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A note on a vibrating beam that made up of smart material ⋆

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Abstract: We study a special beam that is made up of smart material. We show that the beam possesses a number of unusual properties that makes its stability analysis very difficult. However, it does have a nice property that its eigenfunctions form a Riesz basis.

1. INTRODUCTION

Due to the requirements of the light weight, faster operational speed, lower energy consumption in the engineering applications of space technology and robotics (see [4, 5]), smart materials has been widely used in the suppression of vibration of elastic structures. In a lot of applications, such smart materials are used as passive or active controllers ([3]). The model of the smart-material beam that we shall study comes from H.T. Banks [1, 2], who had used finite element approach to study the problem. In [4, 5], a combination of a boundary feedback and an internal damping were used to achieve stability. In this paper, we are aiming at a more subtle investigation. We shall show that even in the simplest case where the external force is null, the smart beam behaves quite different from an ordinary beam. First, its spectrum has two branches and contains a continuous spectrum. Second, its resolvents are not necessarily compact operators. Third, the system operator is not closed under the conventional state space and hence does not generate a semigroup of operators. Fourth, despite all these, the associated eigenfunctions still form a Riesz basis. The model that we shall use is the one after taking the external force away from the beam in [1, 2]:

$$\begin{cases} u_{tt}(t,x) + \alpha u_{xx}(t,x)_{xx} + \beta u_{xxt}(t,x)_{xx} = 0, \\ u(t,0) = u_x(t,0) = 0, \\ \alpha u_{xx}(t,1) + \beta u_{xxt}(t,1) = 0, \\ \alpha u_{xxx}(t,1) + \beta u_{xxxt}(t,1) = 0. \end{cases} \tag{1}$$

Here, $u(t,x)$ denotes the transverse displacement of the beam at the position x and time t, and α and β are positive numbers. This system is usually called an Euler-Bernoulli equation with Kelvin-Voigt damping.

The stability of an elastic system with various kind of dampings deplotted over the whole region or just at the boundary of the region have been studied extensively during the past two decades ([10]). Several methods have become very effective in applications. One is the classical multiplier method, see for instance [8], in the early 1980's. A frequency domain method has also been developed, see Gearhart [6] and Huang [7]. Recently, a Riesz basis approach has evolved to yield the spectrum-determined growth condition together with the stability all in one strike (see [11]-[13]).

For Kelvin-Voigt dampings, either locally or globally distributed on Rayleigh and Euler-Bernoulli beams, were studied in [3], [9] and [10], where well-posedness, regularity and stabilities of the system on the energy space were investigated using a frequency domain method as well as a contrapositive argument with the multiplier method.

In this paper, we shall see that the eigenfunctions do form a Riesz basis but the system operator may not generate a C_0-semigroup of operators, that affect the well-posedness of the problem. We shall also carry out a spectral analysis for system and reveal the complicated nature of its spectrum.

2. A SMART-MATERIAL BEAM

As we have mentioned, the model of the smart-material beam that we shall study comes from H.T. Banks [1] who had used finite element approach to study the problem. In [2], a combination of a boundary feedback and an internal damping were used to achieve stability. Here we take the external force away and show that this simple system is far more complicated than people think:

$$\begin{cases} u_{tt}(t,x) + \alpha u_{xxxx}(t,x) + \beta u_{xxxxt}(t,x) = 0 \\ u(t,0) = 0 \\ u_x(t,0) = 0 \\ \alpha u_{xx}(t,1) + \beta u_{xxt}(t,1) = 0 \\ \alpha u_{xxx}(t,1) + \beta u_{xxxt}(t,1) = 0 \end{cases} \tag{2}$$

where α and β are positive numbers. We shall see that the spectrum of this beam has an accumulation point and its

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resolvents are not compact. To begin, we set the Hilbert space H to be

$$H := \left\{ \left(\begin{array}{c} f \\ g \end{array}\right) \middle| f \in H^2[0,1],\, g \in L^2[0,1],\, f(0) = f'(0) = 0 \right\} \quad (3)$$

with the norm on $(f\ g)$ to be

$$[\int_0^1 |f''(x)|^2 + |g(x)|^2 \, dx]^{\frac{1}{2}}.$$

We now carry out a spectral analysis on the system. To solve the eigen-problem of this beam system, we consider

$$A\left(\begin{array}{c} \phi \\ \psi \end{array}\right) = \lambda \left(\begin{array}{c} \phi \\ \psi \end{array}\right),$$

which is

$$\begin{cases} \psi(x) = \lambda\phi(x), \\ -\alpha\phi''''(x) - \beta\psi''''(x) = \lambda\psi(x), \\ \phi(0) = 0, \\ \phi'(0) = 0, \\ \alpha\phi''(1) + \alpha\psi''(1) = 0, \\ \alpha\phi'''(1) + \alpha\psi'''(1) = 0, \end{cases}$$

and it is equivalent to

$$\begin{cases} \psi(x) = \lambda\phi(x), \\ (\alpha+\beta\lambda)\phi''''(x) + \lambda^2\phi(x) = 0, \\ \phi(0) = 0, \\ \phi'(0) = 0 \\ (\alpha+\beta\lambda)\phi''(1) = 0, \\ (\alpha+\beta\lambda)\phi'''(1) = 0. \end{cases}$$

