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# Representations With Constant System Matrices of Linear Time-Periodic Dynamical Systems

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**Abstract:** In the paper it is investigated, under which conditions a time variable state transformation can be used to change a periodic autonomous system realization into a form with constant real coefficients. Conditions for the transformation to be periodic are further considered, and it is shown that the necessary and sufficient condition to meet the conditions is in that two specific matrices must be similar to each other. The properties of the matrices and the transformation are studied, and the discussion is then extended to input-output systems. An approach to design stabilizing control laws for these kinds of periodic systems is outlined.

Keywords: Time-varying systems; Periodic; Transformation matrices; Lyapunov stability; Controller; Differential equation.

## 1. INTRODUCTION

The analysis and control of linear time-variable periodic systems have been discussed extensively in the literature because of their importance in many practical engineering problems. The key ideas are expressed by the *Floquet-Lyapunov theory* (Montagnier et al. (2004), Deshmukh and Sinha (2004)) which compares the solutions of the original periodic system to those of a structurally simpler system. The main idea is to change the periodic system matrix into a constant matrix by a state transformation; a procedure which makes analysis of the original system simpler as well (Rugh (1993)). It is well-known that a periodic system matrix is always *kinematically similar* or *reducible* to a constant system matrix (Harris and Miles (1980)), (Brockett (1970)), but a deeper analysis of the transformation is usually lacking. For example, it is often stated that the target matrix can be complex-valued. That is true, but it is often possible to choose real target matrices as well, which is far more motivating to a systems engineer or control engineer. It is important to investigate, which real target matrices can be achieved through a periodic transformation.

State transformations are especially useful, if they keep the structural properties, i.e. stability, observability and controllability, unchanged between the original and target systems. It is well-known that stability is preserved, if and only if the transformation matrix is a Lyapunov transformation (Lyapunov (1966)). There have been numerous attempts to analyze the stability properties of general time-variable linear differential systems, see e.g. (Kamen (1988), Neerhoff and van der Kloet, Zenger (2004)), but none of these methods has not - at least so far - lead to any major breakthrough. Special time-varying systems like the important subclass of $T$-periodic systems are easier to attack because they possess some special utilizable characteristics. The fact that they can be changed into a constant coefficient form by a time-periodic transformation is one of these.

The purpose of the paper is twofold. Conditions under which a time-periodic system matrix can be changed into a real constant matrix by a periodic state transformation are studied. It is shown, how a similarity transformation plays the key role and also gives the necessary freedom in choosing the target system matrix. Analysis is then extended to input-output systems, for which the structural properties are investigated under the periodic transformation. A strategy to design stabilizing control laws is outlined.

## 2. REDUCTION OF A TIME-PERIODIC SYSTEM MATRIX INTO A CONSTANT FORM

Consider the two autonomous linear differential systems

$$\dot{x}(t) = A(t)x(t), \quad x(t_0) = x_0 \tag{1}$$

$$\dot{s}(t) = E(t)s(t), \quad s(t_0) = s_0 \tag{2}$$

where $A(\cdot) \in (\Re^{n\times n})^{\Re}$, $E(\cdot) \in (\Re^{n\times n})^{\Re}$ are bounded regulated functions and $x(t_0) = P_0 s(t_0)$ with a fixed $t_0 \in \Re$ and a constant invertible matrix $P_0 \in \Re^{n\times n}$. It is then easy to show (Zenger (2004)) that there exists a linear transformation

$$x(t) = P(t)s(t), \quad P(t_0) = P_0 \tag{3}$$

where the matrix-valued function $P(\cdot) \in (\Re^{n\times n})^{\Re}$ is given by

$$P(t) = \Phi_A(t,t_0)P_0\Phi_E(t,t_0)^{-1} = \Phi_A(t,t_0)P_0\Phi_E(t_0,t) \tag{4}$$

