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Memoryless Control to Drive States of Delayed Continuous-time Systems within the Nonnegative Orthant

Abelaziz Hmamed [*] **Abdellah Benzaouia** [**]
Mustapha Ait Rami, Fernando Tadeo [***]

[*] *LESSI, University Mohamed Ben Abdellah, Faculty of Sciences Dhar El Mhraz, BP 1796, Fes, Morocco (e-mail: hmamed_abdelaziz@yahoo.fr)*
[**] *EACPI, University Cadi Ayyad, Faculty of Science Semlalia, BP 2390, Marrakech, Morocco (e-mail: benzaouia@ucam.ac.ma)*
[***] *Dpto. Ingenieria de Sistemas y Automatica, Universidad de Valladolid, 47005 Valladolid, Spain (e-mail: aitrami@autom.uva.es, fernando@autom.uva.es)*

Abstract: The stabilization problem for the class of linear continuous-time systems with fixed but unknown delay is solved when the additional condition that the states are nonnegative is studied. In particular, the synthesis of state-feedback controllers is solved by giving necessary and sufficient conditions in terms of Linear Programs. The solution is also extended to stabilization by bounded control (including nonnegative control) and stabilization under uncertainty.

Keywords: positive systems, delay systems, stabilization, bounded controls, linear programming.

1. INTRODUCTION

The objective of this paper is to characterize the state-feedback controllers that make the state of feedback systems nonnegative, whenever the initial conditions are nonnegative. In the literature, systems with nonnegative states are referred as positive systems (see [4, 13] for general references). These systems appear in many practical problems, when the states represent physical quantities that have an intrinsically constant sign (Absolute temperatures, levels, heights, concentrations, etc).

For the stabilization of positive systems, some previous works, based on algebraic approaches, can be found in [4, 7]. Note that these works are only concerned with the single-input case. Based on Gersgorin's theorem, a sufficient condition is provided in [12] and formulated as a quadratic programming problem. Recently, in [10] a necessary and sufficient Linear Matrix Inequality (LMI) condition is proposed for the stabilization of positive linear systems, similar to that previously proposed in [5].

In comparison with these previous works, the recent work of two of the authors [1, 2] provides a new treatment for the stabilization of positive linear systems where all the proposed conditions are necessary and sufficient, and expressed in terms of Linear Programming (LP).

This paper proposes to extend those results to systems with delays, presenting a new approach for the stabilization of MIMO positive linear systems with delay by means of state feedback. This extension to systems with delay is prompted by the existence of transport delays in many control problems that involve positive systems (In process control, irrigation systems, thermal systems, etc.). The stabilization problem of this kind of systems is a problem of interest, because the existence of a delay is known to cause instabilities [11]. The stabilization of systems with delays has been extensively studied in the literature (see [16, 9] and references therein), but only a few authors have considered positive systems in this context of time-delay systems: we can cite [14, 18]. Unfortunately, so far, no one has come up with a complete solution to the stabilization problem for delayed positive systems. This paper, for the first time, presents necessary and sufficient conditions, which turn out to be easily checkable and computable in terms of LP. The issue of control limitations is also dealt with: it is shown that the proposed approach can easily be extended to handle other constraints, such as componentwise lower and/or upper bounds on the controls, that includes the interesting case of positiveness of the control. The paper concentrates on memoryless control (that is, no information on the delayed states is used), as they are simple to implement and do not depend on precise knowledge of the delayed state or the length of the delay. However, it is possible to extend the results to controllers with memory, where the delayed state is known. Some previous results on global stabilization via memoryless control laws for general systems with delays have been presented in the literature (see [8, 11, 16] and references therein).

Thus, following the approach proposed in this paper, the problem of stabilization in the nonnegative state space of systems with delay and (maybe) control constraints can be solved. Although these systems might also be studied using general results from stabilization of constrained plants (a problem that has been extensively studied in the literature: see [17, 15] and references therein), the approach in this paper is original and the proposed conditions are simple to check (they are necessary and sufficient conditions expressed in terms of Linear Programs). In fact no previous work has been done on imposing positiveness in constrained systems with delay [8].

