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Tracking Control of Timed Continuous Petri Net Systems under Infinite Servers Semantics

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Abstract: A Lyapunov-function-based control algorithm is proposed for timed continuous Petri Net (contPN) systems working under infinite servers semantics. A timed contPN is a switched linear system and its control signal must be non-negative and upper bounded by a function of system states. An input variable transformation is applied to convert the system to a set of integrators plus static constraints. Then, a low-and-high gain algorithm is proposed for step-tracking. To improve transient performance, planning of the reference target is further discussed.

NOMENCLATURE

$\mathbb{R}^+$: the set of non-negative real numbers; $\mathbb{R}^k$: the Euclidean space of dimension k; $N = \{1, \cdots, n\}$ and n is the number of places; $M = \{1, \cdots, m\}$ and m is the number of transitions; $G = \{1, \cdots, g\}$ and g is the number of nets configurations; $\Omega = \{1, \cdots, \omega\}$ and ω is the number of intermediate states; For a given $\mathbf{a}\backslash\mathbf{a}_b \in \mathbb{R}^k$, $a_i\backslash a_{b,i}$ is the i-th element of $\mathbf{a}\backslash\mathbf{a}_b$; For a given $\mathbf{A}\backslash\mathbf{A}_b \in \mathbb{R}^{k\times l}$, $\mathbf{a}_i\backslash\mathbf{a}_{b,i}$ is the i-th row of $\mathbf{A}\backslash\mathbf{A}_b$ except special indication; Given $\mathbf{a}_1$, $\mathbf{a}_2 \in \mathbb{R}^k$, the i-th element of $\min\{\mathbf{a}_1, \mathbf{a}_2\}$ is $\min\{a_{1,i}, a_{2,i}\}$; $\mathbf{a}_1 \leq \mathbf{a}_2$ means, $\forall i \in \{1, \cdots, k\}$, $a_{1,i} \leq a_{2,i}$; Given a finite set S, $|S|$ is the size of the set S.

1. INTRODUCTION

Petri Nets (PNs) are powerful mathematical tools with appealing graphical representations for discrete even systems. However, discrete PNs suffer from the so called state explosion problem. One way to tackle this problem is to fluidify the discrete models. The resulting *continuous* PN (contPN) systems have the potential for the applications of analytical techniques developed for continuous and hybrid systems. Steady state control for timed contPN systems has been studied in Mahulea et al. [2005]; assuming all transitions can be fired, it can be formulated as a linear programming problem. A Lyapunov-function-based dynamic control method was proposed in Xu et al. [2006] for Join-Free (JF) timed contPNs, which can ensure global convergence of both system states and input signals. However, the application is only limited to JF cases. Hence, dynamic control of a general timed contPN still needs further investigations and great improvements.

The first peculiarity for timed contPN control is that, due to the synchronization, i.e. *minimum* operator used in the flow definition, a timed contPN system under infinite servers semantics switches within different configurations. The switching is completely defined by system states, and it does not depend on time explicitly. Secondly, there are certain input constraints, i.e. the control signal must be non-negative and upper bounded by a function of system states. Hence, the main challenge in our work is to develop control laws under the switched dynamics and the special input constraints to obtain global tracking convergence.

Since switching within stable systems may result in instability, lots of work have been done on the control law design of switched linear systems. However, it is always assumed that the eigenvalues can be arbitrarily assigned for every subsystem. For timed contPN systems, this assumption is not valid due to the input constraints. Hence, how to construct control algorithms for switched linear systems with input constraints is still an open problem. The input constraints can be treated as input saturations which has been thoroughly discussed. However, the common assumptions are that the lower saturation bounds are negative constants and the upper ones are positive constants. To deal with the special input constraints in timed contPNs, a modified LQ-theory-based low-gain controller was proposed in Xu et al. [2006]. Nevertheless, only JF cases were discussed there, which are non-switched linear systems.

In this paper, general timed contPN systems working under infinite servers semantics are considered and a new low-and-high gain control scheme is proposed for tracking step targets. The presented algorithm can ensure global asymptotical convergence of both markings and control signals in presence of the switched dynamics and the input constraints. An input variable transformation is constructed first to convert the system into a set of integrators plus static input constraints. Analogous to the work of Xu et al. [2006], to ensure the global convergence and the smoothness of control signal, a new reference trajectory is developed to take the place of a pure step target. Based on the new model and the modified tracking trajectory, a low-and-high gain controller is proposed. Moreover, to improve the transient performance, the trajectory planning problem is briefly discussed. The paper is organized as follows. Section 2 introduces the basic concepts of contPNs. The control problem is formulated in Section 3. The tracking trajectory design is outlined in Section 4. Section 5 focuses on control the law design and global convergence analysis. An illustrative example is given in Section 6. The tracking reference planning is further discussed in Section 7.

2. CONTINUOUS PETRI NET SYSTEMS

2.1 Untimed Continuous Petri Net Systems

A contPN system is a pair $\langle \mathcal{N}, \mathbf{m}_0 \rangle$, where $\mathcal{N} = \langle \mathbf{P}, \mathbf{T}, \mathbf{Pre}, \mathbf{Post} \rangle$ specifies the net structure ($\mathbf{P}$ and $\mathbf{T}$ are disjoint (finite) sets of places and transitions, and $\mathbf{Pre}$ and $\mathbf{Post}$ are incidence matrices on non-negative real numbers), and $\mathbf{m}_0$ is the initial marking. $\mathcal{N}$ is always assumed to be connected, while $\mathbf{P}$ and $\mathbf{T}$ have n and m elements, respectively. Hence, the marking $\mathbf{m} \in \mathbb{R}^{+n}$, $\mathbf{Pre} \in \mathbb{R}^{\mathbf{n} \times \mathbf{m}}$ and $\mathbf{Post} \in \mathbb{R}^{\mathbf{n} \times \mathbf{m}}$. For $w \in \mathbf{P} \cup \mathbf{T}$, the set of its input and output nodes are denoted as $^\bullet w$, and $w^\bullet$ respectively. A PN is *conservative* iff $\exists \mathbf{y} > \mathbf{0}$, such that $\mathbf{yC} = \mathbf{0}$, where $\mathbf{C} = \mathbf{Post} - \mathbf{Pre}$ is the token flow matrix and $\mathbf{y}$ is called P-semiflow. A PN is *consistent* iff $\exists \mathbf{x} > \mathbf{0}$ such that $\mathbf{Cx} = \mathbf{0}$, where $\mathbf{x}$ is *T-semiflow.* In this work, only conservative and consistent PNs will be considered.

Proposition 1. (Silva and Recalde [2002]) Let $\mathcal{N}$ be a consistent contPN and all its transitions can be fired.

1.1 $\mathbf{m} \geq \mathbf{0}$ is reachable in $\langle \mathcal{N}, \mathbf{m}_0 \rangle$, iff $\exists \boldsymbol{\sigma} \in \mathbb{R}^{+m} \geq \mathbf{0}$ such that $\mathbf{m} = \mathbf{m}_0 + \mathbf{C}\boldsymbol{\sigma}$.

1.2 If $\mathcal{N}$ is consistent and conservative, $\mathbf{m} = \mathbf{m}_0 + \mathbf{C}\boldsymbol{\sigma}$ ($\boldsymbol{\sigma} \geq \mathbf{0}$) is equivalent to $\mathbf{Ym} = \mathbf{Ym}_0$, where $\mathbf{Y}$ is a basis of P-semiflows.