The function is differentiable and pointwise invertible; the functions $\Phi_A(\cdot,\cdot)$ and $\Phi_E(\cdot,\cdot)$ are the state transition matrices of the systems (1) and (2), respectively. The transformation matrix $P(\cdot)$ fulfils the differential equation

$$\begin{aligned}\dot{P}(t) &= \frac{\partial}{\partial t}\left(\Phi_A(t,t_0)\right)P_0\Phi_E(t,t_0)^{-1}\\ &\quad+\Phi_A(t,t_0)\frac{\partial}{\partial t}\left(P_0\Phi_E(t,t_0)^{-1}\right)\\ &= A(t)\Phi_A(t,t_0)P_0\Phi_E(t,t_0)^{-1}\\ &\quad-\Phi_A(t,t_0)P_0\Phi_E(t,t_0)^{-1}E(t)\Phi_E(t,t_0)\Phi_E(t,t_0)^{-1}\\ &= A(t)P(t)-P(t)E(t)\end{aligned} \tag{5}$$

The state transition matrices of $A(\cdot)$ and $E(\cdot)$ are related according to

$$\Phi_A(t,\tau) = P(t)\Phi_E(t,\tau)P^{-1}(\tau) \tag{6}$$

$$\Phi_E(t,\tau) = P^{-1}(t)\Phi_A(t,\tau)P(\tau) \tag{7}$$

as shown in (Zenger (2004)).

Using the well-known properties of the state-transition matrix (see e.g. (Rugh (1993))) it is easy to write the transformation $P(\cdot)$ as

$$\begin{aligned}P(t) &= \Phi_A(t,t_0)P(t_0)\Phi_E^{-1}(t,t_0)\\ &= \Phi_A(t,0)P(0)\Phi_E^{-1}(t,0)\end{aligned} \tag{8}$$

Next, consider an autonomous $T$-periodic system

$$\dot{x}(t) = A(t)x(t), \quad x(t_0) = x_0, \quad A(t+T) = A(t) \tag{9}$$

for all $t$ and some non-negative constant $T$. Consider the possibility to change the system matrix $A(\cdot)$ into a constant matrix $R$ in (1) - (2). That is always possible (Zenger (2004)) by using the transformation (3) where

$$\Phi_A(t,t_0) = P(t)e^{R(t-t_0)}P^{-1}(t_0) \tag{10}$$

and

$$P(t) = \Phi_A(t,t_0)P(t_0)e^{-R(t-t_0)} = \Phi_A(t,0)P(0)e^{-Rt} \tag{11}$$

Note that the result holds for all square matrices $R$ of an appropriate dimension, i.e. a transformation matrix $P(\cdot)$ that changes $A(\cdot)$ to $R$ exists. It is then meaningful to consider only real matrices $R$, so that the transformation is also real.

To study the periodicity of $P(\cdot)$ write

$$\begin{aligned}P(t+T) &= \Phi_A(t+T,0)P(0)e^{-R(t+T)}\\ &= \Phi_A(t+T,T)\Phi_A(T,0)P(0)e^{-RT}e^{-Rt}\\ &= \Phi_A(t+T,T)P(T)e^{-Rt}\end{aligned} \tag{12}$$

But it is easy to prove that

$$\Phi_A(t+T,T) = \Phi_A(t,0)$$

so that

$$P(t+T) = \Phi_A(t,0)P(T)e^{-Rt} = P(t) \tag{13}$$

provided that $P(T) = P(0)$. Note that this does not imply that the solution to the autonomous system $x(t) = \Phi_A(t,t_0)x_0$ would be $T$-periodic.

But it is not at all obvious that $P(T) = P(0)$. The condition to be fulfilled is then

$$e^{RT} = P(0)^{-1}\Phi_A(T,0)P(0) \tag{14}$$

so that the matrices $\exp(RT)$ and $\Phi_A(T,0)$ must be similar through the matrix $P(0)$.