The remainder of the paper is structured as follows: Section 2 deals with the problem statement and some preliminary results. Section 3 presents the main results. Finally, section 4 gives some concluding remarks.

1.1 Notation and definitions

• $\mathbb{R}^n_+$ denotes the non-negative orthant of the n-dimensional real space $\mathbb{R}^n$.
• M^T denotes the transpose of the real matrix M.
• A matrix $M \in \mathbb{R}^{n\times n}$ is called a *Metzler* matrix if its off-diagonal elements are nonnegative. That is, if $M = \{m_{ij}\}^n_{i,j=1}$, M is Metzler if $m_{ij} \geq 0$ when $i \neq j$.
• A matrix M (or a vector) is said to be nonnegative if all its components are nonnegative (by notation $M \geq 0$). It is said to be positive if all its components are positive ($M > 0$).

2. PRELIMINARIES

2.1 Delayed Systems

This paper deals with the following set of governed delayed linear systems with delay:

$$\begin{cases} \dot{x}(t) = Ax(t) + A_1x(t-r) + Bu(t) \\ x(\theta) = \phi(\theta) \in R^n_+, \quad \theta \in [-r, 0] \end{cases} \tag{1}$$

where $x \in R^n$ is the state, $u \in R^m$ is the control vector, $r \in R$ is the delay (fixed but unknown), and the matrices $A \in R^{n\times n}$, $A_1 \in R^{n\times n}$ and $B \in R^{n\times m}$ are supposed to be constant and known (this last assumption will be lifted in section 3.5).

Remark 2.1. It must be pointed out that although for simplicity a single delay is considered in the model, the technique proposed can be easily generalized to systems with multiple delays (see the observation problem for systems with multiple delays in [3]).

The main problem considered in this paper is the following: consider the control law $u \in R^{n\times m}$ as a memoryless state feedback and not restricted in sign. This control law must be designed in such a way that the resulting governed system is positive and asymptotically stable for any $r > 0$. In other words, the main problem reduces to look for a memoryless state feedback law $u(t) = Kx(t)$, leading to the delayed closed-loop system defined by:

$$\dot{x}(t) = (A + BK)x(t) + A_1x(t-r), \tag{2}$$

where the matrix $K \in R^{m\times n}$ has to be determined to satisfy the following problem:

Find necessary and sufficient conditions on matrices $A, A_1 \in R^{n\times n}$ $B \in R^{n\times m}$, such that there exists a matrix $K \in R^{m\times n}$ satisfying:
• Positivity in closed-loop ($A_c = A + BK$ is a Metzler matrix).
• Closed-loop stability.

To solve this problem, some useful results on delayed positive systems are now given.

2.2 Delayed Positive Systems

Definition 2.1. Given any positive initial condition $x(\theta) = \phi(\theta) \in R^n_+$, $\theta \in [-r, 0]$, the delayed system (1) is said to be positive if the corresponding trajectory is never negative: $x(t) \in R^n_+$ for all $t \geq 0$.

According to this definition, we need to find under which condition the delayed system (1) is positive (see for example [6], [14]).

Lemma 2.1. System (1) is positive (i.e.: $x(t) \in R^n_+$) if and only if A is a Metzler matrix and A_1 is a nonnegative matrix.

The following result presents a necessary and sufficient condition for the asymptotic stability of the delayed system (1).