2.2 Timed Continuous Unforced Petri Net Systems

A timed contPN system can be represented as $\langle \mathcal{N}, \boldsymbol{\lambda}, \mathbf{m}_0 \rangle$, where $\boldsymbol{\lambda} \in \mathbb{R}^{+m} > \mathbf{0}$ are the firing rates of transitions. The state equation is $\mathbf{m}(\tau) = \mathbf{m}_0 + \mathbf{C}\boldsymbol{\sigma}(\tau)$, where τ is time. Hence, $\dot{\mathbf{m}}(\tau) = \mathbf{Cf}(\tau)$ can be obtained, where $\mathbf{f}(\tau) = \dot{\boldsymbol{\sigma}}(\tau)$ are the flows of transitions. For notation simplicity, τ will be omitted in the rest of the paper. Different semantics have been introduced for the definition of $\mathbf{f}$ and the most important ones are *infinite servers* and *finite servers.* Define $\mathbf{m}[p]$ as the marking of place p, $\boldsymbol{\lambda}[t]$ as the firing rate of transition t and $\mathbf{Pre}[p, t]$ as the element in $\mathbf{Pre}$ corresponding to place p and transition t. Under infinite server semantics, $\mathbf{f} = \mathbf{\Phi}(\mathbf{m})\mathbf{m}$, where $\mathbf{\Phi}(\mathbf{m})[t, p] = \frac{\boldsymbol{\lambda}[t]}{\mathbf{Pre}[p,t]}$ if $p \in {}^\bullet t$ and $\mathbf{m}[p]/\mathbf{Pre}[p, t]$ is minimum for all $p \in {}^\bullet t$, $\mathbf{\Phi}(\mathbf{m})[t, p] = 0$ otherwise. Thus,

$$\dot{\mathbf{m}} = \mathbf{Cf} = \mathbf{C\Phi}(\mathbf{m})\mathbf{m}. \qquad (1)$$

Note that the value of $\mathbf{\Phi}(\mathbf{m})$ changes when the system switches its configuration. Therefore, an autonomous timed contPN system (1) can be interpreted as a switched linear system. Assume a timed contPN has g configurations. Define $\mathbf{\Phi}_l \in \mathbb{R}^{+m \times n}$, $l \in G$, to denote all the possible values of $\mathbf{\Phi}(\mathbf{m})$. Moreover, let $\mathbf{\Phi}_0 = \mathbf{\Phi}(\mathbf{m}_0)$ and $\mathbf{\Phi}_d = \mathbf{\Phi}(\mathbf{m}_d)$, where $\mathbf{m}_d$ is the desired marking.

Next, a timed contPN system will be given as an illustration example throughout the whole paper.

Example 1: Consider the net in Figure 1 with $\boldsymbol{\lambda} = [3, 1, 1]^T$ *and* $\mathbf{m}_0 = [4, 8, 5, 3]^T$*. Here,*

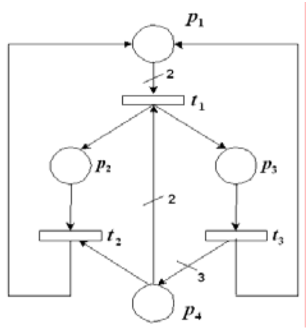

Fig. 1. Timed ContPN System.

$\mathbf{T} = \{t_1, t_2, t_3\}$, $\mathbf{P} = \{p_1, p_2, p_3, p_4\}$*;*
$\mathbf{Pre} = [2, 0, 0; 0, 1, 0; 0, 0, 1; 2, 1, 0]$*;*
$\mathbf{Post} = [0, 1, 1; 1, 0, 0; 1, 0, 0; 0, 0, 3]$*;*
$\mathbf{C} = [-2, 1, 1; 1, -1, 0; 1, 0, -1; -2, -1, 3]$*;*
A basis of P-semiflows: $[1, 1, 1, 0; 1, 0, 4, 1]$*;*
A basis of T-semiflows: $[1; 1; 1]$.
Under infinite semantics, $g = 4$*. Moreover,*
$\mathbf{\Phi}_1 = [1.5, 0, 0, 0; 0, 1, 0, 0; 0, 0, 1, 0]$*;*
$\mathbf{\Phi}_2 = [1.5, 0, 0, 0; 0, 0, 0, 1; 0, 0, 1, 0]$*;*
$\mathbf{\Phi}_3 = [0, 0, 0, 1.5; 0, 1, 0, 0; 0, 0, 1, 0]$*;*
$\mathbf{\Phi}_4 = [0, 0, 0, 1.5; 0, 0, 0, 1; 0, 0, 1, 0]$.
Let $\mathbf{m}_d = [1, 10, 6, 2]^T$*,* $\mathbf{\Phi}_0 = \mathbf{\Phi}_3$ *and* $\mathbf{\Phi}_d = \mathbf{\Phi}_2$. ■

Property 1. Let $\langle \mathcal{N}, \boldsymbol{\lambda}, \mathbf{m}_0 \rangle$ be a timed contPN system with a desired marking $\mathbf{m}_d$. Assume $\mathbf{m}_1$ and $\mathbf{m}_2$ are two reachable markings. Then, (a). $\mathbf{\Phi}(\mathbf{m}_1)\mathbf{m}_1 \leq \mathbf{\Phi}(\mathbf{m}_2)\mathbf{m}_1$, (b). $\min_{l \in G}\{\mathbf{\Phi}_l \min\{\mathbf{m}_0, \mathbf{m}_d\}\} = \min\{\mathbf{\Phi}_0\mathbf{m}_0, \mathbf{\Phi}_d\mathbf{m}_d\}$.

Proof: (a): According to the definition of $\mathbf{\Phi}(\mathbf{m})$, it can be derived straightforwardly.

(b): As $\min\limits_{l \in G}\{\mathbf{\Phi}_l \min\{\mathbf{m}_0, \mathbf{m}_d\}\} = \min\limits_{l \in G}\{\min\{\mathbf{\Phi}_l\mathbf{m}_0, \mathbf{\Phi}_l\mathbf{m}_d\}\}$, from the result of (a), $\forall l \in G$, $\mathbf{\Phi}_0\mathbf{m}_0 \leq \mathbf{\Phi}_l\mathbf{m}_0$ and $\mathbf{\Phi}_d\mathbf{m}_d \leq \mathbf{\Phi}_l\mathbf{m}_d$. Hence, (b) can be obtained. ■

3. PROBLEM FORMULATION

For concise expression, "timed contPN" will be written as "contPN". The control action to PN systems is to slow down their firing flows. From (1), a contPN system with a control action becomes

$$\dot{\mathbf{m}} = \mathbf{C}(\mathbf{\Phi}(\mathbf{m}) - \mathbf{u}) \overset{\triangle}{=} \mathbf{A}(\mathbf{m})\mathbf{m} - \mathbf{Bu}, \qquad (2)$$
$$\mathbf{0} \leq \mathbf{u} \leq \mathbf{f},$$

where $\mathbf{A}(\mathbf{m}) = \mathbf{C\Phi}(\mathbf{m})$, $\mathbf{B} = \mathbf{C}$ and $\mathbf{u} \in \mathbb{R}^{+m}$. Define $\mathbf{w} = \mathbf{\Phi}(\mathbf{m})\mathbf{m} - \mathbf{u}$. (2) can be further rewritten as

$$\dot{\mathbf{m}} = \mathbf{Bw}, \qquad (3)$$
$$\mathbf{0} \leq \mathbf{w} \leq \mathbf{f}. \qquad (4)$$

Considering the definition of $\mathbf{f}$, (4) is equivalent to

$$\mathbf{0} \leq \mathbf{w} \leq \mathbf{\Phi}_l\mathbf{m} \quad (\forall l \in G). \qquad (5)$$

The controller design will be based on (3) and (5). However, $\mathbf{u}$ can be obtained directly from $\mathbf{w}$.

Our control objective is to construct control laws such that both $\mathbf{m}$ and $\mathbf{u}$ converge to a desired reachable marking $\mathbf{m}_d$ and a desired control action $\mathbf{u}_d$ asymptotically. From *Proposition 1*, $\mathbf{m}_d$ must fulfill that $\mathbf{Ym}_d = \mathbf{Ym}_0$. Due to the input constraints, $\mathbf{0} \le \mathbf{u}_d \le \mathbf{\Phi}_d\mathbf{m}_d$ must be satisfied. Define $\mathbf{A}_d = \mathbf{C}\mathbf{\Phi}_d$. As $\mathbf{m}_d$ are constants, according to (2),

$$\mathbf{0} = \mathbf{A}_d\mathbf{m}_d - \mathbf{B}\mathbf{u}_d. \tag{6}$$

Assumption 1. $\forall i \in N$, $m_{0,i} > 0$ and $m_{d,i} > 0$.

$\forall i \in N$, $m_{d,i} > 0$ is usually the case of optimal steady states in practical systems. If some elements of $\mathbf{m}_0$ are zero, either there is a transition that will never be fired (and so it can be removed), or a firing sequence exists such that $\mathbf{m}_0[\boldsymbol{\sigma} > \mathbf{m}$ and $\mathbf{m} > \mathbf{0}$ (Silva and Recalde [2002]).

