane is shown in Fig. 4. Stability boundaries are denoted by thick lines, while the contour plots of the maximal magnitude of the eigenvalues are also presented by thin lines. The diagram was determined via point-by-point evaluation of the monodromy matrix (53) and the associated critical eigenvalues over a 200×200-sized grid of parameters d_1 and d_2. It can be seen that there exist a finite domain of the control parameters (d_1, d_2) where the system is stable. Moreover, numerical analysis of matrix (53) shows that both eigenvalues are zero if the control parameters are $d_1 = 12.2496$ and $d_2 = 7.0721$. Consequently, these parameters results in deadbeat control with deadbeat period $T_{\mathrm{DB}} = 2T = 4$.

$$\Phi^{(2)}(T) = \begin{bmatrix} 15.989523 - 0.521885 d_1 - 0.592188 d_2 & 9.213484 - 0.197396 d_1 - 0.521885 d_2 \\ 27.640452 - 1.739474 d_1 - 2.227752 d_2 & 15.989523 - 0.742584 d_1 - 1.739474 d_2 \end{bmatrix}. \tag{53}$$

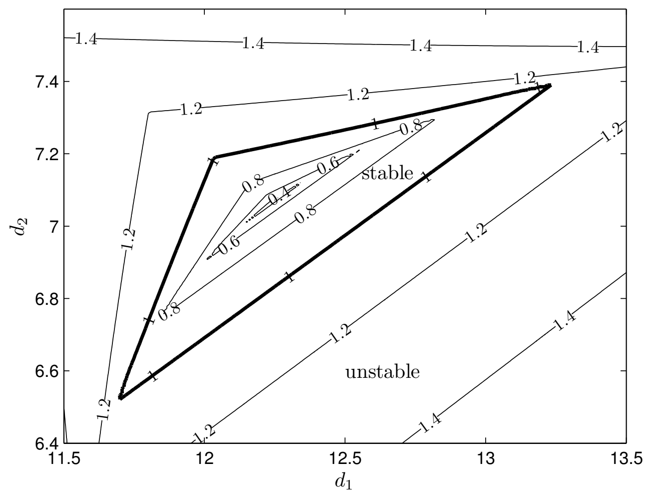

Fig. 4. Stability chart for the act-and-wait controller

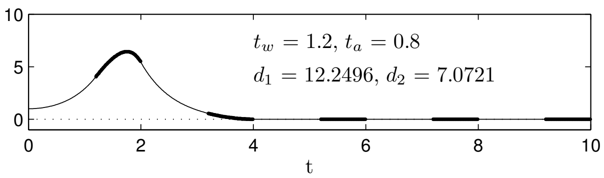

Fig. 5. Time history for deadbeat control

The corresponding time-domain simulations can be seen in Fig. 5. Thick lines denote the periods of acting and thin lines denote waiting. It can be seen that $x(t)$ grows exponentially in the first waiting period $[0, t_w)$ since matrix A is unstable, then, during the first acting period $[t_w, T)$, the growing tendency of $x(t)$ is reversed, and the deadbeat convergence is completed in the next act-and-wait period.

This example presents the efficiency of the act-and-wait control concept. It was shown that a system, which cannot be stabilized by a time invariant feedback due to the feedback delay, can be stabilized by the act-and-wait control concept. Furthermore, it was shown that deadbeat control can be attained.

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