Proceedings of the 17th World Congress
The International Federation of Automatic Control
Seoul, Korea, July 6-11, 2008

GPC Control of a Fractional–Order Plant: Improving Stability and Robustness

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Abstract: This work deals with the use of Generalized Predictive Control (GPC) with fractional order plants. Low integer–order discrete approximations will be used as models to design the controllers. The stability and robustness of the closed loop system will be studied with the Nyquist criterion. Three techniques will be proposed to enhance robustness: the improvement of the model response at low frequencies, the use of the prefilter $T(z^{-1})$, and a new recommendation to choose two of the parameters (the control horizon N_u and the error weighting sequence λ) of the GPC controller.

1. INTRODUCTION

Fractional Calculus can be defined as integration and differentiation of noninteger order. Fractional differentiation (integration) is the generalization of the derivative (integral) operator D^n (D^{-n}) using real or even complex values for the ordinary integer value n (Oldham and Spanier, 1974; Podlubny, 1999a).

Fractional integro–differential calculus generally uses two definitions: (1) Grünwald–Letnikov (GL) and (2) Riemann–Liouville (RL):

$$D^{\alpha} f(t)_{t=kh} = \lim_{h\to 0} h^{-\alpha} \sum_{j=0}^{k} (-1)^j \binom{\alpha}{j} f\left(kh - jh\right) \qquad (1)$$

$$D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-\tau)^{n-\alpha-1} f(\tau) d\tau \qquad (2)$$

with $\alpha > 0$ for derivation and $\alpha < 0$ for integration.

The Laplace domain is frequently used to describe the fractional operations. Expression (3) is given as Laplace transform of the Riemann–Liouville derivative/integral (2) under zero initial conditions (Oldham and Spanier, 1974):

$$L\left\{D^{\pm\alpha} f(t)\right\} = s^{\pm\alpha} F(s) \qquad (3)$$

For a wide class of functions, which appear in real physical and engineering applications, the two definitions GL and RL are equivalent. For this reason, RL is usually used for algebraic manipulations and GL (together with the short memory principle), for numerical integration and simulation (Podlubny, 1999a).

Fractional order controllers have been used to enhance system performance. Typical fractional order controllers include the CRONE control (Oustaloup, *et al.*, 1995) and the $PI^{\lambda}D^{\mu}$ controller (Petráš, 1999; Podlubny, 1999b). More control applications are described in (Vinagre and Chen, 2002; Oustaloup, 2006).

Model–Based Predictive Control (MPC) has been proposed to control plants with fractional dynamics (Romero, *et al.,* 2007). Predictive control has been in use in the process industries during the last 30 years, where it has become an industry standard due to its intrinsic ability to handle input and state constraints for large scale multivariable plants (Maciejowski, 2002; Rossiter, 2003; Camacho and Bordóns, 2004).

In this paper low integer–order discrete approximations will be used as models to design the controllers, so a model–process mismatch will appear. For this reason the stability and robustness of a fractional order plant with a predictive control law will be studied and some methods to improve them will be proposed.

This paper is organized as follows: In section 2 GPC, one of the most representative predictive controllers, is introduced. Section 3 describes how to study the stability and robustness of a GPC control loop with a fractional order plant. In section 4 this study is illustrated with some examples. In section 5 some techniques to improve the robustness are proposed. Finally, section 6 draws the main conclusions of this work.

2. GENERALIZED PREDICTIVE CONTROL

GPC stands for Generalized Predictive Control (Clarke, *et al.*, 1987a, 1987b), one of the most representative predictive controllers due to its success in industrial and academic applications (Clarke, 1988).