4. DESIGN OF TRACKING REFERENCE

To ensure global convergence and the smoothness of control signals, a pure step target $\mathbf{m}_d$ is replaced by the following step-ramp-mixed reference trajectory $\mathbf{m}_r(\tau) \in \mathbb{R}^{+n}$.

$$\mathbf{m}_r(\tau) = \begin{cases} \mathbf{m}_{r0} + \dfrac{\mathbf{m}_d - \mathbf{m}_{r0}}{h}\tau, & \tau \in [0, h] \\ \mathbf{m}_d, & \tau \in [h, \infty) \end{cases}, \tag{7}$$

where $\mathbf{m}_{r0}$ is the initial step, that is, $\mathbf{m}_{r0} = \mathbf{m}_0 + \delta(\mathbf{m}_d - \mathbf{m}_0)$ with $0 \le \delta < 1$, and $h > 0$ is the time when $\mathbf{m}_r(\tau)$ reaches $\mathbf{m}_d$. It has to be proved that, for given $\mathbf{m}_{r0}$ and $\mathbf{m}_d$, control action $\mathbf{w}_r \in \mathbb{R}^m$ exists such that

$$\begin{aligned} \dot{\mathbf{m}}_r = \mathbf{B}\mathbf{w}_r &= \begin{cases} \dfrac{\mathbf{m}_d - \mathbf{m}_{r0}}{h}, & \tau \in [0, h^-] \\ 0, & \tau \in [h^+, \infty) \end{cases} \\ &= \begin{cases} \beta(\mathbf{m}_d - \mathbf{m}_0), & \tau \in [0, h^-] \\ 0, & \tau \in [h^+, \infty) \end{cases}, \end{aligned} \tag{8}$$

$$\mathbf{0} \le \mathbf{w}_r \le \mathbf{\Phi}_l\mathbf{m}_r(\tau) \quad \forall l \in G, \tag{9}$$

where $\beta = \frac{1-\delta}{h}$. Since $\min\{\mathbf{m}_0, \mathbf{m}_d\} \le \mathbf{m}_r(\tau)$ and $\mathbf{\Phi}_l \min\{\mathbf{m}_0, \mathbf{m}_d\} \le \mathbf{\Phi}_l\mathbf{m}_r(\tau)$ ($\forall l \in G$), (9) can be rewritten as follows

$$\begin{cases} \mathbf{0} \le \mathbf{w}_r \le \mathbf{\Phi}_l \min\{\mathbf{m}_0, \mathbf{m}_d\} \quad \forall l \in G, & \tau \in [0, h^-] \\ \mathbf{0} \le \mathbf{w}_r \le \mathbf{\Phi}_d\mathbf{m}_d, & \tau \in [h^+, \infty) \end{cases} \tag{10}$$

Proposition 2. Let $\mathbf{m}_0 > \mathbf{0}$ and let $\mathbf{m}_d > \mathbf{0}$ be a reachable marking. Then, $\beta > 0$ can always be found such that $\mathbf{w}_r$ satisfying (8) and (10) exists.

Choose

$$\mathbf{w}_r = \begin{cases} \beta\boldsymbol{\sigma}, & \tau \in [0, h^-] \\ \mathbf{\Phi}_d\mathbf{m}_d - \mathbf{u}_d, & \tau \in [h^+, \infty). \end{cases} \tag{11}$$

The proof for *Proposition 2* is same as the proof of *Proposition 1* in Xu et al. [2006]. Moreover, to obtain a faster system response, the calculation of β can be formulated as follows (Xu et al. [2006]):

$$\begin{aligned} \max\ \beta \quad s.t. :\ & \mathbf{m}_d = \mathbf{m}_0 + \mathbf{B}\boldsymbol{\sigma} \\ & \mathbf{0} \le \beta\boldsymbol{\sigma} \le \min\{\mathbf{\Phi}_0\mathbf{m}_0, \mathbf{\Phi}_d\mathbf{m}_d\} \\ & \beta > 0. \end{aligned} \tag{12}$$

Remark 1. Because of consistency, there is at least one zero-element in both $\boldsymbol{\sigma}$ and $\min\{\mathbf{\Phi}_0\mathbf{m}_0, \mathbf{\Phi}_d\mathbf{m}_d\} - \beta\boldsymbol{\sigma}$.

For the design of $\mathbf{m}_r(\tau)$, β is calculated first based on (12). Then δ will be further decided. A larger δ leads to a smaller h. However, it also results in a larger initial error, which may destroy the tracking convergence. The design of δ will be addressed in Section 5.3.

5. TRACKING CONTROL OF CONTPN SYSTEMS

The control signal $\mathbf{w}$ for (3) is designed as follows:

$$\mathbf{w} = sat(\mathbf{w}_{lg} + \mathbf{w}_{hg}) + \mathbf{w}_r - \mathbf{\Phi}_d\mathbf{e}, \tag{13}$$

where $\mathbf{w}_{lg}$ is the low-gain part, $\mathbf{w}_{hg}$ is the high-gain term, $\mathbf{w}_r$ is defined as in (11) and $\mathbf{e} = \mathbf{m}_r(\tau) - \mathbf{m}$. $\forall\ \mathbf{a} \in \mathbb{R}^m$, $sat(\mathbf{a}) \triangleq [sat(a_1), \cdots, sat(a_m)]^T$ and $\forall j \in M$, $sat(a_j)$ is defined as $sat(a_j) = \begin{cases} a_{j,upper}, & \textbf{if} \ \ a_j \ge a_{j,upper} \\ a_j, & \textbf{if} \ \ a_{j,lower} < a_j < a_{j,upper} \\ a_{j,lower}, & \textbf{if} \ \ a_j \le a_{j,lower} \end{cases}$, where $a_{j,upper} = \lambda_j \min\limits_{p_i \in {}^\bullet t_j}(\dfrac{m_i}{Pre(p_i, t_j)}) + \boldsymbol{\phi}_{d,j}\mathbf{e} - w_{r,j}$ and $a_{j,lower} = \boldsymbol{\phi}_{d,j}\mathbf{e} - w_{r,j}$. Note that (13) ensures $\mathbf{0} \le \mathbf{w} \le \mathbf{f}$.

$\mathbf{w}_{lg}$ is constructed under the input constraints in such a way that the closed loop system is stable. Hence, $\mathbf{w} = \mathbf{w}_{lg} + \mathbf{w}_r - \mathbf{\Phi}_d\mathbf{e}$ can guarantee the realization of control objective already. Since $\mathbf{0} \le \mathbf{w} \le \mathbf{f}$, the constraints for $\mathbf{w}_{lg}$ are:

$$-\mathbf{w}_r + \mathbf{\Phi}_d\mathbf{e} \le \mathbf{w}_{lg} \le \mathbf{f} - \mathbf{w}_r + \mathbf{\Phi}_d\mathbf{e}. \tag{14}$$

5.1 System Error Dynamics

From (3), (7) and (13), we have

$$\dot{\mathbf{e}} = \begin{cases} \dfrac{\mathbf{m}_d - \mathbf{m}_{r0}}{h} - \mathbf{B}(\mathbf{w}_r - \mathbf{\Phi}_d\mathbf{e}) \\ \quad -\mathbf{B}sat(\mathbf{w}_{lg} + \mathbf{w}_{hg}) & \tau \in [0, h^-] \\ -\mathbf{B}(\mathbf{w}_r - \mathbf{\Phi}_d\mathbf{e}) \\ \quad -\mathbf{B}sat(\mathbf{w}_{lg} + \mathbf{w}_{hg}) & \tau \in [h^+, \infty) \end{cases}. \tag{15}$$

As $\mathbf{w}_r$ is the solution of (8),

$$\dot{\mathbf{e}} = \mathbf{A}_d\mathbf{e} - \mathbf{B}sat(\mathbf{w}_{lg} + \mathbf{w}_{hg}). \tag{16}$$