It is reasonable to emphasize the importance of the above result. The necessary and sufficient condition for the existence of a periodic transformation that changes a periodic system matrix into a constant form is that two specific matrices are similar to each other. The matrix $P(0)$ gives freedom in choosing the target matrix $R$. In the literature it is stated that the target matrix $R$ is often complex (see e.g. Rugh (1993), Brockett (1970), Montagnier et al. (2004), Montagnier et al. (2003)), but this statement can be misleading: if the matrix $P(\cdot)$ is real and the matrices $\exp(RT)$ and $\Phi_A(T,0)$ are similar, the matrix $R$ is real indeed, which is generally a desired property from the system engineering viewpoint.

It is interesting to study, what possibilities we have to choose $R$. For example, it is obvious that if the matrix $\exp(RT)$ has an eigenvalue on the negative real axis, then $R$ cannot be real. Because of similarity, the eigenvalues of $\exp(RT)$ and $\Phi_A(T,0)$ must be the same. Let $\lambda$ be the eigenvalue of $\exp(RT)$ and $\lambda_R$ be the eigenvalue of $R$. Then trivially

$$e^{\lambda_R T} = \lambda \;\Rightarrow\; \lambda_R T = \ln(\lambda) \;\Rightarrow\; \lambda_R = \frac{1}{T}\ln(\lambda) \tag{15}$$

It follows that in order for $\lambda_R$ to be real, $\lambda$ must be real and non-negative.

In summary, $R$ must be chosen such that $e^{RT}$ has the same eigenvalues as $\Phi_A(T,0)$. The transformation

$$P(t) = \Phi_A(t,0)P(0)e^{-Rt} \tag{16}$$

then works, and the matrix $P(\cdot)$ is T-periodic.

The matrix $R$ can be chosen to be a diagonal matrix, if the eigenvectors of $\Phi_A(T,0)$ are linearly independent. Then $P(0)$ contains these vectors as its columns. If $P(0)$ is chosen to commute with $\Phi_A(T,0)$, the standard choice

$$e^{RT} = \Phi_A(T,0) \tag{17}$$

follows. Additionally, from the properties of the similarity transformation it follows that if the eigenvector belonging to the eigenvalue $\lambda$ of $\exp(RT)$ is $x$, then the eigenvector belonging to the same eigenvalue of $\Phi_A(T,0)$ is $P(0)x$.

Even more information can be obtained by noticing that the matrices $\exp(RT)$ and $\Phi_A(T,0)$ must have the same Jordan form, because they are similar. That is a necessary and sufficient condition. Other necessary conditions are easily derived. In addition to the conditions mentioned above it must also hold

$$\begin{aligned}\det\left(e^{RT}\right) &= \det\left(\Phi_A(T,0)\right)\\ \mathrm{tr}\left(e^{RT}\right) &= \mathrm{tr}\left(\Phi_A(T,0)\right)\end{aligned} \tag{18}$$

The determinant condition can be explored further. Because $\exp(Rt)$ is the state transition matrix attached to the system matrix $R$, then according to the theorem of Abel-Jacobi-Liouville (Rugh (1993))

$$\det\left(e^{Rt}\right) = e^{\mathrm{tr}(R)t} \;\Rightarrow\; \det\left(e^{RT}\right) = e^{\mathrm{tr}(R)T} \tag{19}$$

Also

$$\det\left(\Phi_A(T,0)\right) = e^{\int_0^T \mathrm{tr}[A(\tau)]d\tau} \tag{20}$$

so that

$$e^{\mathrm{tr}(R)T} = e^{\int_0^T \mathrm{tr}[A(\tau)]d\tau} \quad \Rightarrow \quad \mathrm{tr}(R) = \frac{1}{T}\int_0^T \mathrm{tr}\left[A\left(\tau\right)\right]d\tau \tag{21}$$

which can be seen as the generalization of the following result: If we choose $\exp(RT) = \Phi_A(T, 0)$ and $A(\cdot)$ commutes with its integral, or in other words

$$\Phi_A(T, 0) = e^{\int_0^T A(\tau)\, d\tau}$$

then

$$R = \frac{1}{T}\int_0^T A(\tau)\, d\tau \tag{22}$$

It is certainly of interest to study, when the solution of the original $T$-periodic autonomous system (1) is periodic as well. According to the literature (see e.g. (Rugh (1993))), given $t_0$ there exists an initial state $x(t_0)$ such that the solution is $T$-periodic exactly, when at least one eigenvalue of $e^{RT}$ is unity. Let us look at this closer.