All predictive controllers share a common methodology: at each "present" instant t, future process outputs $y(t+k|t)$ are predicted for a certain time window, $k = 1, 2, \ldots, N$, using the process model. The optimal control law is obtained by minimizing a given cost function (4) subject to a set of constraints:

$$J(\Delta u,t) = E\left\{ \sum_{j=N_1}^{N_2} \gamma(j)\left[r(t+j\,|\,t) - y(t+j\,|\,t)\right]^2 + \sum_{j=1}^{N_u} \lambda(j)\left[\Delta u(t+j-1\,|\,t)\right]^2 \right\} \quad (4)$$
$$\textit{Subject to}: \quad H\Delta u \le h$$

where $E\{\cdot\}$ is the expectation operator, Δ is the increment operator, N_1 and N_2 are the minimum and maximum costing horizons, respectively, N_u represents the control horizon, γ is a future error weighting sequence, λ is a control weighting sequence, and H and h are a matrix and a vector, respectively. In the minimization process, it is usually assumed that the control signal $u(t)$ remains constant from time instant $t + N_u$ (Maciejowski, 2002; Rossiter, 2003; Camacho and Bordóns, 2004).

GPC uses CARIMA (Controlled Auto–Regressive Integrated Moving–Average) models to describe the system dynamics:

$$A(z^{-1})y(t) = B(z^{-1})u(t) + \frac{T(z^{-1})}{\Delta}\xi(t) \quad (5)$$

where $B(z^{-1})$ and $A(z^{-1})$ are the numerator and denominator of the transfer function, respectively, and $\xi(t)$ represents uncorrelated zero–mean white noise. In practice, $T(z^{-1})$ is not considered a model parameter but a controller parameter, *i.e.* a (pre)filter that is chosen to improve the system robustness rejecting disturbance and noise.

If constraints are not defined, the minimization of (4) leads to a linear time invariant (LTI) control law that can be pre–computed in advance.

Figure 1 shows equivalent control loop, where R and S are constant polynomials obtained from the model polynomials A and B and the controller parameters N_1, N_2, N_u, γ, and λ (Clarke, *et al.*, 1987a). (The actual plant polynomials A_0 and B_0 are generally different from the model polynomials A and B used to define the GPC controller, as no mathematical model can represent a physical system perfectly.)

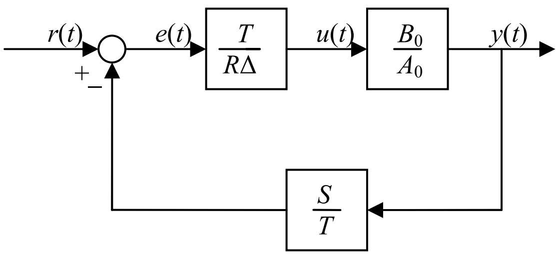

Fig. 1. Closed loop schema.

The task of finding the parameters N_1, N_2, N_u, γ, and λ is critical, as they determine the closed loop stability. However, thumb–rules exist that help the user to find initial guesses of their values quickly. It is usually accepted that $N_1 = 1$, $N_2 = 10$, $\lambda = 10^{-6}$, $\gamma = 1$, and N_u equal to the number unstable or badly-damped poles of the system are adequate for a wide range of applications (Clarke, *et al.*, 1987a).

3. STABILITY AND ROBUSTNESS

The denominator of the transfer function of a fractional system

$$H(s) = a_n s^{\alpha_n} + a_{n-1}s^{\alpha_{n-1}} + \ldots + a_0 s^{\alpha_0}, \quad \alpha_i \in \mathbb{R}^+ \quad (6)$$

is a multievaluated function of the complex variable s. Its domain is defined by a Riemann surface that has an infinite number of sheets in the general case. The main sheet is defined by $-\pi < \arg(s) < \pi$.

$H(s) = 0$ has an infinite number of roots, but only a finite number of them will be in the main sheet of the Riemann surface and the system stability depends only on them (Vinagre, *et al.*, 2002). Moreover, the discretization of fractional order plants has an infinite dimension (Vinagre, 2001), and there are no polynomial techniques (such as Routh or Jury) to analyse the stability of fractional systems. For this reason, in the following the Nyquist criterion will be used to study the stability of the control loop depicted in Figure 1, as A_0 and B_0 define a fractional order plant.