ContPNs with at least one P-semiflow are non-controllable and a transformation matrix can be constructed to separate it into controllable and non-controllable parts. The transformation matrix is chosen as $\mathbf{H} \in \mathbb{R}^{n\times n}$, where the first r rows is a basis of P-semiflows, i.e. $\mathbf{Y}$, and the remaining rows are completed with elementary vectors such that $\mathbf{H}$ is full rank (Mahulea et al. [2005]). Define $\bar{\mathbf{e}} = \mathbf{H}\mathbf{e}$. The error dynamics becomes $\dot{\bar{\mathbf{e}}} = \bar{\mathbf{A}}_d\bar{\mathbf{e}} - \bar{\mathbf{B}}sat(\mathbf{w}_{lg} + \mathbf{w}_{hg})$, where $\bar{\mathbf{A}}_d = \mathbf{H}\mathbf{A}_d\mathbf{H}^{-1}$ and $\bar{\mathbf{B}} = \mathbf{H}\mathbf{B}$. Considering $\mathbf{Y}\mathbf{C} = \mathbf{0}$, $\dot{\bar{e}}_i = 0$ $(i = 1, \cdots, r)$ can be derived. From $\mathbf{Y}\mathbf{m}_d = \mathbf{Y}\mathbf{m}_{r0}$, $\forall i \in \{1, \cdots, r\}$, $\bar{e}_i(0) = 0$ can be obtained. Therefore, $\forall \tau \in [0, \infty)$, $\bar{e}_i(\tau) = 0$ $(i = 1, \cdots, r)$ and the errors of the uncontrollable part are always zero. Moreover, the controllable part can be rewritten as:

$$\dot{\bar{\mathbf{e}}}_c = \bar{\mathbf{A}}_{dc}\bar{\mathbf{e}}_c - \bar{\mathbf{B}}_c sat(\mathbf{w}_{lg} + \mathbf{w}_{hg}), \tag{17}$$

where $\bar{\mathbf{e}}_c \triangleq [\bar{e}_{r+1}, \cdots, \bar{e}_n]^T \in \mathbb{R}^{n-r}$, $\bar{\mathbf{A}}_{dc} \in \mathbb{R}^{(n-r)\times(n-r)}$ and $\bar{\mathbf{B}}_c \in \mathbb{R}^{(n-r)\times m}$. It is valid that

$$\mathbf{e} = \mathbf{S}\bar{\mathbf{e}}_c, \tag{18}$$

where $\mathbf{S} \in \mathbb{R}^{n\times(n-r)}$ is $\mathbf{H}^{-1}$ without first r columns.

Example 2: For the contPN system in Figure 1, $\mathbf{H} = [1, 1, 1, 0; 1, 0, 4, 1; 0, 0, 1, 0; 0, 0, 0, 1]$*;* $\bar{\mathbf{A}}_{dc} = [-1, 1.5; 3, -4]$*;* $\bar{\mathbf{B}}_c = [1, 0, -1; -2, -1, 3]$. ■

5.2 Design of $\mathbf{w}_{lg}$

From (18), the definitions of $\mathbf{f}$ and $\mathbf{w}_r$ and $\min\{\mathbf{m}_0, \mathbf{m}_d\} \leq \mathbf{m}_d$, if $\tau \in [0, h^-]$, (14) can be rewritten as follows:

$$-\beta\boldsymbol{\sigma} \leq \mathbf{w}_{lg} - \boldsymbol{\Phi}_d\mathbf{S}\bar{\mathbf{e}}_c \ \text{ and}$$
$$\mathbf{w}_{lg} + (\boldsymbol{\Phi}_l - \boldsymbol{\Phi}_d)\mathbf{S}\bar{\mathbf{e}}_c \leq -\beta\boldsymbol{\sigma} + \boldsymbol{\Phi}_l \min\{\mathbf{m}_0, \mathbf{m}_d\} \ \ l \in G.$$

Similarly, if $\tau \in [h^+, \infty)$, we have

$$-(\boldsymbol{\Phi}_d\mathbf{m}_d - \mathbf{u}_d) \leq \mathbf{w}_{lg} - \boldsymbol{\Phi}_d\mathbf{S}\bar{\mathbf{e}}_c \ \text{ and}$$
$$\mathbf{w}_{lg} + (\boldsymbol{\Phi}_l - \boldsymbol{\Phi}_d)\mathbf{S}\bar{\mathbf{e}}_c \leq (\boldsymbol{\Phi}_l - \boldsymbol{\Phi}_d)\mathbf{m}_d + \mathbf{u}_d \ \ l \in G.$$

Hence, the constraints of $\mathbf{w}_{lg}$ can be rewritten as follows:

$$-\min\{\beta\boldsymbol{\sigma}, \boldsymbol{\Phi}_d\mathbf{m}_d - \mathbf{u}_d\} \leq \mathbf{w}_{lg} - \boldsymbol{\Phi}_d\mathbf{S}\bar{\mathbf{e}}_c \ \text{ and}$$
$$\mathbf{w}_{lg} + (\boldsymbol{\Phi}_l - \boldsymbol{\Phi}_d)\mathbf{S}\bar{\mathbf{e}}_c \leq \min\{-\beta\boldsymbol{\sigma} + \boldsymbol{\Phi}_l \min\{\mathbf{m}_0, \mathbf{m}_d\},$$
$$(\boldsymbol{\Phi}_l - \boldsymbol{\Phi}_d)\mathbf{m}_d + \mathbf{u}_d\} \ \ l \in G. \quad (19)$$

Define two instrumental vectors: $\mathbf{c}_1 = \min\{\beta\boldsymbol{\sigma}, \boldsymbol{\Phi}_d\mathbf{m}_d - \mathbf{u}_d\}$ and $\mathbf{c}_{2l} = \min\{\boldsymbol{\Phi}_l \min\{\mathbf{m}_0, \mathbf{m}_d\} - \beta\boldsymbol{\sigma}, (\boldsymbol{\Phi}_l - \boldsymbol{\Phi}_d)\mathbf{m}_d + \mathbf{u}_d\}$, $(l \in G)$. $\mathbf{w}_{lg}$ has to verify

$$\mathbf{w}_{lg} - \boldsymbol{\Phi}_d\mathbf{S}\bar{\mathbf{e}}_c \geq -\mathbf{c}_1 \text{ and } \mathbf{w}_{lg} + (\boldsymbol{\Phi}_l - \boldsymbol{\Phi}_d)\mathbf{S}\bar{\mathbf{e}}_c \leq \mathbf{c}_{2l} \quad (20)$$

$\mathbf{w}_{lg}$ is designed to minimize $J(\bar{\mathbf{e}}_c(0)) = \int_0^\infty (\bar{\mathbf{e}}_c^T\mathbf{Q}\bar{\mathbf{e}}_c + \gamma\mathbf{u}^T\mathbf{R}\mathbf{u})dt$, where $\mathbf{Q}$ is a diagonal positive definite matrix, $\mathbf{R} = diag(r_1, \cdots, r_m)$ is positive definite and $\gamma > 0$. Let $\mathbf{w}_{lg} = \mathbf{K}\bar{\mathbf{e}}_c$ where $\mathbf{K}$ is the feedback gain and is calculated according to the properties of $\mathbf{c}_1$ and $\mathbf{c}_{2l}$ $(l \in G)$.

From *Remark 1*, $\mathbf{c}_1$ has at least one zero-element and there always exists $k \in G$ such that $\mathbf{c}_{2k}$ has at least one zero-element. Define $G_0 \subset G$ represent the set of $\mathbf{c}_{2l}$ with at least one zero-element. Moreover, if $c_{2l,z} = 0$ ($l \in G_0$ and $z \in M$), from the definition of $\mathbf{c}_{2l}$, considering $\boldsymbol{\Phi}_d\mathbf{m}_d \leq \boldsymbol{\Phi}_l\mathbf{m}_d$ (*Property 1.a*) and *Property 1.b*, it can be derived that $\boldsymbol{\phi}_{l,z} = \boldsymbol{\phi}_{0,z}$ or $\boldsymbol{\phi}_{l,z} = \boldsymbol{\phi}_{d,z}$.

To clearly explain the basic idea, the following case will be considered here: 1) only one element of $\mathbf{c}_1$, i.e. $\mathbf{c}_{1,z_1}$ ($z_1 \in M$), is zero; 2) $\forall l \in G_0$, the zero-element in $\mathbf{c}_{2l}$ is the z_2-th element ($z_2 \in M$). However, the extension to general cases is straightforward.

Let $z_1 \neq z_2$ and the design $\mathbf{K}$ is classified into three cases.