If $(\alpha + \beta\lambda) = 0$, then $\phi(x) = \psi(x) = 0$, so $\lambda = -\frac{\alpha}{\beta}$ is not an eigenvalue. Furthermore the eigenvalue problem is equivalent to

$$\begin{cases} (\alpha+\beta\lambda)\phi''''(x) + \lambda^2\phi(x) = 0, \\ \phi(0) = 0, \\ \phi'(0) = 0, \\ \phi''(1) = 0, \\ \phi'''(1) = 0, \end{cases}$$

which has a nonzero solution. Let ρ be a fourth root of $\frac{\lambda^2}{\alpha+\beta\lambda}$, i.e. $\rho^4 = \frac{\lambda^2}{\alpha+\beta\lambda}$, and we look for solutions of the form

$$\phi(x) = C_1 e^{\rho\omega_1 x} + C_2 e^{\rho\omega_2 x} + C_3 e^{\rho\omega_3 x} + C_4 e^{\rho\omega_4 x}$$

with ω_i $(i = 1,2,3,4)$ being the fourth roots of -1, Substituting it into the initial conditions, the equation

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ \rho\omega_1 & \rho\omega_2 & \rho\omega_3 & \rho\omega_4 \\ (\rho\omega_1)^2 e^{\rho\omega_1} & (\rho\omega_2)^2 e^{\rho\omega_2} & (\rho\omega_3)^2 e^{\rho\omega_3} & (\rho\omega_4)^2 e^{\rho\omega_4} \\ (\rho\omega_1)^3 e^{\rho\omega_1} & (\rho\omega_2)^3 e^{\rho\omega_2} & (\rho\omega_3)^3 e^{\rho\omega_3} & (\rho\omega_4)^3 e^{\rho\omega_4} \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

would have a nonzero solution if and only if

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ \rho\omega_1 & \rho\omega_2 & \rho\omega_3 & \rho\omega_4 \\ (\rho\omega_1)^2 e^{\rho\omega_1} & (\rho\omega_2)^2 e^{\rho\omega_2} & (\rho\omega_3)^2 e^{\rho\omega_3} & (\rho\omega_4)^2 e^{\rho\omega_4} \\ (\rho\omega_1)^3 e^{\rho\omega_1} & (\rho\omega_2)^3 e^{\rho\omega_2} & (\rho\omega_3)^3 e^{\rho\omega_3} & (\rho\omega_4)^3 e^{\rho\omega_4} \end{vmatrix} = 0.$$

Setting $\omega := \omega_1$ and express the other roots of unity in terms of ω, we seek to solve for ρ from

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ \omega & i\omega & -\omega & -i\omega \\ ie^{\rho\omega} & -ie^{i\rho\omega} & ie^{-\rho\omega} & -ie^{-i\rho\omega} \\ i\omega e^{\rho\omega} & \omega e^{i\rho\omega} & -i\omega e^{-\rho\omega} & -\omega e^{-i\rho\omega} \end{vmatrix} = 0.$$