The Nyquist criterion applies the argument principle to the contour generally known as the Nyquist path. It states that, for discrete systems represented in the z–plane,

$$Z = N + P \quad (7)$$

where Z is the number of unstable poles in closed loop, P is the number of unstable poles of the open loop transfer function, and N is the number of clockwise encirclements of the point –1 for a contour evaluation of the open loop transfer function.

4. STABILITY ANALYSIS OF A FRACTIONAL PLANT CONTROLLED BY GPC

In the following we shall consider the control of the fractional order integrator of order 0.4 and gain $k = 1$:

$$G(s) = \frac{k}{s^{0.4}} \quad (8)$$

Firstly, a discrete approximation of the fractional order plant (8) is needed to design the GPC controller. To do so many techniques exist (Vinagre, *et al.*, 2000; Vinagre, 2001; Chen, *et al.*, 2003; Dorcák, 2003). In this paper, two methods based on Chebyshev polynomials will be used due to their accuracy: the so–called Chebyshev–Padé –CP– (de Madrid, *et al.,* 2006) and Rational Chebyshev –RC– (Romero, *et al.,* 2006) approximations.

Expressions (9) and (10) show the third order CP and RC approximations, respectively, with sampling time T_s equal to 0.1 seconds. (The backward rule has been used as generating function $\omega(z^{-1})$). Both of them fulfill the following conditions: be rational functions, be stable, be minimum phase and have a zero–pole interlacing along $z \in (-1\ 1)$ (Vinagre, 2001; Chen, *et al.,* 2003; Valério, 2005). Figure 2 shows the step response of both approximations compared with the actual fractional plant.

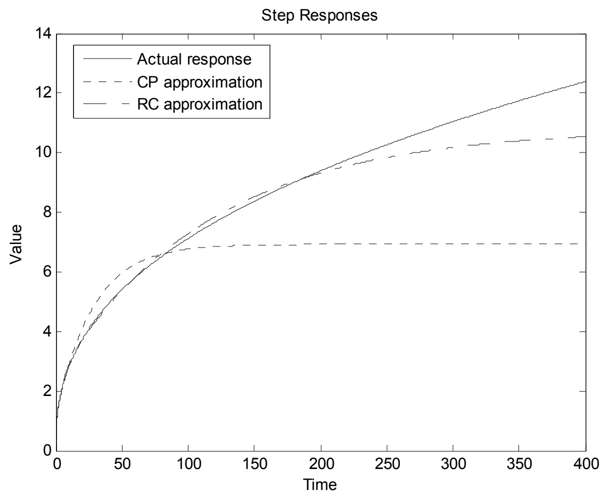

Fig. 2. Step responses of the actual fractional order integrator and the Chebyshev based approximations.

$$G_{CP}(z^{-1}) = \frac{0.226352 - 0.411845z^{-1} + 0.206593z^{-2} - 0.019978z^{-3}}{0.568564 - 1.261632z^{-1} + 0.864601z^{-2} - 0.171371z^{-3}} \quad (9)$$

$$G_{RC}(z^{-1}) = \frac{0.395274 - 0.897793z^{-1} + 0.626306z^{-2} - 0.123733z^{-3}}{1 - 2.679155z^{-1} + 2.364368z^{-2} - 0.685208z^{-3}} \quad (10)$$

Now, GPC controllers will be calculated using the previous approximations as models and the default settings. Both CP and RC approximations are stable in the following $N_u = 1$, if not stated otherwise.

It is well known that the prefilter T can improve the system robustness against the model–process mismatch. In (Yoon and Clarke, 1995) some guidelines about how to chose T are given. However, for these initial simulations $T(z^{-1}) = 1$.

The actual response of the fractional order plant will be computed with the GL short memory approximation (1), with $k = 100$. A unit step at $t = 0$ will be used as the reference $r(t)$ in the simulations.