Case 1. $\forall l \in G_0$**,** $c_{2l,z_2} = 0 \Rightarrow \boldsymbol{\phi}_{l,z_2} = \boldsymbol{\phi}_{d,z_2}$

Define $\Delta\boldsymbol{\Phi}_d \in \mathbb{R}^{m\times n}$ and $\Delta\bar{\mathbf{B}}_c \in \mathbb{R}^{(n-1)\times m}$. The z_1-th rows of $\Delta\boldsymbol{\Phi}_d$ and $\boldsymbol{\Phi}_d$ are same and the remaining ones of $\Delta\boldsymbol{\Phi}_d$ are $\mathbf{0}$. The z_1-th and the z_2-th columns of $\Delta\bar{\mathbf{B}}_c$ are same as those of $\bar{\mathbf{B}}_c$ and the remaining ones of $\Delta\bar{\mathbf{B}}_c$ are $\mathbf{0}$.

Part A. $(\bar{\mathbf{A}}_{dc} - \bar{\mathbf{B}}_c\Delta\boldsymbol{\Phi}_d\mathbf{S},\ \bar{\mathbf{B}}_c - \Delta\bar{\mathbf{B}}_c)$ *is stabilizable*

Let $\mathbf{K} = \mathbf{K}_1 + \mathbf{K}_2$, where $\mathbf{K}_1 = \frac{1}{\gamma}\mathbf{R}^{-1}(\bar{\mathbf{B}}_c - \Delta\bar{\mathbf{B}}_c)^T\mathbf{W}$ and $\mathbf{K}_2 = \Delta\boldsymbol{\Phi}_d\mathbf{S}$. The solution of $\mathbf{W}$ is found from

$$\mathbf{W}(\bar{\mathbf{A}}_{dc} - \bar{\mathbf{B}}_c\Delta\boldsymbol{\Phi}_d\mathbf{S}) + (\bar{\mathbf{A}}_{dc} - \bar{\mathbf{B}}_c\Delta\boldsymbol{\Phi}_d\mathbf{S})^T\mathbf{W}$$
$$-\frac{1}{\gamma}\mathbf{W}(\bar{\mathbf{B}}_c - \Delta\bar{\mathbf{B}}_c)\mathbf{R}^{-1}(\bar{\mathbf{B}}_c - \Delta\bar{\mathbf{B}}_c)^T\mathbf{W} + \mathbf{Q} = \mathbf{0}. \quad (21)$$

From the definition of $\Delta\bar{\mathbf{B}}_c$ and $\Delta\boldsymbol{\Phi}_d$, it can be derived that $\mathbf{k}_{z_1} = \boldsymbol{\phi}_{d,z_1}\mathbf{S}$ and $\mathbf{k}_{z_2} = \mathbf{0}$.

Part B. $(\bar{\mathbf{A}}_{dc} - \bar{\mathbf{B}}_c\Delta\boldsymbol{\Phi}_d\mathbf{S},\ \bar{\mathbf{B}}_c - \Delta\bar{\mathbf{B}}_c)$ *is not stabilizable*

Let $\mathbf{K} = \frac{1}{\gamma}\mathbf{R}^{-1}\bar{\mathbf{B}}_c^T\mathbf{W}$ and $\mathbf{W}$ is found from

$$\mathbf{W}\bar{\mathbf{A}}_{dc} + \bar{\mathbf{A}}_{dc}^T\mathbf{W} - \frac{1}{\gamma}\mathbf{W}\bar{\mathbf{B}}_c\mathbf{R}^{-1}\bar{\mathbf{B}}_c^T\mathbf{W} + \mathbf{Q} = \mathbf{0}. \quad (22)$$

Case 2. $\forall l \in G_0$**,** $c_{2l,z_2} = 0 \Rightarrow \boldsymbol{\phi}_{l,z_2} = \boldsymbol{\phi}_{0,z_2}$

Define $\Delta\boldsymbol{\Phi} = \Delta\boldsymbol{\Phi}_d' - \Delta\boldsymbol{\Phi}_0$. The z_1-th and the z_2-th rows of $\Delta\boldsymbol{\Phi}_d'$ are same as those of $\boldsymbol{\Phi}_d$ and the remaining ones are $\mathbf{0}$. The z_2-th row of $\Delta\boldsymbol{\Phi}_0$ is same as that of $\boldsymbol{\Phi}_0$ and the remaining ones are $\mathbf{0}$. Replacing $\Delta\boldsymbol{\Phi}_d$ by $\Delta\boldsymbol{\Phi}$, the design of $\mathbf{K}$ in this case is exactly same as that in **Case 1**.

Case 3. $\exists\ l_1 \in G_0$**,** $l_2 \in G_0$ **so that** $c_{2l_1,z_2} = 0 \Rightarrow \boldsymbol{\phi}_{l_1,z_2} = \boldsymbol{\phi}_{0,z_2}$**,** $c_{2l_1,z_2} = 0 \Rightarrow \boldsymbol{\phi}_{l_1,z_2} = \boldsymbol{\phi}_{d,z_2}$ **and** $\boldsymbol{\phi}_{0,z_2} \neq \boldsymbol{\phi}_{d,z_2}$**.**

Same as the *Part B* of **Case 1**.

Finally, if $z_1 = z_2$, it is same as the *Part B* of **Case 1**.

5.3 Design of δ

$\epsilon(\mathbf{W}, \rho) = \{\bar{\mathbf{e}}_c : \bar{\mathbf{e}}_c^T\mathbf{W}\bar{\mathbf{e}}_c \leq \rho\}$, where $\rho = \bar{\mathbf{e}}_c^T(0)\mathbf{W}\bar{\mathbf{e}}_c(0)$. δ is designed off-line such that $\forall\bar{\mathbf{e}}_c \in \epsilon(\mathbf{W}, \rho)$,

$$\mathbf{K}'\bar{\mathbf{e}}_c \geq -\mathbf{c}_1 \ \text{ and } \ \mathbf{K}_l''\bar{\mathbf{e}}_c \leq \mathbf{c}_{2l} \ (l \in G), \quad (23)$$

where $\mathbf{K}' \triangleq \mathbf{K} - \boldsymbol{\Phi}_d\mathbf{S}$, and $\mathbf{K}_l'' \triangleq \mathbf{K} + (\boldsymbol{\Phi}_l - \boldsymbol{\Phi}_d)\mathbf{S}$. Note that (23) is equivalent to (20).

Proposition 3. Let $\langle\mathcal{N}, \boldsymbol{\lambda}, \mathbf{m}_0\rangle$ be conservative and consistent. Given $\mathbf{Q}$, $\mathbf{R}$ and γ. Define $\mathbf{W}$ and $\mathbf{K}$ as in Subsection 5.2. Then it is possible to find δ such that $\forall\bar{\mathbf{e}}_c \in \epsilon(\mathbf{W}, \rho)$, $\mathbf{K}'\bar{\mathbf{e}}_c \geq -\mathbf{c}_1$ and $\mathbf{K}_l''\bar{\mathbf{e}}_c \leq \mathbf{c}_{2l}$ for all $l \in G$.

Proof: $\forall j \in M$, $\max\limits_{\bar{\mathbf{e}}_c \in \epsilon(\mathbf{W},\rho)} |\mathbf{k}_j'\bar{\mathbf{e}}_c| = \sqrt{\rho}(\mathbf{k}_j'\mathbf{W}^{-1}\mathbf{k}_j'^T)^{1/2}$ and $\max\limits_{\bar{\mathbf{e}}_c \in \epsilon(\mathbf{W},\rho)} |\mathbf{k}_{l,j}''\bar{\mathbf{e}}_c| = \sqrt{\rho}(\mathbf{k}_{l,j}''\mathbf{W}^{-1}\mathbf{k}_{l,j}''^T)^{1/2}$ $(\forall l \in G)$ are valid (Wredenhage and Bélanger [1994]). Hence, we have to prove

$$\sqrt{\rho}(\mathbf{k}_j'\mathbf{W}^{-1}\mathbf{k}_j'^T)^{1/2} \leq c_{1,j}, \quad (24)$$
$$\sqrt{\rho}(\mathbf{k}_{l,j}''\mathbf{W}^{-1}\mathbf{k}_{l,j}''^T)^{1/2} \leq c_{2l,j} \ \ \forall l \in G. \quad (25)$$

Let $z_1 \neq z_2$. The design of δ also has three cases.