Since

$$\begin{aligned}
&\begin{vmatrix} 1 & 1 & 1 & 1 \\ \omega & i\omega & -\omega & -i\omega \\ ie^{\rho\omega} & -ie^{i\rho\omega} & ie^{-\rho\omega} & -ie^{-i\rho\omega} \\ i\omega e^{\rho\omega} & \omega e^{i\rho\omega} & -i\omega e^{-\rho\omega} & -\omega e^{-i\rho\omega} \end{vmatrix} \\
&= i\begin{vmatrix} 1 & 1 & 1 & 1 \\ \omega & i\omega & -\omega & -i\omega \\ e^{\rho\omega} & -e^{i\rho\omega} & e^{-\rho\omega} & -e^{-i\rho\omega} \\ i\omega e^{\rho\omega} & \omega e^{i\rho\omega} & -i\omega e^{-\rho\omega} & -\omega e^{-i\rho\omega} \end{vmatrix} \\
&= \begin{vmatrix} 1 & 1 \\ \omega & i\omega \end{vmatrix} \begin{vmatrix} e^{-\rho\omega} & -e^{-i\rho\omega} \\ -i\omega e^{-\rho\omega} & -\omega e^{-i\rho\omega} \end{vmatrix} \\
&\quad - \begin{vmatrix} 1 & 1 \\ \omega & -\omega \end{vmatrix} \begin{vmatrix} -e^{i\rho\omega} & -e^{-i\rho\omega} \\ \omega e^{i\rho\omega} & -\omega e^{-i\rho\omega} \end{vmatrix} \\
&\quad + \begin{vmatrix} 1 & 1 \\ \omega & -i\omega \end{vmatrix} \begin{vmatrix} e^{i\rho\omega} & -e^{-\rho\omega} \\ -i\omega e^{i\rho\omega} & -\omega e^{-\rho\omega} \end{vmatrix} \\
&\quad + \begin{vmatrix} 1 & 1 \\ i\omega & -\omega \end{vmatrix} \begin{vmatrix} e^{\rho\omega} & -e^{-i\rho\omega} \\ i\omega e^{\rho\omega} & -\omega e^{-i\rho\omega} \end{vmatrix} \\
&\quad - \begin{vmatrix} 1 & 1 \\ i\omega & -i\omega \end{vmatrix} \begin{vmatrix} e^{\rho\omega} & -e^{-\rho\omega} \\ i\omega e^{\rho\omega} & -i\omega e^{-\rho\omega} \end{vmatrix} \\
&\quad + \begin{vmatrix} 1 & 1 \\ -\omega & -i\omega \end{vmatrix} \begin{vmatrix} e^{\rho\omega} & -e^{i\rho\omega} \\ i\omega e^{\rho\omega} & \omega e^{i\rho\omega} \end{vmatrix} \\
&= (i\omega - \omega)(-\omega + i\omega)e^{\rho(-\omega - i\omega)} - (-\omega-\omega)(\omega+\omega) \\
&\quad + (-i\omega - \omega)(-\omega + i\omega)e^{\rho(-\omega + i\omega)} \\
&\quad + (-\omega - i\omega)(\omega + i\omega)e^{\rho(\omega - i\omega)} \\
&\quad + (-i\omega - i\omega)(-i\omega - i\omega) + (-i\omega + \omega)(\omega + i\omega)e^{\rho(\omega + i\omega)} \\
&= 2ie^{-\sqrt{2}i\rho} + 4i + 2ie^{-\sqrt{2}\rho} + 2ie^{\sqrt{2}\rho} + 4i + 2ie^{\sqrt{2}i\rho} \\
&= 2i(e^{-\sqrt{2}i\rho} + e^{-\sqrt{2}\rho} + e^{\sqrt{2}\rho} + e^{\sqrt{2}i\rho} + 4),
\end{aligned}$$

it follows that

$$e^{-\sqrt{2}i\rho} + e^{-\sqrt{2}\rho} + e^{\sqrt{2}\rho} + e^{\sqrt{2}i\rho} = -4.$$

Setting $\sqrt{2}\rho := a + bi$, then

$$\begin{cases} \cosh b \cos a + \cosh a \cos b = -2 \\ \sinh a \sin b - \sinh b \sin a = 0. \end{cases}$$

Combining the second equation with the first one, we get $a = \pm b$. From the first equation, we then have $\cosh a \cos a = -1$, i.e. $\cos a = -1/\cosh a = O(e^{-|a|})$. Hence

$$a = n\pi + \frac{\pi}{2} + O(e^{-|n|}).$$

Since $\rho^4 = \frac{\lambda^2}{\alpha+\beta\lambda}$, we have

$$\begin{aligned}
&\frac{\lambda^2}{\alpha+\beta\lambda} \\
&= (\frac{1}{\sqrt{2}}a(1 \pm i))^4 = -a^4 \\
&= -(n\pi + \frac{\pi}{2} + O(e^{-|n|}))^4 = -(n\pi + \frac{\pi}{2})^4 + O(n^3 e^{-|n|}).
\end{aligned}$$

From this, we see that the eigenvalue are

$$\begin{aligned}
\lambda_n &= \frac{-a^4\beta+\sqrt{a^8\beta^2-4a^4\alpha}}{2} = \frac{-4a^4\alpha}{2(a^4+\sqrt{a^8\beta^2-4a^4\alpha})} \\
&= \frac{-2\alpha}{\beta+\sqrt{\beta^2-\frac{4\alpha}{a^4}}} = -\frac{\alpha}{\beta}+\frac{-2\alpha}{\beta+\sqrt{\beta^2-\frac{4\alpha}{a^4}}}+\frac{\alpha}{\beta} \\
&= -\frac{\alpha}{\beta}+\frac{\alpha}{\beta}\left(\frac{-\beta+\sqrt{\beta^2-\frac{4\alpha}{a^4}}}{\beta+\sqrt{\beta^2-\frac{4\alpha}{a^4}}}\right) \\
&= -\frac{\alpha}{\beta}+\frac{\alpha}{\beta}\left(\frac{-\frac{4\alpha}{a^4}}{\left(\beta+\sqrt{\beta^2-\frac{4\alpha}{a^4}}\right)^2}\right) \\
&= -\frac{\alpha}{\beta}+O(\frac{1}{a^4}) = -\frac{\alpha}{\beta}+O(\frac{1}{n^4}),
\end{aligned} \tag{4}$$

or

$$\begin{aligned}
\lambda_n &= \frac{-a^4\beta-\sqrt{a^8\beta^2-4a^4\alpha}}{2} \\
&= -a^4\beta-\frac{-a^4\beta+\sqrt{a^8\beta^2-4a^4\alpha}}{2} \\
&= -\beta(n\pi+\frac{\pi}{2})^4+O(n^3e^{-|n|})-\frac{\alpha}{\beta}+O(\frac{1}{n^4}) \\
&= -\beta(n\pi+\frac{\pi}{2})^4-\frac{\alpha}{\beta}+O(\frac{1}{n^4}).
\end{aligned} \tag{5}$$

Summarizing these calculations, we have the following theorem.