The performance of the control system obtained using the CP approximation as a model and the default settings is illustrated in Figure 3. The system output seems smooth and stable. Closed loop equations (see Figure 1) are as follows:

$$\begin{aligned}
sys_2 &= \frac{S}{T} = \frac{34.097475 - 68.181907z^{-1} + 43.303567z^{-2} - 8.219135z^{-3}}{1} \\
sys_3 &= \frac{T}{R\Delta} = \frac{1}{0.889948 - 13.506336z^{-1} + 21.780722z^{-2} - 10.120500z^{-3} + 0.956166z^{-4}} \quad (11) \\
sys_1 &= \frac{k}{s^{0.4}} = \frac{k}{\omega(z^{-1})^{0.4}} = \frac{k}{\left[\frac{1}{T_s}(1 - z^{-1})\right]^{0.4}}
\end{aligned}$$

In order to check the stability of this control system, the Nyquist criterion will be used. The open loop transfer function is $sys_1 \cdot sys_2 \cdot sys_3$,[1] which has an unstable pole at $z = 13.4148$, a double pole at $z = 1$ and therefore $P = 2$. Figure 4 shows the corresponding Nyquist plot, where the contour encircles the point $-1+0j$ counterclockwise twice, *i.e.* $N = -2$. Thus $Z = 0$. The Nyquist criterion establishes that this system is stable in closed loop, with gain margin $k \in$ (0.9579 1.0183).

[1] sys_1 has a structural root in $z = 1$.

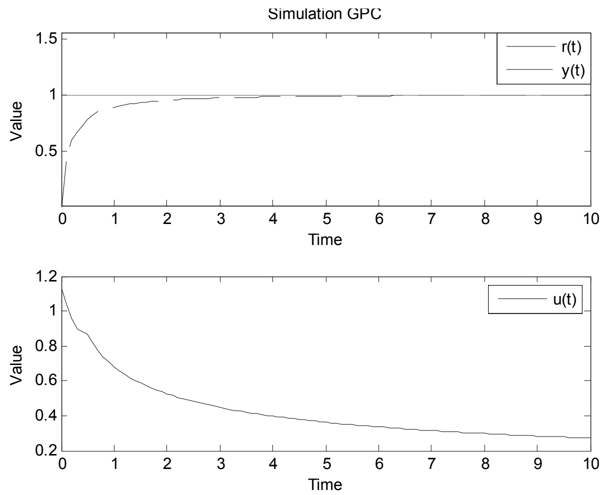

Fig. 3. GPC response with model = CP and default settings.

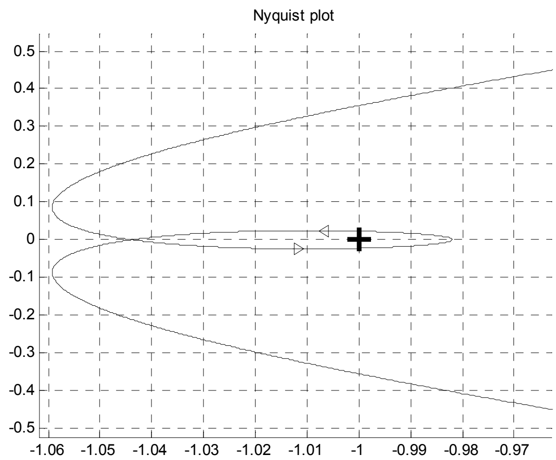

Fig. 4. Nyquist plot using the CP approximation as model.

However, if the RC approximation is used as the system model the output becomes unstable. In this case, the open loop transfer function is given by (12). It has an unstable pole at $z = 28.1224$, a double pole at $z = 1$ and therefore $P = 2$.

$$\begin{aligned}
sys_2 &= \frac{S}{T} = \frac{67.955465 - 163.108264z^{-1} + 131.592300z^{-2} - 35.439501z^{-3}}{1} \\
sys_3 &= \frac{T}{R\Delta} = \frac{1}{0.896252 - 26.795719z^{-1} + 53.573269z^{-2} - 34.073371z^{-3} + 6.399568z^{-4}} \quad (12) \\
sys_1 &= \frac{k}{s^{0.4}} = \frac{k}{\omega(z^{-1})^{0.4}} = \frac{k}{\left[\frac{1}{T_s}(1 - z^{-1})\right]^{0.4}}
\end{aligned}$$

The contour (Figure 5) does not encircle the point $-1+0j$: $N = 0$. Thus $Z = 2$, *i.e.*, the system is unstable in closed loop.