Case 1. $\forall l \in G_0$**,** $c_{2l,z_2} = 0 \Rightarrow \boldsymbol{\phi}_{l,z_2} = \boldsymbol{\phi}_{d,z_2}$

Part A. $(\bar{\mathbf{A}}_{dc} - \bar{\mathbf{B}}_c\Delta\boldsymbol{\Phi}_d\mathbf{S},\ \bar{\mathbf{B}}_c - \Delta\bar{\mathbf{B}}_c)$ *is stabilizable*

$\mathbf{k}_j'\mathbf{W}^{-1}\mathbf{k}_j'^T$ and $\mathbf{k}_j''\mathbf{W}^{-1}\mathbf{k}_j''^T$ do not depend on δ. Smaller δ will lead to smaller $\bar{\mathbf{e}}_c(0)$ and ρ. Thus, as $c_{1,j}$ $(j \neq z_1)$ and $c_{2l,j}$ (when $l \in G_0$, $j \neq z_2$) are strictly positive constants, $\delta > 0$ small enough can always be found such that (24) (when $j \neq z_1$) and (25) (when $l \in G_0$, $j \neq z_2$) are valid.

Since $\mathbf{k}_{z_1}' = \mathbf{k}_{lg,z_1} - \boldsymbol{\phi}_{d,z_1}\mathbf{S}$, considering $\mathbf{k}_{lg,z_1} = \boldsymbol{\phi}_{d,z_1}\mathbf{S}$, $\mathbf{k}_{z_1}' = \mathbf{0}$ can be derived. On the other hand, $\forall l \in G_0$, $c_{2l,z_2} = \mathbf{0}$ and $\Delta\boldsymbol{\phi}_{l,z_2} - \boldsymbol{\phi}_{d,z_2} = \mathbf{0}$ can be derived. Then, as $\mathbf{k}_{lg,z_2}$ is $\mathbf{0}$, $\mathbf{k}_{l,z_2}'' = \mathbf{k}_{lg,z_2} + (\Delta\boldsymbol{\phi}_{l,z_2} - \boldsymbol{\phi}_{d,z_2})\mathbf{S} = \mathbf{0}$.

Therefore, $\forall j \in M$, (24) and (25) are valid.

Part B. $(\bar{\mathbf{A}}_{dc} - \bar{\mathbf{B}}_c\Delta\boldsymbol{\Phi}_d\mathbf{S},\ \bar{\mathbf{B}}_c - \Delta\bar{\mathbf{B}}_c)$ *is not stabilizable*

Let $\delta = 0$. (24) and (25) are valid for all $j \in M$.

Case 2. $\forall l \in G_0$**,** $c_{2l,z_2} = 0 \Rightarrow \boldsymbol{\phi}_{l,z_2} = \boldsymbol{\phi}_{0,z_2}$

Replacing $\Delta\mathbf{\Phi}_d$ by $\Delta\mathbf{\Phi}$, the analysis is same as **Case 1**.

Case 3. $\exists\, l_1 \in G_0$, $l_2 \in G_0$ **so that** $c_{2l_1,z_2} = 0 \Rightarrow \boldsymbol{\phi}_{l_1,z_2} = \boldsymbol{\phi}_{0,z_2}$, $c_{2l_1,z_2} = 0 \Rightarrow \boldsymbol{\phi}_{l_1,z_2} = \boldsymbol{\phi}_{d,z_2}$ **and** $\boldsymbol{\phi}_{0,z_2} \neq \boldsymbol{\phi}_{d,z_2}$.

Same as the *Part B* of **Case 1**, i.e. $\delta = 0$

Finally, if $z_1 = z_2$, it is same as the *Part B* of **Case 1**. ■

Remark 2. Let us re-consider **Case 1**. As $c_{1,z_1} = 0$, to meet (24), $\rho = 0$ or $\mathbf{k}'_{z_1} = \mathbf{0}$, which needs $\delta = 0$ or $\mathbf{k}_{z_1} = \boldsymbol{\phi}_{d,z_1}\mathbf{S}$. Hence, to obtain $\delta > 0$, $\mathbf{k}_{z_1} = \boldsymbol{\phi}_{d,z_1}\mathbf{S}$. Similarly, as $c_{2l,z_2} = 0$ ($l \in G_0$), to achieve $\delta > 0$, $\mathbf{k}_{z_2} = \mathbf{0}$. If $(\bar{\mathbf{A}}_{dc} - \bar{\mathbf{B}}_c\Delta\mathbf{\Phi}_d\mathbf{S},\ \bar{\mathbf{B}}_c - \Delta\bar{\mathbf{B}}_c)$ is stabilizable, $\mathbf{K}$ with $\mathbf{k}_{z_1} = \boldsymbol{\phi}_{d,z_1}\mathbf{S}$ and $\mathbf{k}_{z_2} = \mathbf{0}$ can be found and hence $\delta > 0$. Otherwise, δ can only be zero.

To achieve faster system response, the design of δ can be formulated as follows:

$$
\begin{aligned}
&\max\ \ \delta &(26)\\
&s.t. : 0 < \delta \leq 1, \gamma > 0,\\
&\quad \bar{\mathbf{e}}_c^T(0)\mathbf{W}\bar{\mathbf{e}}_c(0)\mathbf{k}'_j\mathbf{W}^{-1}\mathbf{k}'^T_j \leq c^2_{1,j} \quad \forall j \in M,\\
&\quad \bar{\mathbf{e}}_c^T(0)\mathbf{W}\bar{\mathbf{e}}_c(0)\mathbf{k}''_{l,j}\mathbf{W}^{-1}\mathbf{k}''^T_{l,j} \leq c^2_{2l,j} \quad j \in M, \forall l \in G &(27)\\
&\quad \mathbf{W}(\bar{\mathbf{A}}_{dc} - \bar{\mathbf{B}}_c\Delta\mathbf{\Phi}^*\mathbf{S}) + (\bar{\mathbf{A}}_{dc} - \bar{\mathbf{B}}_c\Delta\mathbf{\Phi}^*\mathbf{S})^T\mathbf{W}\\
&\qquad -\frac{1}{\gamma}\mathbf{W}(\bar{\mathbf{B}}_c - \Delta\bar{\mathbf{B}}_c)\mathbf{R}^{-1}(\bar{\mathbf{B}}_c - \Delta\bar{\mathbf{B}}_c)^T\mathbf{W} + \mathbf{Q} = \mathbf{0}\\
&\quad \mathbf{K} = \frac{1}{\gamma}\mathbf{R}^{-1}(\bar{\mathbf{B}}_c - \Delta\bar{\mathbf{B}}_c)^T\mathbf{W} + \Delta\mathbf{\Phi}^*\mathbf{S}\\
&\quad \mathbf{K}' = \mathbf{K} - \mathbf{\Phi}_d\mathbf{S}, \mathbf{K}'' = \mathbf{K} + (\mathbf{\Phi}_l - \mathbf{\Phi}_d)\mathbf{S} &(28)
\end{aligned}
$$

where $\Delta\mathbf{\Phi}^*$ stands for either $\Delta\mathbf{\Phi}_d$ (in **Case 1**) or $\Delta\mathbf{\Phi}$ (in **Case 2**). For (27), we only need to consider the cases when $\boldsymbol{\phi}_{l,j}$ ($l \in G$ and $j \in M$) are different from each other and the number of the constraints can be reduced accordingly.

5.4 Calculation of $\mathbf{w}_{hg}$

$\mathbf{w}_{hg} = k_{hg}\bar{\mathbf{B}}_c^T\mathbf{W}\bar{\mathbf{e}}_c$, where $k_{hg} > 0$.

5.5 Asymptotical Convergence Analysis

Theorem 1. Let $\langle \mathcal{N}, \boldsymbol{\lambda}, \mathbf{m}_0 \rangle$ be conservative and consistent. Given $\mathbf{m}_0$, $\mathbf{m}_d$ and $\mathbf{u_d}$. The proposed low-and-high gain algorithm can ensure the global asymptotical convergence of both the system markings and the control signals.

Proof: From *Proposition 3*, δ exists such that, $\forall \bar{\mathbf{e}}_c \in \epsilon(\mathbf{W}, \rho)$, $\mathbf{k}'_j\bar{\mathbf{e}}_c \geq -c_{1,j}$ and $\mathbf{k}''_{l,j}\bar{\mathbf{e}}_c \leq c_{2l,j}$ ($\forall l \in G$).