From (10) it follows that

$$\begin{aligned}\Phi_A(t+T, \tau) &= P(t+T)e^{R(t+T-\tau)}P^{-1}(\tau) \\ &= P(t)e^{R(t-\tau)}e^{RT}P^{-1}(\tau)\end{aligned} \tag{23}$$

so that

$$\begin{aligned}x(t+T) &= \Phi_A(t+T, \tau)x(\tau) \\ &= P(t)e^{R(t-\tau)}e^{RT}P^{-1}(\tau)x(\tau)\end{aligned} \tag{24}$$

The result indicates that if the matrix $e^{RT}P^{-1}(\tau)$ has a unity eigenvalue, then choosing the corresponding eigenvector $x(\tau)$ as the initial value results in a periodic solution.

For stability considerations note that since $P(\cdot)$ is continuous and bounded, it is a *Lyapunov transformation*, indicating that the stability properties of the original and transformed systems are the same (Lyapunov (1966)).

**Example:** Consider the system ( 1) with

$$A(t) = \begin{bmatrix} -1 & 0 \\ -\cos(t) & 0 \end{bmatrix}$$

and $x(0) = x_0$=0 (Rugh (1993)). Clearly, the system is $T$-periodic with $T = 2\pi$. The state transition matrix is

$$\Phi_A(t, 0) = \begin{bmatrix} e^{-t} & 0 \\ -1/2 + 1/2 \cdot e^{-t}(\cos(t) - \sin(t)) & 1 \end{bmatrix}$$

so that

$$\Phi_A(2\pi, 0) = \begin{bmatrix} e^{-2\pi} & 0 \\ -1/2 + 1/2 \cdot e^{-2\pi} & 1 \end{bmatrix}$$

and

$$\lambda\left(\Phi_A\left(2\pi, 0\right)\right) = \left\{e^{-2\pi}, 1\right\}, \quad \lambda_R = \{-1, 0\}$$

It is then possible to choose e.g.

$$R = \begin{bmatrix} -1 & 0 \\ -1/2 & 0 \end{bmatrix}$$

or

$$R = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}$$

For easier calculations choose the latter one and calculate

$$\begin{aligned}P(0) &= \begin{bmatrix} p_{11}(0) & p_{12}(0) \\ p_{21}(0) & p_{22}(0) \end{bmatrix} = \Phi_A(T, 0)P(0)e^{-RT} \\ &= \begin{bmatrix} e^{-2\pi} & 0 \\ -1/2 + 1/2 \cdot e^{-2\pi} & 1 \end{bmatrix} \begin{bmatrix} p_{11}(0) & p_{12}(0) \\ p_{21}(0) & p_{22}(0) \end{bmatrix} \\ &\cdot \begin{bmatrix} e^{2\pi} & 0 \\ 0 & 1 \end{bmatrix}\end{aligned}$$

The elements on the equation become then

$$\begin{cases} p_{11}(0) = p_{11}(0) \\ p_{12}(0) = p_{12}(0)e^{-2\pi} \\ p_{21}(0) = 1/2 p_{11}(0) \\ p_{22}(0) = -1/2 \cdot p_{12}(0)\left(1 - e^{-2\pi}\right) + p_{22}(0) \end{cases}$$

and

$$\begin{aligned}&p_{12}(0) = 0,\ p_{11}(0) \in \Re \neq 0, \\ &p_{21}(0) = 1/2 \cdot p_{11}(0),\ p_{22}(0) \in \Re \neq 0\end{aligned}$$

$$P(0) = \begin{bmatrix} p_1 & 0 \\ 1/2 \cdot p_1 & p_2 \end{bmatrix}, \quad p_1 \neq 0,\ p_2 \neq 0$$