Theorem 1. The spectrum of system (2) consists of two branches of eigenvalues, the one (4) has an accumulation $-\frac{\alpha}{\beta}$ and the other (5) goes to infinity. All the resolvents of A are not compact.

3. RIESZ BASIS PROPERTY

In this section we will show that the eigenvectors of this system form a Riesz basis [14].

Lemma 3.1. Let L be as before $\{\rho_n, n\in N\}$ and $\{\phi_n; n\in N\}$ be its spectrum and eigenfunctions respectively. Define vectors $\Phi_n \in \mathcal{H}$ by

$$\Phi_{1,n} = \frac{1}{\sqrt{\beta\rho_n}}\begin{pmatrix}\phi_n\\ \lambda_{1,n}\phi_n\end{pmatrix}, \quad \Phi_{2,n} = \frac{1}{\beta\rho_n}\begin{pmatrix}\phi_n\\ \lambda_{2,n}\phi_n\end{pmatrix}, \tag{6}$$

and

$$\Psi_{1,n} = \frac{1}{\sqrt{\beta\rho_n}}\begin{pmatrix}\frac{-\alpha}{\overline{\lambda_{1,n}}}\phi_n\\ \phi_n\end{pmatrix}, \quad \Psi_{2,n} = \begin{pmatrix}\frac{-\alpha}{\overline{\lambda_{2,n}}}\phi_n\\ \phi_n\end{pmatrix}. \tag{7}$$

Then we have

$$(\Phi_{j,m}, \Psi_{i,n}) = \delta_{i,n,j,m}, \quad \forall, i,j=1,2, \quad n,m\in N \tag{8}$$

and

$$\mathcal{A}\Phi_{j,n} = \lambda_{j,n}\Phi_{j,n}, \quad j=1,2, \forall n\in N \tag{9}$$

$$\mathcal{A}^*\Psi_{j,n} = \overline{\lambda}_{j,n}\Psi_{j,n}, \quad j=1,2, \forall n\in N \tag{10}$$

Proof Obviously, (9)–(10) hold and we only verify equalities (8).

$$\begin{aligned}
&(\Phi_{1,m}, \Psi_{1,n}) \\
&= \frac{1}{\beta\rho_n}\int_0^1\left[\phi_m''(x)\overline{\left(\frac{-\alpha}{\overline{\lambda_{1,n}}}\phi_n''(x)\right)}+\lambda_{1,m}\phi_m\overline{\phi_n}\right]dx \\
&= \frac{1}{\beta\rho_n}\left[\frac{-\alpha}{\lambda_{1,n}}\int_0^1\phi_m(x)\overline{\phi_n^{(4)}(x)}+\lambda_{1,m}\frac{1}{\beta\rho_n}\int_0^1\phi_m\overline{\phi_n}\right]dx \\
&= \frac{1}{\beta\rho_n}\left(\frac{-\alpha}{\lambda_{1,n}}\rho_n+\lambda_{1,m}\right)\delta_{n,m} \\
&= \frac{-\alpha\rho_n+\lambda_{1,n}\lambda_{1,m}}{\beta\rho_n\lambda_{1,n}}\delta_{n,m} = \delta_{1,1}\delta_{n,m}.
\end{aligned}$$

$$\begin{aligned}
(\Phi_{1,m}, \Psi_{2,n}) &= \int_0^1\left[\phi_m''(x)\overline{\left(\frac{-\alpha}{\overline{\lambda_{2,n}}}\phi_n''(x)\right)}+\lambda_{1,m}\phi_m\overline{\phi_n}\right]dx \\
&= \frac{-\alpha}{\lambda_{2,n}}\int_0^1\phi_m(x)\overline{\phi_n^{(4)}(x)}dx+\lambda_{1,m}\int_0^1\phi_m\overline{\phi_n}dx \\
&= \left(\frac{-\alpha}{\lambda_{2,n}}\rho_n+\lambda_{1,m}\right)\delta_{n,m} \\
&= \frac{-\alpha\rho_n+\lambda_{2,n}\lambda_{1,m}}{\lambda_{2,n}}\delta_{n,m} = 0 = \delta_{1,2}\delta_{n,m}.
\end{aligned}$$