Theorem 2.1. Assume that system (1) is positive (or equivalently that the matrix A is a Metzler matrix and A_1 is a nonnegative matrix); then the delayed system (1) is asymptotically stable for any $r > 0$ and initial condition $x(\theta) = \phi(\theta) \in R^n_+$, $\theta \in [-r, 0]$, if and only if there exists $\lambda \in R^n$ such that,

$$(A + A_1)\lambda < 0, \quad \lambda > 0. \tag{3}$$

Proof:
(Necessity): Assume that system (1) is asymptotically stable. Since system (1) is linear and any initial condition can be expressed as the difference of two positive vectors, then consider $x_0 > 0$. Then, by integrating the terms of the system (1), after simple manipulation, it is possible to obtain

$$x(t) - x_0 = A\int_0^t x(\tau)d\tau + A_1\int_0^t x(\tau - r)d\tau. \tag{4}$$

If t goes to infinity, as $x(t)$ converges to 0, then

$$-x_0 - A_1\int_{-\tau}^0 x(t)dt = (A + A_1)\int_0^\infty x(t)dt. \tag{5}$$

By taking into account that $x_0 > 0$ and the positiveness of the initial condition of (1), this leads to

$$-x_0 - A_1\int_{-\tau}^0 x(t)dt = (A + A_1)\int_0^\infty x(t)dt < 0. \tag{6}$$

Consequently, condition (3) holds by defining $\lambda = \int_0^\infty x(t)dt$, which is positive by construction and the fact that $x(t) \in R^n_+$.

(Sufficiency): Knowing that the dual system $\dot{x}(t) = A^Tx(t) + A_1^Tx(t-r)$ is positive and asymptotically stable if and only if system (1) is positive and asymptotically stable, it suffices to prove that condition (3) implies the asymptotic stability of the dual system. For this, consider the following functional:

$$V(x(t)) = x^T(t)\lambda + \int_{t-r}^t x^T(s)A_1\lambda ds, \lambda > 0 \tag{7}$$

which is obviously definite positive, and also fulfills $V(x(t)) = 0$ if and only if $x(t) = 0$: to prove that $V(x(t)) = 0$ implies $x(t) = 0$ it is only necessary to check that $V(x(t)) = 0$ is equivalent to

$$-x^T(t)\lambda = \int_{t-r}^t x^T(s)A_1\lambda ds. \tag{8}$$

The left side of this equation is nonpositive (because $\lambda > 0$ and $x(t) \geq 0$) and the right side is nonnegative (because $\lambda > 0$, $x(t) \geq 0$ and $A_1 \geq 0$). Thus, the only possibility is that $x(t) = 0$.

Computing the rate of variation of $V(x(t))$ gives

$$\dot{V}(x(t)) = \dot{x}^T(t)\lambda + x^T(t)A_1\lambda - x^T(t-r)A_1\lambda. \tag{9}$$

Substituting $\dot{x}(t) = Ax(t) + A_1x(t-r)$ leads to:

$$\dot{V}(x(t)) = x^T(t)(A + A_1)\lambda. \tag{10}$$

Recalling that $x(t) \geq 0$, condition (3) implies that $\dot{V}(x(t)) < 0$, consequently the delayed system (1) is asymptotically stable. □

The following Lemma will be used in the sequel:

Lemma 2.2. Consider the autonomous delayed system (1): for a given $\bar{x} > 0$ we have $0 \leq x(t) \leq \bar{x}$, for any condition satisfying $0 \leq \phi(\theta) \leq \bar{x}$, $\theta \in [-r, 0]$ if and only if

$$\text{A is Metzler} \tag{11}$$

$$(A + A_1)\bar{x} < 0 \tag{12}$$

Proof:
(Sufficiency): Let us present the solution for system (1) in the following form:

$$\begin{aligned} x(t) &= e^{At}x_0 + \int_0^t e^{A(t-\tau)}A_1x(\tau - r)d\tau \\ &= e^{At}x_0 + \int_0^t e^{A\nu}A_1x(t-\nu-r)d\nu. \end{aligned}$$