Finally, the transformation is

$$P(t) = \Phi_A(t, 0)P(0)e^{-Rt} = \begin{bmatrix} p_1 & 0 \\ 1/2 \cdot p_1\left(\cos(t) - \sin(t)\right) & p_2 \end{bmatrix}$$

The inverse matrix is

$$P^{-1}(t) = \frac{1}{p_1 p_2}\begin{bmatrix} p_2 & 0 \\ -1/2 \cdot p_1\left(\cos(t) - \sin(t)\right) & p_1 \end{bmatrix}$$

and it is obvious that for $p_1 \neq 0$, $p_2 \neq 0$ the matrix $P(\cdot)$ is a Lyapunov transformation.

## 3. INPUT-OUTPUT SYSTEMS

From the control engineering viewpoint the input-output system and its structural properties are of interest. To that end consider the input-state-output realization of a system

$$\begin{aligned}\dot{x}(t) &= A(t)x(t) + B(t)u(t), \quad x(t_0) = x_0 \\ y(t) &= C(t)x(t) + D(t)u(t)\end{aligned} \tag{25}$$

where the state $x(\cdot) \in (\Re^n)^{\Re}$, input $u(\cdot) \in (\Re^m)^{\Re}$ and output $y(\cdot) \in (\Re^r)^{\Re}$, coefficients $A(\cdot) \in (\Re^{n\times n})^{\Re}$, $B(\cdot) \in (\Re^{n\times m})^{\Re}$, $C(\cdot) \in (\Re^{r\times n})^{\Re}$ and $D(\cdot) \in (\Re^{r\times m})^{\Re}$ are bounded regulated functions. A target system can be written as

$$\begin{aligned}\dot{s}(t) &= E(t)s(t) + F(t)v(t), \quad s(t_0) = s_0 \\ z(t) &= G(t)s(t) + H(t)v(t)\end{aligned} \tag{26}$$

with obvious dimensions. Introduce the time-varying transformations

$$x(t) = P(t)s(t) \quad u(t) = U(t)v(t) \quad y(t) = Y(t)z(t) \tag{27}$$

so that the original system changes into the form (Zenger (2004))

$$\begin{aligned}\dot{s}(t) &= P^{-1}(t)\left[A(t)P(t) - \dot{P}(t)\right]s(t) \\ &\quad + P^{-1}(t)B(t)U(t)v(t) \\ z(t) &= Y^{-1}(t)C(t)P(t)s(t) + Y^{-1}(t)D(t)U(t)v(t)\end{aligned} \tag{28}$$

with

$$\begin{aligned} &E(t) = P^{-1}(t)\left[A(t)P(t) - \dot{P}(t)\right], \\ &F(t) = P^{-1}(t)B(t)U(t), \\ &G(t) = Y^{-1}(t)C(t)P(t), \quad H(t) = Y^{-1}(t)D(t)U(t) \end{aligned} \tag{29}$$

The solution to the output of the original system is

$$y_A(t) = C(t)\Phi_A(t,t_0)x_0 + \int_{t_0}^{t} g_A(t,\tau)u(\tau)d\tau + D(t)u(t) \tag{30}$$

where $g_A(t,\tau) = C(t)\Phi_A(t,\tau)B(\tau)$ is the weighting function. For the target system

$$z_E(t) = G(t)\Phi_E(t,t_0)s_0 + \int_{t_0}^{t} g_E(t,\tau)v(\tau)d\tau + H(t)v(t) \tag{31}$$

$$\begin{aligned} g_E(t,\tau) &= G(t)\Phi_E(t,\tau)F(\tau) \\ &= Y^{-1}(t)C(t)\underbrace{P(t)\Phi_E(t,\tau)P^{-1}(\tau)}_{\Phi_A(t,\tau)}B(\tau)U(\tau) \\ &= Y^{-1}(t)g_A(t,\tau)U(\tau) \end{aligned} \tag{32}$$