$$\begin{aligned}
&(\Phi_{2,m}, \Psi_{1,n}) \\
&= \frac{1}{\beta\rho_m\sqrt{\beta\rho_n}}\int_0^1\left[\phi_m''(x)\overline{\left(\frac{-\alpha}{\overline{\lambda_{1,n}}}\phi_n''(x)\right)}+\lambda_{2,m}\phi_m\overline{\phi_n}\right]dx \\
&= \frac{1}{\beta\rho_m\sqrt{\beta\rho_n}}\left(\frac{-\alpha}{\lambda_{1,n}}\int_0^1\phi_m(x)\overline{\phi_n^{(4)}(x)}+\frac{\lambda_{2,m}}{\beta\rho_m}\int_0^1\phi_m\overline{\phi_n}\right)dx \\
&= \frac{1}{\beta\rho_m\sqrt{\beta\rho_n}}\left(\frac{-\alpha}{\lambda_{1,n}}\rho_n+\lambda_{2,m}\right)\delta_{n,m} \\
&= \frac{-\alpha\rho_n+\lambda_{1,n}\lambda_{2,m}}{\beta\rho_m\lambda_{1,n}\sqrt{\beta\rho_n}}\delta_{n,m} = 0 = \delta_{2,1}\delta_{n,m}.
\end{aligned}$$

$$\begin{aligned}
&(\Phi_{2,m}, \Psi_{2,n}) \\
&= \frac{1}{\beta\rho_m}\int_0^1\left[\phi_m''(x)\overline{\left(\frac{-\alpha}{\overline{\lambda_{2,n}}}\phi_n''(x)\right)}+\lambda_{2,m}\phi_m\overline{\phi_n}\right]dx \\
&= \frac{1}{\beta\rho_m}\left(\frac{-\alpha}{\lambda_{2,n}}\int_0^1\phi_m(x)\overline{\phi_n^{(4)}(x)}+\lambda_{2,m}\frac{1}{\beta\rho_m}\int_0^1\phi_m\overline{\phi_n}\right)dx \\
&= \frac{1}{\beta\rho_m}\left(\frac{-\alpha}{\lambda_{2,n}}\rho_n+\lambda_{2,m}\right)\delta_{n,m} \\
&= \frac{-\alpha\rho_n+\lambda_{2,n}\lambda_{2,m}}{\beta\rho_m\lambda_{2,n}}\delta_{n,m} = \delta_{2,2}\delta_{n,m}.
\end{aligned}$$

Direct calculating the norm of the vectors, we get

$$\begin{aligned}
\|\Phi_{1,n}\|^2 &= \frac{1}{(\beta\rho_n)}\int_0^1[\phi_n''(x)\phi_n''(x)+|\lambda_{1,n}|^2\phi_n(x)\phi_n(x)]dx \\
&= \frac{\lambda_{1,n}^2+\rho_n}{(\beta\rho_n)} \approx \frac{1}{\beta},
\end{aligned}$$

$$\|\Phi_{2,n}\|^2 = \frac{1}{(\beta\rho_n)^2}\int_0^1[\phi_n''(x)\phi_n''(x)+|\lambda_{2,n}|^2\phi_n(x)\phi_n(x)]dx$$

$$= \frac{\lambda_{2,n}^2 + \rho_n}{(\beta\rho_n)^2} \approx 1,$$

$$\|\Psi_{1,n}\|^2 = \frac{1}{\beta\rho_n}\left[\frac{\alpha^2}{|\lambda_{1,n}|^2}\int_0^1 \phi_n''(x)\phi_n''(x)dx + \int_0^1 \phi_n(x)\phi_n(x)dx\right] = \frac{1}{\beta\rho_n}\left[\frac{\alpha^2\rho_n}{|\lambda_{1,n}|^2} + 1\right] \approx \frac{\beta}{\alpha},$$

$$\|\Psi_{2,n}\|^2 = \left[\frac{\alpha^2}{|\lambda_{2,n}|^2}\int_0^1 \phi_n''(x)\phi_n''(x)dx + \int_0^1 \phi_n(x)\phi_n(x)dx\right] = \left[\frac{\alpha^2\rho_n}{|\lambda_{2,n}|^2} + 1\right] = \left[\frac{\alpha^2\rho_n + \lambda_{2,n}^2}{|\lambda_{2,n}|^2}\right] = \frac{\alpha^2\rho_n - \alpha\rho_n - \beta\rho_n\lambda_{2,n}}{|\lambda_{2,n}|^2} \approx 1.$$

Lemma 3.2. Let $\{\Phi_n; n \in N\}$ and $\{\Psi_n; n \in N\}$ be defined as Lemma 3.1. Then $\{\Phi_n; n \in N\}$ form a Riesz basis in $\mathcal{H}$.

Proof For each $F = (f, g) \in \mathcal{H}$, we have

$$(F, \Psi_{1,n}) = \frac{1}{\sqrt{\beta\rho_n}}\int_0^1 \left[f''(x)\overline{\left(\frac{-\alpha}{\overline{\lambda_{1,n}}}\phi_n''(x)\right)}dx + g(x)\overline{\phi_n(x)}\right]dx$$
$$= \frac{1}{\sqrt{\beta\rho_n}}\left(\frac{-\alpha}{\overline{\overline{\lambda_{1,n}}}}\right)\int_0^1 f(x)\overline{\phi_n^{(4)}(x)}dx + \frac{1}{\sqrt{\beta\rho_n}}\int_0^1 g(x)\overline{\phi_n(x)}dx$$
$$= \frac{1}{\sqrt{\beta\rho_n}}\left(\frac{-\alpha\rho_n}{\overline{\lambda_{1,n}}}\right)\int_0^1 f(x)\overline{\phi_n(x)}dx + \frac{1}{\sqrt{\beta\rho_n}}\int_0^1 g(x)\overline{\phi_n(x)}dx$$