Let,

$$0 \leq x(\theta) \leq \bar{x}, \theta \in [-r, 0]. \tag{13}$$

By using (13) and the fact that $e^{At} \geq 0$ ($\forall t \geq 0$), we obtain that

$$0 \leq x(t) \leq e^{At}\bar{x} + \int_0^t e^{A\nu}A_1x(t-\nu-r)d\nu.$$

Using the well known formula $e^{At} - I = \int_0^t e^{A\nu}Ad\nu$ (given for example in [8]), it follows that

$$[e^{At} - I]\bar{x} = \int_0^t e^{A\nu}A\bar{x}d\nu, \ \forall t \geq 0. \tag{14}$$

Thus, it is possible to deduce that

$$0 \leq x(t) \leq \bar{x} + \int_0^t e^{A\nu}[A\bar{x} + A_1x(t-\nu-r)d\nu]. \tag{15}$$

From (12), the following equation can be written:

$$(A + A_1)\bar{x} = -\rho_1, \ \rho_1 \geq 0.$$

Using the facts that $A_1 \geq 0$, and $e^{A\nu} \geq 0$ ($\forall \nu \geq 0$) it is possible to obtain that

$$0 \leq x(t) \leq \bar{x} + \int_0^t e^{A\nu}A_1[x(t-\nu-r) - \bar{x}]d\nu.$$

By using the same reasoning as [8], it follows that for $0 \leq \nu \leq t$ and $0 \leq t \leq r$, we can obtain $-r \leq t-\nu-r \leq 0$. Thus, using (13), the integral is not positive, so it follows that $0 \leq x(t) \leq \bar{x}$ for $t \in [0, r]$. In the same way, it is possible to obtain that $0 \leq x(t) \leq \bar{x}$ in the intervals $[r,\ 2r], [2r,\ 3r]$, *etc.*

(Necessity): Assume that the delayed system (1) is positive and condition (12) is not satisfied while $0 \leq x(t) \leq \bar{x}$ for any initial condition satisfying $0 \leq \phi(t) \leq \bar{x}$, $t \in [0, r]$. That is, there exists a subscript i, such that

$$\sum_{j=1}^{n} a(i,j)\bar{x}_j + \sum_{j=1}^{n} a_1(i,j)\bar{x}_j > 0, \tag{16}$$

where $a(i,j)$ represents the element (i,j) of the corresponding matrix A. Consider the following positive vectors: $\xi(t) = \bar{x}$ and $\xi(t-r) = \bar{x}$. It follows that $\dot{\xi}(t) = A\xi(t) + A_1\xi(t-r)$, with the i^{th} component given by $\dot{\xi}_i(t) = \sum_{j=1}^{n} a(i,j)\bar{x}_j + \sum_{j=1}^{n} a_1(i,j)\bar{x}_j > 0$. According to (16), $\dot{\xi}_i(t) > 0$, which implies that $\xi_i(t+\tau) > \bar{x}_i$, for any $\tau > 0$. This contradicts the assumption, consequently $(A + A_1)\bar{x} \leq 0$. □

Remark 2.2. : Condition (12) can also be seen as the necessary and sufficient condition for the set $\{x \in R^n, 0 \leq x \leq \bar{x}\}$ to be positively invariant with respect to the delayed system (1). Thus, this condition can be obtained as a particular case of the results of [8].

3. MAIN RESULTS

This section contains the main results: first the stabilization problem for general systems is studied and solved. After that, the result will be extended to solve related problems, when some signals are bounded or the state matrices are uncertain.

3.1 Controller Synthesis

In this subsection, necessary and sufficient conditions for positive asymptotic stabilization are presented for memoryless feedback control when the control is not bounded.