Case I. $\delta > 0$

Define $V = \bar{\mathbf{e}}_c^T\mathbf{W}\bar{\mathbf{e}}_c$. Hence,

$$\dot{V} = \dot{\bar{\mathbf{e}}}_c^T\mathbf{W}\bar{\mathbf{e}}_c + \bar{\mathbf{e}}_c^T\mathbf{W}\dot{\bar{\mathbf{e}}}_c. \tag{29}$$

The error dynamics (17) can be rewritten as

$$
\begin{aligned}
\dot{\bar{\mathbf{e}}}_c &= [(\bar{\mathbf{A}}_{dc} - \bar{\mathbf{B}}_c\Delta\mathbf{\Phi}^*\mathbf{S}) - (\bar{\mathbf{B}}_c - \Delta\bar{\mathbf{B}}_c)\mathbf{K}_1]\bar{\mathbf{e}}_c - \bar{\mathbf{B}}_c\\
&\quad (sat(\mathbf{w}_{lg} + \mathbf{w}_{hg}) - \mathbf{K}_1\bar{\mathbf{e}}_c) - \Delta\bar{\mathbf{B}}_c\mathbf{K}_1\bar{\mathbf{e}}_c + \bar{\mathbf{B}}_c\Delta\mathbf{\Phi}^*\mathbf{S}\bar{\mathbf{e}}_c\\
&\triangleq [(\bar{\mathbf{A}}_{dc} - \bar{\mathbf{B}}_c\Delta\mathbf{\Phi}^*\mathbf{S}) - (\bar{\mathbf{B}}_c - \Delta\bar{\mathbf{B}}_c)\mathbf{K}_1]\bar{\mathbf{e}}_c - \bar{\mathbf{B}}_c\mathbf{v}, \quad (30)
\end{aligned}
$$

where $\mathbf{v} = sat(\mathbf{w}_{lg} + \mathbf{w}_{hg}) - \mathbf{K}\bar{\mathbf{e}}_c$. Substituting (30) into (29) and considering the relationship of (21), we have

$$\dot{V} \leq -\bar{\mathbf{e}}_c^T\mathbf{Q}\bar{\mathbf{e}}_c - 2\sum_{j=1}^{m}\bar{\mathbf{e}}_c^T\mathbf{W}\bar{\mathbf{b}}_{c,j}v_j \tag{31}$$

where $v_j = sat(w_{lg,j} + w_{hg,j}) - \mathbf{k}_j\bar{\mathbf{e}}_c$. $\forall j \in M$, let us discuss the term $\bar{\mathbf{e}}_c^T\mathbf{W}\bar{\mathbf{b}}_{c,j}v_j$ in (31).

(I). $a_{j,lower} < w_{lg,j} + w_{hg,j} < a_{j,upper}$

$$
\begin{aligned}
\bar{\mathbf{e}}_c^T\mathbf{W}\bar{\mathbf{b}}_{c,j}v_j &= \bar{\mathbf{e}}_c^T\mathbf{W}\bar{\mathbf{b}}_{c,j}(\mathbf{k}_j\bar{\mathbf{e}}_c + k_{hg}\bar{\mathbf{b}}_{c,j}^T\mathbf{W}\bar{\mathbf{e}}_c - \mathbf{k}_{lg,j}\bar{\mathbf{e}}_c)\\
&= k_{hg}(\bar{\mathbf{e}}_c^T\mathbf{W}\bar{\mathbf{b}}_{c,j})^2 \geq 0. \qquad (32)
\end{aligned}
$$

(II). $w_{lg,j} + w_{hg,j} \leq a_{j,lower}$

$$
\begin{aligned}
\bar{\mathbf{e}}_c^T\mathbf{W}\bar{\mathbf{b}}_{c,j}v_j &= \bar{\mathbf{e}}_c^T\mathbf{W}\bar{\mathbf{b}}_{c,j}(\boldsymbol{\phi}_{d,j}\mathbf{S}\bar{\mathbf{e}}_c - w_{r,j} - \mathbf{k}_j\bar{\mathbf{e}}_c)\\
&= \bar{\mathbf{e}}_c^T\mathbf{W}\bar{\mathbf{b}}_{c,j}(-w_{r,j} - \mathbf{k}'_j\bar{\mathbf{e}}_c). \qquad (33)
\end{aligned}
$$

From the definitions of $c_{1,j}$ and $\mathbf{w}_r$, $c_{1,j} \leq w_{r,j}$. Hence,

$$\mathbf{k}'_j\bar{\mathbf{e}}_c \geq -c_{1,j} \geq -w_{r,j} \Rightarrow -w_{r,j} - \mathbf{k}'_j\bar{\mathbf{e}}_c \leq 0. \tag{34}$$

On the other hand, $w_{lg,j} + w_{hg,j} \leq a_{j,lower}$ leads to

$$k_{hg}\bar{\mathbf{b}}_{c,j}^T\mathbf{W}\bar{\mathbf{e}}_c \leq -w_{r,j} - \mathbf{k}'_j\bar{\mathbf{e}}_c. \tag{35}$$

From (34) and (35), $\bar{\mathbf{b}}_{c,j}^T\mathbf{W}\bar{\mathbf{e}}_c < 0$. Hence, according to (33) and (34), $\bar{\mathbf{e}}_c^T\mathbf{W}\bar{\mathbf{b}}_{c,j}v_j > 0$ can be derived.

(III). $w_{lg,j} + w_{hg,j} \geq a_{j,upper}$

Similar to *(II)*, $\bar{\mathbf{e}}_c^T\mathbf{W}\bar{\mathbf{b}}_{c,j}v_j > 0$ can be ensured.

Therefore, $\dot{V} \leq -\bar{\mathbf{e}}_c^T\mathbf{Q}\bar{\mathbf{e}}_c$ and $\epsilon(\mathbf{W}, \rho)$ is an invariant region. $\bar{\mathbf{e}}_c(0) \in \epsilon(\mathbf{W}, \rho)$ ensures $\bar{\mathbf{e}}_c(\tau) \in \epsilon(\mathbf{W}, \rho)$ ($\tau \geq 0$). Thus, $\bar{\mathbf{e}}_c$, $\bar{\mathbf{e}}$ and $\mathbf{e}$ converge to zero. The convergence of $\bar{\mathbf{e}}_c$ leads to the convergence of $\mathbf{w}$ and $\mathbf{u}$ to $\mathbf{w}_r$ and $\mathbf{u}_d$.

Case II. $\delta = 0$

The convergence analysis is similar to **Case I**. The only difference is that (22) is considered instead of (21). ■

Given $\mathbf{m}_0$, $\mathbf{m}_d$ and $\mathbf{u}_d$. The design steps are:

Step 1. According to (12), calculate the value of β.

Step 2. Calculate $\mathbf{c}_1$ and $\mathbf{c}_{2l}$. Find δ and γ to satisfy (24) and (25). If $\delta > 0$, δ and γ can be obtained from (28).

Step 4. $\mathbf{w}_r$ is calculated according to (11).

Step 5. Design $\mathbf{w}_{lg}$ and $\mathbf{w}_{hg}$ as in Subsections 5.2 and 5.4.

6. ILLUSTRATIVE EXAMPLE

For *Example 1*, to maximize the flows of the steady state, $\mathbf{u}_d = [0, 0.5, 4.5]^T$. The solutions of (12) are $\boldsymbol{\sigma} = [2, 0, 1]^T$ and $\beta = \frac{3}{4}$. Calculate $\mathbf{c}_1$ and $\mathbf{c}_{2l}$, it can be found that this design belongs to **Case 1** and $(\bar{\mathbf{A}}_{dc} - \bar{\mathbf{B}}_c\Delta\mathbf{\Phi}_d\mathbf{S}, \bar{\mathbf{B}}_c - \Delta\bar{\mathbf{B}}_c)$ is stabilizable. Let $\mathbf{Q} = \mathbf{I}_{2\times2}$ and $\mathbf{R} = \mathbf{I}_{3\times3}$. Solving (26), $\delta = 0.2126$ and $\gamma = 0.3487$. Then, $h = 1.0499s$. Choose $k_{hg} = 6$. Figure 2 and 3 show the convergence of markings and control signals respectively.