$$(F, \Psi_{2,n}) = \int_0^1 \left[f''(x)\overline{\left(\frac{-\alpha}{\overline{\lambda_{2,n}}}\phi_n''(x)\right)}dx + g(x)\overline{\phi_n(x)}\right]dx$$
$$= \left(\frac{-\alpha}{\overline{\overline{\lambda_{2,n}}}}\right)\int_0^1 f(x)\overline{\phi_n^{(4)}(x)}dx + \int_0^1 g(x)\overline{\phi_n(x)}dx$$
$$= \left(\frac{-\alpha\rho_n}{\overline{\overline{\lambda_{2,n}}}}\right)\int_0^1 f(x)\overline{\phi_n(x)}dx + \int_0^1 g(x)\overline{\phi_n(x)}dx$$

$$(F, \Phi_{1,n}) = \frac{1}{\sqrt{\beta\rho_n}}\int_0^1 \left[f''(x)\overline{\phi_n''(x} + g(x)\overline{\lambda_{1,n}\phi_n(x)}\right]dx$$
$$= \frac{1}{\sqrt{\beta\rho_n}}\left[\int_0^1 f(x)\overline{\phi_n^{(4)}(x)}dx + \overline{\lambda_{1,n}}\int_0^1 g(x)\overline{\phi_n(x)}dx\right]$$
$$= \frac{1}{\sqrt{\beta\rho_n}}\left[\rho_n\int_0^1 f(x)\overline{\phi_n(x)}dx + \overline{\lambda_{1,n}}\int_0^1 g(x)\overline{\phi_n(x)}dx\right]$$

$$(F, \Phi_{2,n}) = \frac{1}{\beta\rho_n}\int_0^1 \left[f''(x)\overline{\phi_n''(x} + g(x)\overline{\lambda_{2,n}\phi_n(x)}\right]dx$$
$$= \frac{1}{\beta\rho_n}\left[\int_0^1 f(x)\overline{\phi_n^{(4)}(x)}dx + \overline{\lambda_{2,n}}\int_0^1 g(x)\overline{\phi_n(x)}dx\right]$$
$$= \frac{1}{\beta\rho_n}\left[\rho_n\int_0^1 f(x)\overline{\phi_n(x)}dx + \overline{\lambda_{2,n}}\int_0^1 g(x)\overline{\phi_n(x)}dx\right]$$

If for all $n \in N$, $(F, \Phi_{j,n}) = 0, j = 1, 2,$, then

$$\int_0^1 f(x)\phi_n(x)dx = 0, \int_0^1 g(x)\phi_n(x)dx = 0,$$

and hence $f = g = 0$, i.e., $F = 0$. Similarly, if $(F, \Phi_{j,n}) = 0, j = 1, 2,$ for all $n \in N$, then we can deduce $F = 0$. Therefore $\{\Phi_{j,n}, j = 1, 2, n \in N\}$ and $\{\Psi_{j,n}; j = 1, 2, n \in N\}$ are complete in $\mathcal{H}$.

Now we calculate the Gram matrix $G = ((\Phi_{i,n}, \Phi_{j,m}))$ given by

$$G = \left(\begin{pmatrix} (\Phi_{1,n}, \Phi_{1,m}) & (\Phi_{1,n}, \Phi_{2,m}) \\ (\Phi_{2,n}, \Phi_{1,m}) & (\Phi_{2,n}, \Phi_{2,m}) \end{pmatrix}_{n,m}\right).$$

$$(\Phi_{1,m}, \Phi_{2,n}) = \frac{1}{\beta\rho_n\sqrt{\beta\rho_m}}\int_0^1 \left[\phi_m''(x)\overline{\phi_n''(x)} + \lambda_{1,m}\phi_m(x)\overline{\lambda_{2,n}\phi_n(x)}\right]dx$$
$$= \frac{1}{\beta\rho_n\sqrt{\beta\rho_m}}[\int_0^1 \phi_m(x)\overline{\phi_n^{(4)}(x)}dx + \lambda_{1,m}\overline{\lambda_{2,n}}\int_0^1 \phi_m(x)\overline{\phi_n(x)}dx]$$
$$= \frac{1}{\beta\rho_n\sqrt{\beta\rho_n}}\left[\rho_n + \lambda_{1,m}\overline{\lambda_{2,n}}\right]\delta_{n,m}$$

$$(\Phi_{1,n}, \Phi_{2,n}) = \frac{1}{\beta\rho_n\sqrt{\beta\rho_n}}\left[\rho_n + \lambda_{1,n}\overline{\lambda_{2,n}}\right] = \frac{1+\alpha}{\beta\sqrt{\beta\rho_n}}$$