Theorem 3.1. The delayed system (1) under feedback $u = Kx$ is asymptotically stable for any $r > 0$ and the closed loop states are nonnegative if and only if $A_1 \geq 0$ and there exist vectors $d = [d_1, \ldots, d_n]^T \in R^n$ and $y_1, \ldots, y_n \in R^m$ such that

$$(A + A_1)d + B\sum_{i=1}^{n} y_i < 0 \tag{17}$$

$$d > 0 \tag{18}$$

$$a_{ij}d_j + b_iy_j \geq 0 \ \ i,j = 1, \ldots, n\ , i \neq j, \tag{19}$$

with $A = [a_{ij}]$ and $B^T = [b_1^T, \ldots, b_n^T]$.
Moreover, the gain matrix K is given by:

$$K = [d_1^{-1}y_1, \ldots, d_n^{-1}y_n] \tag{20}$$

Proof. Assume that condition (17) holds and define the matrix $K = [k_1, \ldots, k_n]$ with $k_i = d_i^{-1}y_i$. It is easy to see that $A + BK$ is a Metzler matrix, since condition (19) implies for for $i, j = 1, \ldots, n$ and $i \neq j$ that

$$a_{ij} + b_id_j^{-1}y_j = a_{ij} + b_ik_j = (A + BK)_{ij} \geq 0 \tag{21}$$

Then, computing $BKd = B\sum_{i=1}^{n} y_i$ and using condition (17) leads to $(A + A_1 + BK)d < 0$. Since $A + BK$ is a Metzler matrix, A_1 is nonnegative and $d > 0$, using Theorem 2.1, we can conclude that the delayed closed-loop system (2) is positive asymptotically stable for any $r > 0$. □

Remark 3.1. The result of Theorem 3.1 has been developed for systems where A is not assumed to be a Metzler matrix (only A_1 is assumed to be nonnegative). This makes the proposed result interesting for problems where the original system is not positive (think of an electrical system, where the input current might be positive or negative), that must be made positive and stable by feedback (in the electrical device, the current can be required to be always positive). This is shown in the following numerical example.

Remark 3.2. The necessity that A_1 is nonnegative can be lifted using other feedback laws, such as the state-feedback law with memory $u(t) = Kx(t) + Fx(t - r)$.

3.2 Example: State-feedback Stabilization with Nonnegative States

Consider a delayed system described by (1), with the following system matrices:

$$A = \begin{bmatrix} -1 & -0.5 \\ -0.3 & -0.7 \end{bmatrix},\ A_1 = \begin{bmatrix} 0.1 & 0.2 \\ 0.3 & 0.1 \end{bmatrix},\ B = \begin{bmatrix} -0.4 \\ -0.2 \end{bmatrix}.$$

It can be seen that the open-loop system is not positive (Although A_1 is nonnegative, there are off-diagonal negative elements in A). The objective is to design a state feedback controller $u = kx$ that stabilizes the system and makes the closed-loop states nonnegative for any value of the delay r (starting from any nonnegative initial condition). For this, the conditions of Theorem 3.1 must be fulfilled. The gain of a stabilizing control is given by any feasible solution to the above LP problem, for example: $K = [\,-1.5481\ -1.3269\,]$. If can be seen that with this controller, the feedback system is positive and the state evolution for the system remains always within the nonnegative orthant. For example, the state trajectories from several random initial positive conditions can be seen in Figure 1.

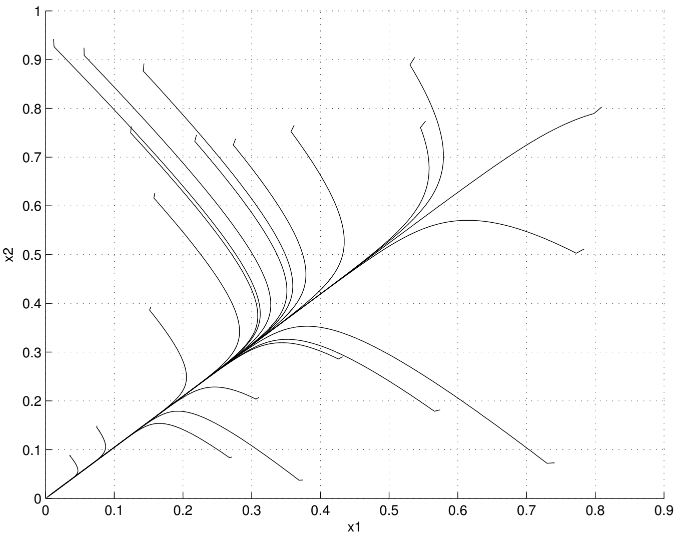

Fig. 1. Trajectories of the states from random initial values (unbounded control example)