5. ROBUSTNESS IMPROVEMENT

In this section, three techniques will be proposed to improve the system robustness against the model–process mismatch induced by the discrete models used to compute the GPC controller. Only the RC approximation will be considered, as it led to an unstable control loop.

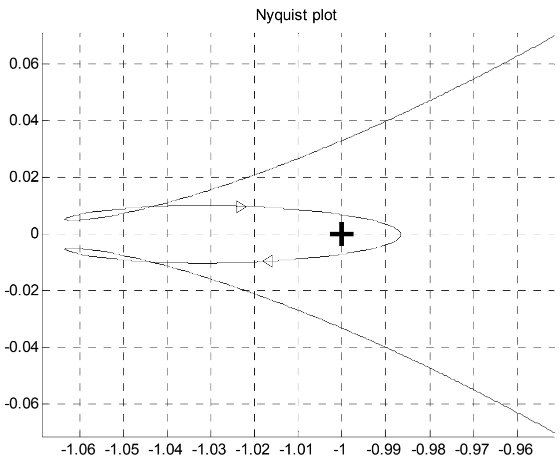

Fig. 5. Nyquist plot using the RC approximation as model.

5.1 Improvement of the model response at low frequencies

Both CP and RC approximations lack precision when low frequencies are considered. In order to solve this, the fractional order integrator of order 0.4 (8) can be expressed as a conventional integer–order integrator multiplied by a fractional derivator of order 0.6:

$$sys_1 = \frac{1}{s^{0.4}} = \frac{1}{s} s^{0.6} . \tag{13}$$

The term $s^{0.6}$ is approximated using RC,

$$s^{0.6} \approx \frac{3.980108 - 9.532780z^{-1} + 7.283315z^{-2} - 1.730421z^{-3}}{1 - 1.794500z^{-1} + 0.869685z^{-2} - 0.069682z^{-3}} \tag{14}$$

On the other hand, the integer–order integrator is given by

$$\frac{1}{s} = \frac{T_s}{1 - z^{-1}} . \tag{15}$$

Hence the new RC approximation of the fractional order integrator is

$$\frac{1}{s^{0.4}} \approx \frac{0.398010 - 0.953278z^{-1} + 0.728331z^{-2} - 0.173042z^{-3}}{1 - 2.794500z^{-1} + 2.664185z^{-2} - 0.939367z^{-3} + 0.069682z^{-4}} \tag{16}$$

Figure 6 shows the frequency response comparison between the initial (10) and the new (16) RC approximations. Both of them are quite similar at high frequencies but only the new one keeps the integration effect at the low ones.

The GPC controller using the new RC approximation as model and default settings is now given by:

$$\begin{aligned} sys_2 &= \frac{S}{T} = \frac{75.348084 - 1.892166z^{-1} + 1.656154z^{-2} - 0.547281z^{-3}}{1} \\ sys_3 &= \frac{T}{R\Delta} = \frac{1}{0.884846 - 29.923045z^{-1} + 63.390326z^{-2} - 44.238901z^{-3} + 9.886773z^{-4}} \\ sys_1 &= \frac{k}{s} s^{0.6} = \frac{k}{\omega(z^{-1})} \omega(z^{-1})^{0.6} = \frac{k}{\frac{1}{T_s}(1 - z^{-1})} \left[\frac{1}{T_s}(1 - z^{-1}) \right]^{0.6} \end{aligned} \tag{17}$$

In this case, the open loop transfer function has an unstable pole at z = 31.599830, a double pole at z = 1 and thus P = 2. The corresponding Nyquist plot is shown in Figure 7, with $N = -2$ and therefore $Z = 0$: the system is stable in closed loop. The gain margin is $k \in$ (0.991 1.005). Figure 8 shows a simulation using the new GPC controller (17).