$$(\Phi_{2,m}, \Phi_{2,n}) = \frac{1}{\beta\rho_n\beta\rho_m}\int_0^1 \left[\phi_m''(x)\overline{\phi_n''(x)} + \lambda_{2,m}\phi_m(x)\overline{\lambda_{2,n}\phi_n(x)}\right]dx$$
$$= \frac{1}{\beta\rho_n\beta\rho_m}[\int_0^1 \phi_m(x)\overline{\phi_n^{(4)}(x)}dx + \lambda_{2,m}\overline{\lambda_{2,n}}\int_0^1 \phi_m(x)\overline{\phi_n(x)}dx]$$
$$= \frac{1}{\beta\rho_n\beta\rho_m}\left[\rho_n + \lambda_{2,m}\overline{\lambda_{2,n}}\right]\delta_{n,m}$$

$$(\Phi_{2,n}, \Phi_{2,n}) = \frac{|\lambda_{2,n}|^2 + \rho_n}{(\beta\rho_n)^2} \approx 1$$

$$\begin{aligned}
&(\Phi_{1,m}, \Phi_{1,n}) \\
&= \frac{1}{\sqrt{\beta\rho_n}\sqrt{\beta\rho_m}} \int_0^1 [\phi''_m(x)\overline{\phi''_n(x)} \\
&\quad + \lambda_{1,m}\phi_m(x)\overline{\lambda_{1,n}\phi_n(x)}]dx \\
&= \left[\frac{1}{\sqrt{\beta\rho_n}\sqrt{\beta\rho_m}}\right] \cdot \\
&\quad \left[\int_0^1 \phi_m(x)\overline{\phi_n^{(4)}(x)}dx + \lambda_{1,m}\overline{\lambda_{1,n}} \int_0^1 \phi_m(x)\overline{\phi_n(x)}dx\right] \\
&= \frac{1}{\sqrt{\beta\rho_n}\sqrt{\beta\rho_m}} \left[\rho_n + \lambda_{1,m}\overline{\lambda_{1,n}}\right] \delta_{n,m}
\end{aligned}$$

$$(\Phi_{1,n}, \Phi_{1,n}) = \frac{|\lambda_{1,n}|^2 + \rho_n}{(\beta\rho_n)} \approx \frac{1}{\beta}$$

So G is a block diagonal matrix

$$G = \mathrm{diag}\left(\begin{pmatrix} (\Phi_{1,n}, \Phi_{1,n}) & (\Phi_{1,n}, \Phi_{2,n}) \\ (\Phi_{2,n}, \Phi_{1,n}) & (\Phi_{2,n}, \Phi_{2,n}) \end{pmatrix}_n\right).$$

$$\begin{aligned}
\Delta &= (\Phi_{1,n}, \Phi_{1,n})(\Phi_{2,n}, \Phi_{2,n}) - |(\Phi_{1,n}, \Phi_{2,n})|^2 \\
&= \frac{|\lambda_{1,n}|^2 + \rho_n}{(\beta\rho_n)} \frac{|\lambda_{2,n}|^2 + \rho_n}{(\beta\rho_n)^2} - \left|\frac{1+\alpha}{\beta\sqrt{\beta\rho_n}}\right|^2 \\
&\approx \frac{1}{\beta} - O(\rho_n^{-1/2})
\end{aligned}$$

Note that

$$\begin{aligned}
&\begin{pmatrix} (\Phi_{1,n}, \Phi_{1,n}) & (\Phi_{1,n}, \Phi_{2,n}) \\ (\Phi_{2,n}, \Phi_{1,n}) & (\Phi_{2,n}, \Phi_{2,n}) \end{pmatrix}^{-1} \\
&= \frac{1}{\Delta}\begin{pmatrix} (\Phi_{2,n}, \Phi_{2,n}) & -(\Phi_{1,n}, \Phi_{2,n}) \\ -(\Phi_{2,n}, \Phi_{1,n}) & (\Phi_{1,n}, \Phi_{1,n}) \end{pmatrix} \\
&= \frac{1}{\Delta}\begin{pmatrix} \frac{|\lambda_{2,n}|^2 + \rho_n}{(\beta\rho_n)^2} & -\frac{1+\alpha}{\beta\sqrt{\beta\rho_n}} \\ -\frac{1+\alpha}{\beta\sqrt{\beta\rho_n}} & \frac{|\lambda_{1,n}|^2 + \rho_n}{(\beta\rho_n)} \end{pmatrix} \\
&\approx \begin{pmatrix} \beta & -O(\rho_n^{-1/2}) \\ -O(\rho_n^{-1/2}) & 1 \end{pmatrix}
\end{aligned}$$

Therefore

$$G^{-1} = \mathrm{diag}\left(\begin{pmatrix} (\Phi_{1,n}, \Phi_{1,n}) & (\Phi_{1,n}, \Phi_{2,n}) \\ (\Phi_{2,n}, \Phi_{1,n}) & (\Phi_{2,n}, \Phi_{2,n}) \end{pmatrix}_n^{-1}\right).$$

is a bounded operator in ℓ^2. According to Theorem in [14, pp32, Theorem 9], $\{\Phi_{j,n}; j = 1, 2, n \in N\}$ forms a Riesz basis for $\mathcal{H}$.