3.3 Synthesis with Bounded Controls

This section studies the problem of closed-loop stabilization and positiveness for bounded positive controls. Consider the following constrained system:

$$\begin{cases} \dot{x}(t) = Ax(t) + A_1x(t-r) + Bu(t), \\ x(t) \geq 0 \\ 0 \leq u(t) \leq \bar{u}. \end{cases} \tag{22}$$

That is, the trajectory of the system is positive and the input is constrained to be positive and bounded by a given value $\bar{u}$.

The aim here is to address the following problem:
Given $\bar{u} > 0$ find $\bar{x} > 0$ corresponding to the set of initial conditions $\Gamma = \{x(0) \in \mathbb{R}^{n\times n} \mid 0 \leq x(0) \leq \bar{x}\}$ for which a nonnegative and bounded state feedback control law $0 \leq u = Kx(t) \leq \bar{u}$ can be determined, such that the following closed-loop system is positive and asymptotically stable:

$$\dot{x}(t) = (A + BK)x(t) + A_1x(t - r) \tag{23}$$

Now, we state the main result of this section:

Theorem 3.2. For system (1), with $A_1 \geq 0$, consider the following LP problem in the variables $\bar{x} = [\bar{x}_1\ \ldots\ \bar{x}_n]^T \in \mathbb{R}^n$ and $y_1, \ldots, y_n \in \mathbb{R}^p$:

$$(LP) \begin{cases} (A + A_1)\bar{x} + B\sum_{i=1}^{n} y_i < 0, \\ \bar{x} > 0, \\ y_i \geq 0, i = 1, \ldots, n, \\ \sum_{i=1}^{n} y_i \leq \bar{u}, \\ a_{ij}\bar{x}_j + b_iy_j \geq 0,\ \ i \neq j = 1, \ldots, n. \end{cases} \tag{24}$$

Then, the closed-loop system (23) is positive and asymptotically stable for any $r > 0$ and any initial condition $x(\theta)$ satisfying $0 \leq x(\theta) \leq \bar{x}$, under the state-feedback bounded control law $0 \leq u = Kx \leq \bar{u}$, with $K = [\bar{x}_1^{-1}y_1\ \ldots\ \bar{x}_n^{-1}y_n]$.

Proof 1. Take any $\bar{x} = [\bar{x}_1\ \ldots\ \bar{x}_n]^T$ and $y_1, \ldots, y_n$ that solve (24) and define $K = [\bar{x}_1^{-1}y_1\ \ldots\ \bar{x}_n^{-1}y_n]$. Then, since for $i \neq j = 1, \ldots, n$,

$$a_{ij} + b_i\bar{x}_j^{-1}y_j = a_{ij} + b_ik_j = (A + BK)_{ij} \geq 0,$$

we have that matrix $A + BK$ is Metzler. The inequality $(A + A_1)\bar{x} + B\sum_{i=1}^{n} y_i < 0$ is equivalent to $(A + A_1 + BK)\bar{x} < 0$. Since $\bar{x} > 0$ and A_1 is nonnegative, then by using Theorem 2.1, we can conclude that $A+A_1+BK$ is a stable nonnegative matrix (in the continuous-time sense). Further, by Lemma 2.2, the trajectory of the system (23) is such that $0 \leq x(t) \leq \bar{x}$ from any initial condition satisfying $0 \leq x(0) \leq \bar{x}$. Using this fact and recalling the inequalities