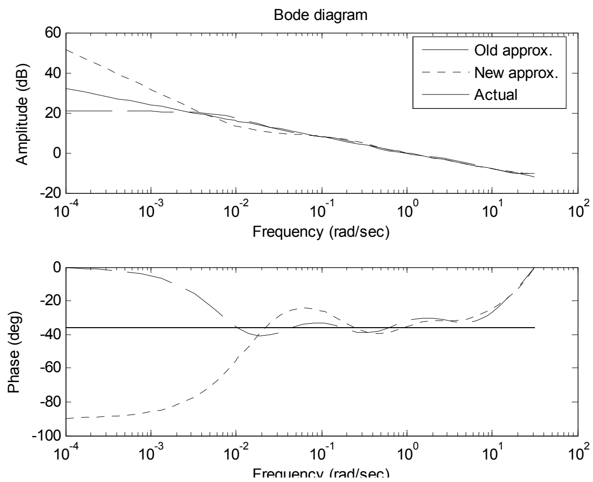

Fig. 6. Frequency response comparison between the initial and the new –with improved response at low frequencies– RC approximations.

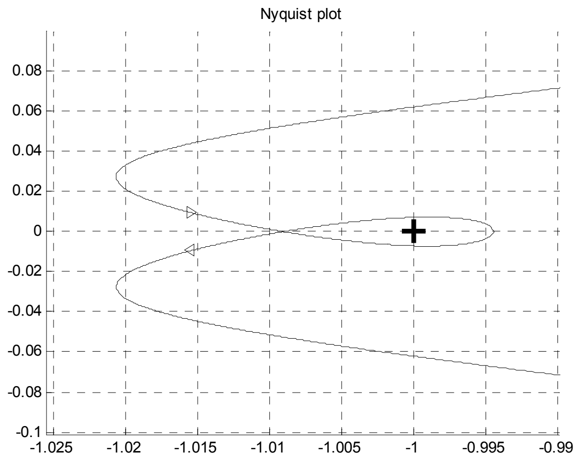

Fig. 7. Nyquist plot using the new RC approximation –with improved response at low frequencies– as model.

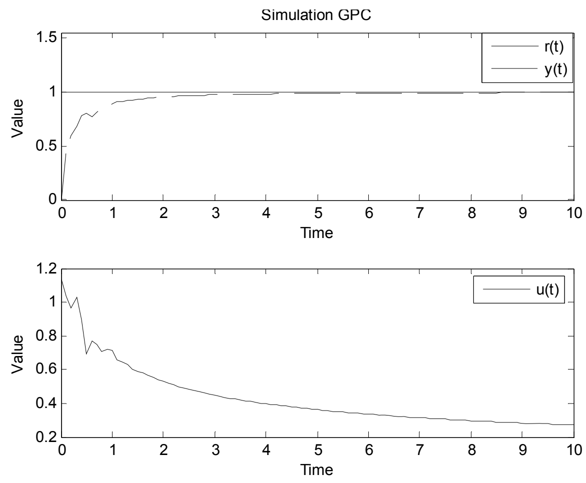

Fig. 8. GPC response using the new RC approximation –with improved response at low frequencies– as model.