Though the eigenvectors forms a Riesz basis, the operator $\mathcal{A}$ cannot generate a strongly continuous semigroup because it is not closed.

Lemma 3.3. Operator A is not closed and so it cannot generate a strongly continuous semigroup.

Proof

See that the set $H_0^4[0,1] := \{f \in H^4[0,1] \,|\, f(0) = f'(0) = 0\}$ is dense in the space $H_0^2[0,1] := \{f \in H^2[0,1] \,|\, f(0) = f'(0) = 0\}$ under the norm

$$\|f\|_2^2 := \int_0^1 |f(x)|^2 dx.$$

So, we can choose a sequence $f_n \in H_0^4[0,1]$ that converges to some $f \in H_0^2[0,1]$ but $f \notin H_0^4[0,1]$. Consider the vector

$$\mathbf{v}_n = \begin{pmatrix} f_n \\ -\frac{\alpha}{\beta} f_n \end{pmatrix}.$$

It is easily to check that $\mathbf{v}_n$ is in $D(\mathcal{A})$ and $\mathbf{v}_n$ converges to $\mathbf{v}$ with

$$\mathbf{v} = \begin{pmatrix} f \\ -\frac{\alpha}{\beta} f \end{pmatrix} \in \mathcal{H}.$$

We also see that

$$\mathcal{A}\mathbf{v}_n = \begin{pmatrix} -\frac{\alpha}{\beta} f_n \\ 0 \end{pmatrix}$$

which also converges in $\mathcal{H}$. But, $\mathbf{v}$ is not in the domain of $\mathcal{A}$. So, $\mathcal{A}$ is not closed, and hence, $\mathcal{A}$ cannot generate a strongly continuous semigroup by the Hille-Yosida Theorem.

4. CONCLUSION

We have exhibited a number of unusual properties for the smart material beam. Despite these difficulties, this beam system can be made into an exponentially stable system and we leave these details to a forthcoming full version of this paper.

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REFERENCES

[1] H.T. Banks, G.A. Pinter, L.K. Potter, M.J. Gaitens and L.C. Yanyo, Modeling of nonlinear hysteresis in elastomers under uniaxial tension, *Journal of Intelligent Material Systems and Structures*, 10(2)(1999), 116-134.

[2] H.T. Banks, R.C. Smith and Y. Wang, *Smart Material Structures: Modeling, Estimation and Control*, Masson, Paris, 1996.

[3] S.P. Chen, K.S. Liu and Z.Y. Liu, Spectrum and stability for elastic systems with global or local Kelvin-Voigt damping, *SIAM J. Appl. Math.*, 59(2)(1999), 651-668.

[4] S.S. Ge, T.H. Lee and J.Q. Gong, A robust distributed controller of a single!link SCARA/Cartesian smart materials robot, *Mechatronics*, 9(1999), 65-93.

[5] S.S. Ge, T.H. Lee and G. Zhu, Asymptotically stable end-point regulation of a flexible SCARA/Cartesian robot, *IEEE/ASME Transactions on mechatronics*, 3(2)1998, 138-144.

[6] L.M. Gearhart, Spectral theory for contraction semigroups on Hilbert space, *Trans. Amer. Math. Soc.*, 236(1978), 385-394.

[7] F.L. Huang, Characteristic condition for exponential stability of linear dynamic systems in Hilbert space, *Ann. Differential Euqations*, 1(1985), 43-56.
[8] V. Komornik, *Exact Controllability and Stabilization (the Multiplier Method)*, RAM: Research in Applied Mathematics, Masson, Paris; John Wiley & Sons, Ltd., Chichester, 1994.
[9] K.S. Liu, Locally distributed control and damping for the conservative systems, *SIAM J. Control Optim.*, 35(5)(1997), 1574-1590.
[10] K.S. Liu and Z.Y. Liu, Exponential decay of energy of the Euler-Bernoulli beam with locally distributed Kelvin-Voigt damping, *SIAM J. Control Optim.*, 36(3)(1998), 1086-1098.
[11] J.M. Wang, G.Q. Xu and S.P. Yung, Siu-Pang Exponential stability of variable coefficients Rayleigh beams under boundary feedback controls: a Riesz basis approach, *Systems Control Lett.*, 51(1)(2004), 33-50.
[12] J.M. Wang, G.Q. Xu and S.P. Yung, Riesz basis property, exponential stability of variable coefficient Euler-Bernoulli beams with indefinite damping, *IMA J. Appl. Math.*, 70(3)(2005), 459-477.
[13] J.M. Wang, G.Q. Xu and S.P. Yung, Exponential stabilization of laminated beams with structural damping and boundary feedback controls, *SIAM J. Control Optim.*, 44(5)(2005), 1575-1597.
[14] R.M. Young, *An Introduction to Nonharmonic Fourier Series*, Academic Press, Inc., London, 2001.