Note that

$$\begin{aligned} &G(t)\Phi_E(t,t_0)s_0 \\ &\quad = Y^{-1}(t)C(t)P(t)P^{-1}(t)\Phi_A(t,t_0)P(t_0)s_0 \\ &\quad = Y^{-1}(t)C(t)\Phi_A(t,t_0)x_0 \end{aligned} \tag{33}$$

$$z_E(t) = Y^{-1}(t)y_A(t) \tag{34}$$

as expected. Choosing $U$ and $Y$ to be identity matrices the weighting functions and impulse responses are the same.

The controllability gramian of the original system is

$$W_A(t_0,t_1) = \int_{t_0}^{t_1} \Phi_A(t_0,t)B(t)B^T(t)\Phi_A^T(t_0,t)dt \tag{35}$$

which for the target system is $(U = I)$

$$\begin{aligned} W_E(t_0,t_1) &= \int_{t_0}^{t_1} \Phi_E(t_0,t)F(t)F^T(t)\Phi_E^T(t_0,t)dt \\ &= \int_{t_0}^{t_1} \left\{P^{-1}(t_0)\Phi_A(t_0,t)B(t)B^T(t)\Phi_A^T(t_0,t)\left(P^T(t_0)\right)^{-1}\right\}dt \\ &= P^{-1}(t_0)W_A(t_0,t_1)\left(P^T(t_0)\right)^{-1} \end{aligned} \tag{36}$$

Because the matrix has full rank, the definiteness of the gramians $W_A$ and $W_E$ is the same. Controllability remains thus invariant in the transformation. A similar calculation shows that observability is invariant with respect to the transformation, because for the observability gramians $(Y = I)$

$$M_A(t_0,t_1) = \int_{t_0}^{t_1} \Phi_A^T(t,t_0)C^T(t)C(t)\Phi_A(t,t_0)dt \tag{37}$$

$$M_E(t_0,t_1) = P^T(t_0)M_A(t_0,t_1)P(t_0) \tag{38}$$

For input-state-output systems, where the system matrix $A(\cdot)$ is $T$-periodic the structural properties are preserved in the state transformation. That is because controllability and observability remain unchanged generally, and the change of the state variable is done by a Lyapunov transformation, which guarantees the preservation of stability. For example, if the matrix $E = R$ is asymptotically stable, so is $A(\cdot)$; input output-stability follows.

Next, let us investigate the possibility of finding a periodic stabilizing state control law for an input-output system, which has a $T$-periodic system matrix $A$ and a $T$-periodic control matrix $B$. Let $A(t+T) = A(t)$ in (25) and assume that $\Phi_A(T,0)$ has no eigenvalues on the negative real axis. Then use (8) to change the representation into (26) with $E(t) = R$ (constant matrix). The eigenvalues of $R$ show, whether the original system is stable or not. The transformation $x(t) = P(t)s(t)$ leads to

$$\dot{s}(t) = Rs(t) + P(t)^{-1}B(t)u(t) \tag{39}$$

for which the control law $u(t) = -L(t)s(t)$ can be used, which gives the closed-loop equation

$$\dot{s}(t) = \left(R - P(t)^{-1}B(t)L(t)\right)s(t) = S(t)s(t) \tag{40}$$

That has a $T$-periodic system matrix provided that $L(t)$ is $T$-periodic.

A sufficient condition for the existence of a stabilising $T$-periodic control matrix $L(t)$ is that there exists a stable constant matrix $R_2$ and an invertible square matrix $P_2(0)$ such that

$$e^{R_2 T} = P_2(0)^{-1}\Phi_S(T,0)P_2(0) \tag{41}$$

If $L(t)$ can assign all eigenvalues of $\Phi_S(T,0)$ to be the same as with the matrix $e^{R_2 T}$, a stabilizing control law has been found. A general algorithmic procedure for doing this is less obvious, however.