5.2 Use of prefilter $T(z^{-1})$

With the initial RC approximation (10) as model, now we shall use default settings and a simple prefilter $T = A$ (the denominator of the approximation). The following expressions give the new GPC control loop:

$$
\begin{aligned}
sys_2 &= \frac{S}{T} = \frac{1-2.679155z^{-1}+2.364368z^{-2}-0.685208z^{-3}}{1-2.679155z^{-1}+2.364368z^{-2}-0.685208z^{-3}} \\
sys_3 &= \frac{T}{R\Delta} = \frac{1-2.679155z^{-1}+2.364368z^{-2}-0.685208z^{-3}}{0.896252-2.731163z^{-1}+2.897622z^{-2}-1.186444z^{-3}+0.123733z^{-4}} \\
sys_1 &= \frac{k}{s^{0.4}} = \frac{k}{\omega(z^{-1})^{0.4}} = \frac{k}{\left[\frac{1}{T_s}(1-z^{-1})\right]^{0.4}}
\end{aligned}
\tag{18}
$$

The system has a double pole at $z = 1$ in open loop and therefore $P = 1$. Figure 9 shows the corresponding Nyquist plot, where $N = -1$ and thus $Z = 0$. The system is stable in closed loop, with a gain margin $k \in (0\ 3.8595)$. Finally, figure 10 shows the response of this controller.

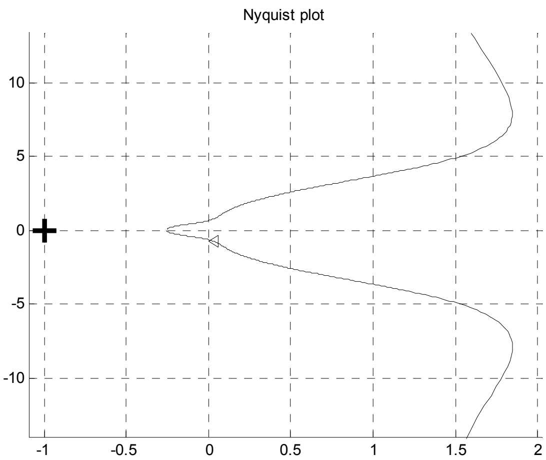

Fig. 9. Nyquist plot using the initial RC approximation and the prefilter $T = A$.

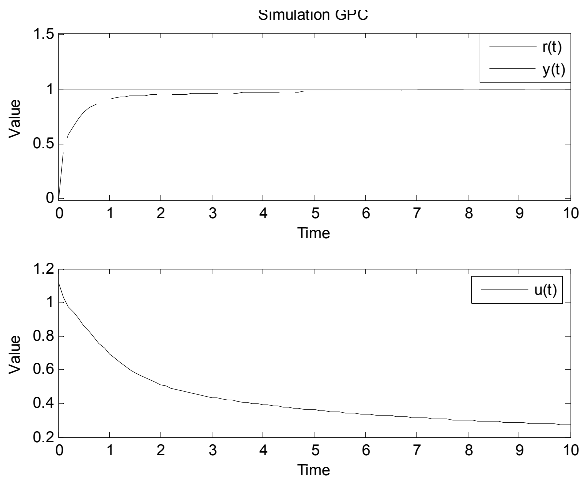

Fig. 10. GPC response using the initial RC approximation and the prefilter $T = A$.

5.3 New N_u and λ settings

In the case of the initial RC approximation, the guideline given in (Clarke, *et al.*, 1987a) has led to instability. For this reason we propose a different recommendation for choosing the control horizon N_u and the error weighting sequence λ:

- Take N_u equal to the order of the transfer function used as a model, even in the case that it is stable. Increasing the value of N_u gives rise to a tighter control action (Clarke, *et al.*, 1987a, 1987b).
- Take λ equal to the binomial coefficients that appear, in a natural way, in the Grünwald–Letnikov approach:

$$\lambda(j) = \left| \begin{pmatrix} \alpha \\ N_u - j \end{pmatrix} \right|, \quad j = 1, 2, \ldots, N_u \tag{19}$$

  In this way, we get a sequence $\lambda(k) < \lambda(k+1)$ that gives rise to smooth control (Camacho and Bordóns, 2004) that counteracts the tight action due to N_u (on the other hand, a sequence $\lambda(k) > \lambda(k+1)$ would produce tighter controls).