$\sum_{i=1}^{n} y_i \leq \bar{u}$, $y_i \geq 0$ for $i = 1, \ldots, n$, (or, equivalently, $K \geq 0$ and $K\bar{x} \leq \bar{u}$), it is easy to see that the state-feedback control $u = Kx$ is such that $0 \leq u(t) \leq K\bar{x} \leq \bar{u}$ for any initial state satisfying $0 \leq x(0) \leq \bar{x}$. $\square$

3.4 Example: Delayed System under Bounded Control

Consider a delayed system described by (1) with the same system matrices as the previous example:

$$A = \begin{bmatrix} -1 & -0.5 \\ -0.3 & -0.7 \end{bmatrix},\ A_1 = \begin{bmatrix} 0.1 & 0.2 \\ 0.3 & 0.1 \end{bmatrix},\ B = \begin{bmatrix} 0.4 \\ 0.2 \end{bmatrix}$$

Using a state feedback control, we want to impose for the controller to stabilize the system, that the closed-loop system is positive, and that the control signal is nonnegative and with a value always smaller than $\bar{u} = 10$. Thus, the conditions of Theorem 3.3 must be fulfilled. One feasible solution to the LP problem (24) provides $K = [\,0.0127\ 0.7884\,]$, with $\bar{x} = [\,97.8633\ 9.7391\,]$

It can be seen that the feedback system is positive, that the states are bounded by $\bar{x}$ and the control signal is always nonnegative and smaller than $\bar{u} = 10$. For example, the state evolution from some random initial positive conditions (smaller than $\bar{x}$ can be seen in Figure 2. The corresponding control signals are shown in Figure 3, where it can be seen that the imposed bounds on u are fulfilled.

3.5 Synthesis for uncertain system

An important extension of the proposed approach is the possibility of handling the case when the dynamics of the system are not exactly known, as is now presented in this subsection.

Consider the following delayed uncertain system:

$$\begin{aligned} \dot{x}(t) \quad &= \bar{A}x(t) + \bar{A}_1 x(t-r) + \bar{B}u(t), \\ x(\phi(\theta)) = \quad &\phi(\theta) \in \mathbb{R}^n_+, \theta \in [-r, 0]. \end{aligned} \tag{25}$$

Matrices $\bar{A}, \bar{A}_1 \in \mathbb{R}^{n\times n}$, $\bar{B} \in \mathbb{R}^{n\times p}$ are supposed to be not exactly determined, but it is assumed that they belong to the following convex set:

$$\mathcal{P} := \Big\{ \sum_{i=1}^{l} \alpha_i [A^i \ \ A_1^i \ \ B^i] \ \mid \ \sum_{i=1}^{l} \alpha_i = 1, \ \ \alpha_i \geq 0 \Big\}, \tag{26}$$

where $[A^1 \ \ A_1^1 \ \ B^1], \ldots, [A^l \ \ A_1^l \ \ B^l]$ are given matrices.

The proposed robust synthesis problem consists in determining the set of matrices K, such that the following closed-loop system is positive and asymptotically stable for every $[\bar{A} \ \ \bar{A}_1 \ \ \bar{B}] \in \mathcal{P}$:

$$\dot{x}(t) = (\bar{A} + \bar{B}K)x(t) + (\bar{A}_1)x(t-r) \tag{27}$$

Theorem 3.3. There exists a robust state-feedback law $u = Kx$ such that the resulting closed-loop system (27) is asymptotically stable and the closed-loop states are nonnegative for every $[\bar{A} \ \ \bar{A}_1 \ \ \bar{B}] \in \mathcal{P}$, if the following LP problem in the variables $d = [d_1 \ \ \ldots \ \ d_n]^T \in \mathbb{R}^n$ and $y_1, \ldots, y_n \in \mathbb{R}^p$ is feasible:

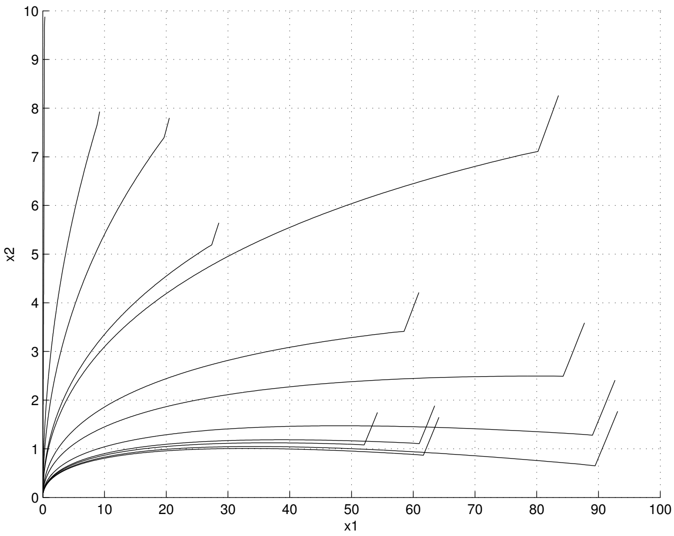

Fig. 2. State Trajectories from random initial values (bounded control example)

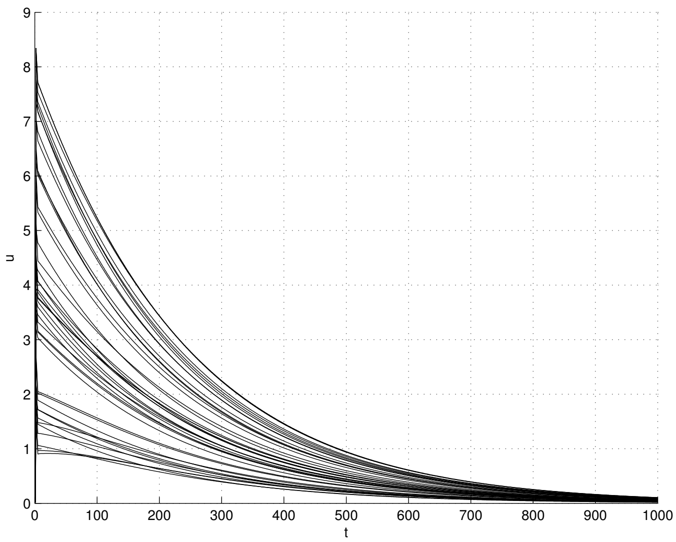

Fig. 3. Control evolution from random initial values (bounded control example)

$$(LP) \begin{cases} (A^k + A_1^k)d + B^k \sum_{i=1}^{n} y_i < 0 \text{ for } k = 1, \ldots, l, \\ d > 0, \\ a_{ij}^k d_j + b_i^k y_j \geq 0 \text{ for } i \neq j = 1, \ldots, n; k = 1, \ldots, l, \end{cases} \tag{28}$$

with $A^k = [a_{ij}^k]$, $A_1^k = [(a_1^k)_{ij}]$ and ${B^k}^T = [{b_1^k}^T \ \ \ldots \ \ {b_n^k}^T]$, $k = 1, \ldots, l$.

Moreover, the gain matrix K of the robust controller can be computed as

$$K = [d_1^{-1} y_1 \ \ \ldots \ \ d_n^{-1} y_n],$$

where d, $y_1 \ldots, y_n$ correspond to any feasible solution of the LP problem (3.3).

Proof 2. It is straightforward, following the proof of Theorem 3.1 and using a simple convexity argument.
$\square$

4. CONCLUSIONS

This paper has solved the problem of imposing nonnegativeness to closed-loop states using state feedback for continuous-time systems with unknown delay. First, necessary and sufficient conditions, using linear programming, have been proposed for system with unbounded controls and states. The same idea was then followed to solve the same problem in the presence of uncertainty or bounded control. It has been pointed out throughout the paper that these results are easy to obtain, as they are based on simple Linear Programming problems. Some examples have illustrated the the proposed approach, showing its feasibility and simplicity.

5. ACKNOWLEDGMENTS

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