There are numerous application areas where the analysis and synthesis problems of periodic time-varying systems play an important role. One such class of systems is active vibration control of rotor movements in electrical machines (Rao (2000), Tammi (2007), Knospe et al. (1997)). When the rotor is driven at the so-called critical frequency (or its harmonics) the unbalancing factors in the rotor cause a resonance, which is detected as oscillation. To avoid this the motor must either be driven at subcritical speed or alternatively an active vibration control method must be developed. One such new idea is to design an extra coil in the stator slots, and control the current through it in such a way that the produced magnetic field creates a counterforce to dampen the vibration.

In Fig.1 the frequency response of a rotor system has been presented. A finite element model of the process has been built and low-order state-space models of it have been constructed by using prediction error and subspace identification. The results show a pretty good match between the FE model and the PE approximation, except at higher frequencies. It is quite extraordinary that the rotating forces in the system can be described by a time-invariant model (by using suitable coordinate transformations). However, in more accurate modelling this is not possible anymore, and systems of the form

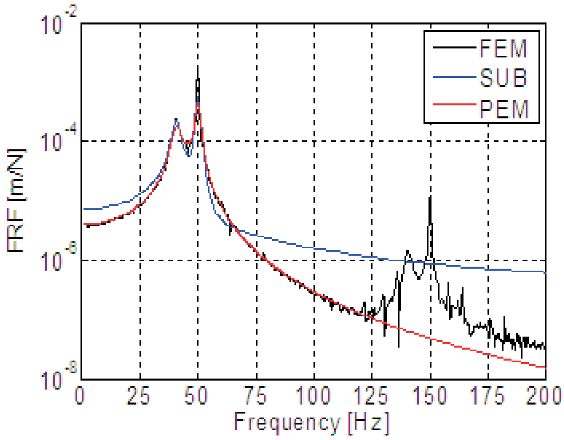


Fig. 1. Gain plot of the PEM model

$$\frac{d}{dt}\begin{pmatrix}\xi\\ \eta\\ i\end{pmatrix} = \begin{bmatrix} -2\Omega\Xi & \Phi_{rc}^T P_{em}(t)\Phi_{rc} & \Phi_{rc}^T C_{em} \\ I & 0 & 0 \\ S_{em}\Phi_{rc} & Q_{em}\Phi_{rc} & A_{em}\end{bmatrix}\begin{pmatrix}\xi\\ \eta\\ i\end{pmatrix} + \begin{bmatrix}0\\ 0\\ B_{em}\end{bmatrix} v + \begin{bmatrix}\Phi_{rc}^T\\ 0\\ 0\end{bmatrix} f_{ex} \tag{42}$$

$$u_{rc} = [\,0\ \ \Phi_{rc}\ \ 0\,]\begin{pmatrix}\xi\\ \eta\\ i\end{pmatrix}$$

follow. The state variables are currents in the system (divided into real and imaginary units), $v$ is the control current in the new actuator, $f_{ex}$ is the disturbance due to the rotor unbalance, and $u_{rc}$ is the rotor vibration (in two dimensions). The term $P_{em}(t)$, which was a constant matrix in a more elementary model, is now periodic. Control design according to the above ideas for this system are currently being developed.

## 4. CONCLUSION

It is well-known that linear periodic autonomous differential systems are reducible, i.e. they can be changed into a constant form by a suitable state transformation. In this paper it has been shown under which conditions the transformation is periodic and what freedom do we have in choosing the system matrix of the target system. Specifically, it was found out that the target matrix can in most cases be chosen to be real; a result which is believed to be new. Discussion was then extended to study input-output systems and their behaviour under state transformations. The structural properties were shown to remain invariant, if the original system matrix was periodic. A strategy to find a stabilizing control law was briefly outlined and an application was presented.

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