Following this new recommendation, we shall use $N_u = 3$, $\lambda(1) = 0.12$, $\lambda(2) = 0.4$, and $\lambda(3) = 1$, which produce the following GPC control loop:

$$
\begin{aligned}
sys_2 &= \frac{S}{T} = \frac{15.465331-32.810550z^{-1}+24.690691z^{-2}-6.345472z^{-3}}{1} \\
sys_3 &= \frac{T}{R\Delta} = \frac{1}{0.701164-6.114224z^{-1}+10.571944z^{-2}-6.304733z^{-3}+1.145848z^{-4}} \\
sys_1 &= \frac{k}{s^{0.4}} = \frac{k}{\omega(z^{-1})^{0.4}} = \frac{k}{\left[\frac{1}{T_s}(1-z^{-1})\right]^{0.4}}
\end{aligned}
\tag{20}
$$

Figure 11 shows the step response of this control system, the system output $y(t)$ is stable and smooth with gain margin $k \in (0.899\ \ 1.037)$. Finally, Figure 12 depicts the corresponding Nyquist plot.

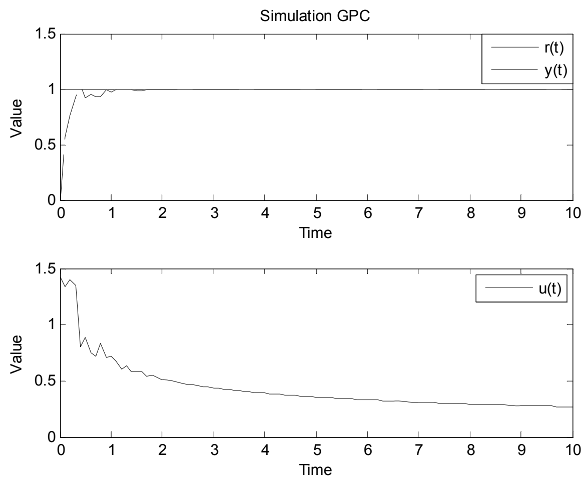

Fig. 11. GPC performance with the initial RC approximation and the new N_u and λ settings.

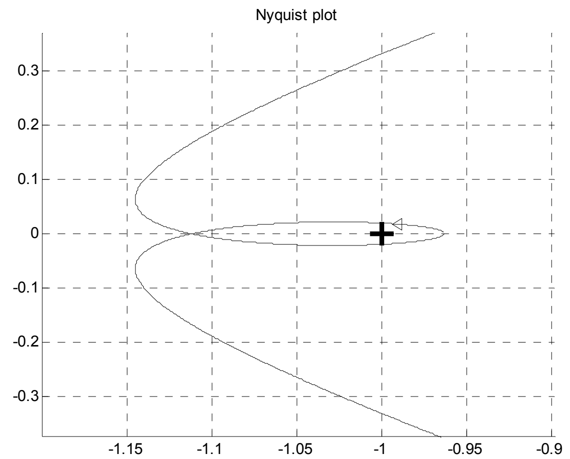

Fig. 12. Nyquist plot with the initial RC approximation and the new N_u and λ settings.

6. CONCLUSIONS

This paper has focused on the stability of fractional systems controlled by GPC. The use of low integer–order discrete approximations as models to design the controllers has led to an intrinsic model–process mismatch. It has been shown that, even with the same GPC tuning, different approximations can lead to very different closed loop behaviours: an accurate approximation is not a sufficient condition to guarantee the stability. For these reasons, the study of the robust stability of the system is needed. As there are no polynomial techniques (similar to Routh or Jury) to do it, the use of Nyquist techniques has been proposed.

Finally, three techniques have been suggested in order to improve the robust stability: an improvement at low frequencies of the approximated model itself, the use of a prefilter, and a new guideline to choose two of the controller parameters.

To sum up, the predictive controller GPC has proved to be a versatile and valuable tool to deal with fractional order plants.

ACKNOWLEDGMENTS

The authors wish to acknowledge the economical support of the Spanish *Ministerio de Educación y Ciencia*, under projects DPI2004–05903 and DPI2005–07980–C03–03.

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