7. TRACKING REFERENCE PLANNING AND CONTROL LAW DESIGN

In Section 4, the new reference design mainly depends on $\mathbf{m}_0$ and $\mathbf{m}_d$, and hence the ramp part directly goes from

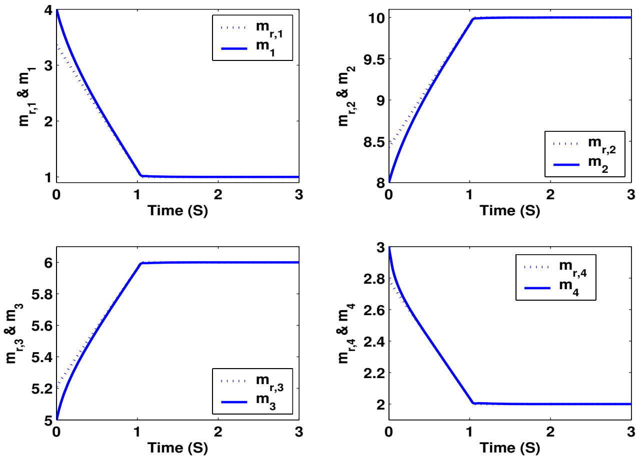

Fig. 2. Convergence of markings.

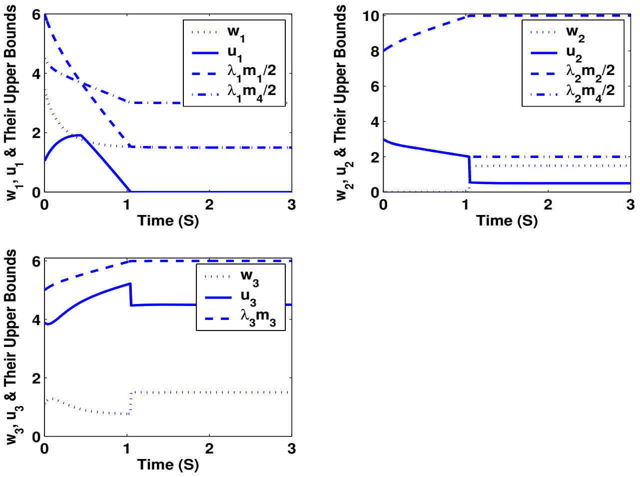

Fig. 3. Control signals.

$\mathbf{m}_{r0}$ to $\mathbf{m}_d$. This ramp limits the response speed. In order to improve system transient performance, intermediate states can be added to the tracking reference. Denote the intermediate states as $\mathbf{m}_{int,q}$, where $q \in \Omega$. $\mathbf{m}_{int,q}$ is designed such that the time it takes from $\mathbf{m}_0 \to \mathbf{m}_{int,1} \to \cdots \to \mathbf{m}_{int,\omega} \to \mathbf{m}_d$ is less than the time it takes from $\mathbf{m}_0 \to \mathbf{m}_d$ directly. The control algorithm proposed in Section 5 will be implemented to track the intermediate states consequently and the controller parameters are denoted with subscript q. It means $\mathbf{m}_{int,1}$ will be the first tracking target and a controller can be constructed accordingly. When the error between $\mathbf{m}$ and $\mathbf{m}_{int,1}$ is small enough, $\mathbf{m}_{int,2}$ will be applied as the second tracking target. Step by step, $\mathbf{m}_d$ will the final tracking target.

By employing a larger ω, more intermediate states can be introduced, which leads to faster system response. However, as one intermediate state results in one discontinuous point in the control signal, a larger ω implies more discontinuous points. Hence, there is a tradeoff between system settling time and the smoothness of the control signal. Let $\delta_q = 0$ $(q = 1, \cdots, \omega+1)$, the time that it takes from $\mathbf{m}_0 \to \mathbf{m}_d$ is $h = \sum_{q=1}^{\omega+1} \frac{1}{\beta_q}$. Based on this h, δ_q $(q = 1, \cdots, \omega+1)$ can be designed to further fastern the system response. Therefore, to determine ω, calculate $\sum_{q=1}^{\omega+1} \frac{1}{\beta_q}$ with different ω as follows:

Table 1. Control with Intermediate States.

ω	$\mathbf{m}_{int,q}$	δ_q	γ_q	t_{settle}
0	*None*	$\delta_1 = 0.05$	$\gamma = 1$	$4.4s$
1	$\mathbf{m}_{int,1} = [2.7, 8.9, 5.4, 2.7]^T$	$\delta_1 = 0$ $\delta_2 = 0.31$	$\gamma_1 = 1$ $\gamma_2 = 1$	$2.2s$
2	$\mathbf{m}_{int,1} = [7.6, 5.9, 3.5, 5.4]^T$ $\mathbf{m}_{int,2} = [3.0, 9.0, 5.0, 4.0]^T$	$\delta_1 = 0.17$ $\delta_2 = 0.19$ $\delta_3 = 0.07$	$\gamma_1 = 0.9$ $\gamma_2 = 10$ $\gamma_3 = 9$	$1.6s$

$$\min \sum_{q=1}^{\omega+1} \frac{1}{\beta_q} \tag{36}$$

$$\begin{aligned} s.t.: \mathbf{m}_{int,1} &= \mathbf{m}_0 + \mathbf{B}\boldsymbol{\sigma}_1 \\ \mathbf{m}_{int,q} &= \mathbf{m}_{int,q-1} + \mathbf{B}\boldsymbol{\sigma}_q \quad (q = 2, \cdots, \omega) \\ \mathbf{m}_d &= \mathbf{m}_{int,\omega} + \mathbf{B}\boldsymbol{\sigma}_{\omega+1} \\ \mathbf{0} &\le \beta_1\boldsymbol{\sigma}_1 \le \min\{\boldsymbol{\Phi}_0\mathbf{m}_0, \boldsymbol{\Phi}(\mathbf{m}_{int,1})\mathbf{m}_{int,1}\} \\ \mathbf{0} &\le \beta_q\boldsymbol{\sigma}_q \le \min\{\boldsymbol{\Phi}(\mathbf{m}_{int,q-1})\mathbf{m}_{int,q-1}, \\ &\qquad \boldsymbol{\Phi}(\mathbf{m}_{int,q})\mathbf{m}_{int,q}\} \ (q = 2, \cdots, \omega) \\ \mathbf{0} &\le \beta_{\omega+1}\boldsymbol{\sigma}_{\omega+1} \le \min\{\boldsymbol{\Phi}(\mathbf{m}_{int,\omega}), \boldsymbol{\Phi}_d\mathbf{m}_d\} \end{aligned}$$

Then, considering system performance requirments, an approporiate ω can be chosen.

Based on Theorem 1, the global convergence of $\mathbf{m}$ and $\mathbf{u}$ can be derived. However, the details will not be given here.

7.1 Illustrative examples

Consider Fig. 1 again with $\mathbf{m}_0 = [13, 3, 1, 10]^T$, but same $\mathbf{Q}$, $\mathbf{R}$ and k_{hg}. Let $\omega = 0, 1$ and 2. The controller parameters and control performance are given in Table 1. As ω increases, the settling time (t_{settle} in Table 1) becomes smaller. However, the control signals are less smooth.

8. CONCLUSION

The step-tracking control for general timed contPN systems under infinite server semantics has been studied. The proposed control approach can guarantee the global tracking convergence in presence of the switched system dynamics and the special input constraints. By introducing intermediate states, a trajectory planning algorithm has been given to further improve the transient performance.

REFERENCES

Mahulea, C., A. Ramírez, L. Recalde, and M. Silva Steady state control, zero valued poles and token conservation laws in continuous net systems. In: *Proceedings of the International Workshop on Control of Hybrid and Discrete Event Systems*, Miami, USA, 2005.

Silva, M. and L. Recalde. Petri Nets and integrality relaxations a view of continuous Petri Nets, volum 32, number 4 of *IEEE Trans on Systems Man and Cybernetics*, pages 314–327, 2002.

Wredenhage, G. F. and P. R. Bélanger. Piecewise-linear LQ control for systems with input constraints. *Automatica* volume 30, number 3, pages 403–416.

Xu, J., L. Recalde, and M. Silva. Tracking control of join–free timed continuous petri net systems. In *2nd IFAC Conference on Analysis and Design of Hybrid Systems*, Alghero, Italy, 2006.