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New delay-dependent H_∞ control for systems with a time-varying delay ⋆

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Abstract: This paper focuses on H_∞ controller design for systems with a time-varying delay. By taking the relationship among the time-varying delay, its upper bound and their difference into account, an improved delay-dependent bounded real lemma (BRL) is proposed. The design method for H_∞ controllers is given using a modified cone complementary linearisation (CCL) algorithm with a new stopping condition. Numerical examples are given to demonstrate the effectiveness and the benefits of the proposed method.

1. INTRODUCTION

During the last decade, considerable attention has been devoted to the problem of delay-dependent stability, stabilization and H_∞ controller design for time-delay systems Park [1999], Moon *et al.* [2001], Fridman & Shaked [2003], Gu *et al.* [2003], Gao & Wang [2003], Lee *et al.* [2004], Han [2004], He *et al.* [2004a,b], Wu *et al.* [2004a], Jiang & Han [2005], Lin *et al.* [2006], Xu *et al.* [2006], He *et al.* [2007a,b]. For systems with time-varying delay, most of the delay-dependent criteria are based on four model transformations of the original system Fridman & Shaked [2003]. Recently, a free-weighting matrix method has been proposed to improve the delay-dependent stability results for systems with time-varying delay He *et al.* [2004b], Wu *et al.* [2004a], in which the bounding techniques on some cross product terms are not involved. Xu *et al.* [2006] employed this method to design the H_∞ controller. However, as pointed out in He *et al.* [2007a,b], some useful terms are ignored in the derivative of the Lyapunov-Krasovskii functional in the existing literatures such as Fridman & Shaked [2003], Han [2004], He *et al.* [2004b], Wu *et al.* [2004a] and Xu *et al.* [2006]. Although He *et al.* [2007a,b] retained these terms and proposed an improved delay-dependent stability criterion for systems with time-varying delay, there is room for further investigation. For instance, in Fridman & Shaked [2003], Han [2004], He *et al.* [2004b], Wu *et al.* [2004a], Xu *et al.* [2006] and He *et al.* [2007a,b], the delay term $d(t)$ with $0 \leq d(t) \leq h$ was often enlarged as h. Another term $h - d(t)$ was also regarded as h in He *et al.* [2007a,b]. That is, $h = d(t) + (h - d(t))$ was enlarged as $2h$, which may lead to conservativeness.

On the other hand, the obtained delay-dependent stabilization and H_∞ controller design conditions with memoryless state feedback using some improved methods cannot be expressed as strict LMI ones. Moon *et al.* [2001] modified the cone complementary linearisation (CCL) algorithm presented in Ghaoui *et al.* [1997] and proposed an LMI-based iterative algorithm to solve the problem of delay-dependent state feedback stabilization controller design. Later, this algorithm was extended to H_∞ state feedback control design in Gao & Wang [2003], Lee *et al.* [2004], Xu *et al.* [2006]. However, the modified CCL algorithm employed in Moon *et al.* [2001], Gao & Wang [2003], Lee *et al.* [2004] and Xu *et al.* [2006] has a drawback that its stopping conditions for iteration are very strict. The gain matrix and the other Lyapunov matrices derived in the previous step of iteration must satisfy one or more matrix inequalities in Moon *et al.* [2001], Gao & Wang [2003], Lee *et al.* [2004], Xu *et al.* [2006]. In fact, once the gain matrix is derived, the delay-dependent stabilization conditions devised using these methods are reduced to LMIs. Thus, the iteration can be stopped if the LMIs with the given gain matrix are feasible, in which the other Lyapunov matrices are the decision variables instead of the given ones.

In this paper, an improved delay-dependent bounded real lemma (BRL) is first presented for systems with a time-varying delay without ignoring any terms in the derivative of Lyapunov-Krasovskii functional. The obtained result takes the relationship between $d(t)$, h and $h - d(t)$ into account. Based on the obtained BRL, the H_∞ controller is designed using a modified CCL algorithm with a new stopping condition. Numerical examples are given to demonstrate the effectiveness and the merits of the proposed method.

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Notation: Throughout this paper, the superscripts '-1' and 'T' stand for the inverse and transpose of a matrix, respectively; $\mathcal{R}^n$ denotes the n-dimensional Euclidean space; $\mathcal{R}^{n\times m}$ is the set of all $n \times m$ real matrices; $P > 0$ means that the matrix P is positive definite; I is an appropriately dimensioned identity matrix; diag$\{\cdots\}$ denotes a block-diagonal matrix; and the symmetric terms in a symmetric matrix are denoted by $\star$, e.g., $\begin{bmatrix} X & Y \\ \star & Z \end{bmatrix} = \begin{bmatrix} X & Y \\ Y^T & Z \end{bmatrix}$.

2. PROBLEM FORMULATION

Consider the following linear system with time-varying delay:

$$\begin{cases} \dot{x}(t) = Ax(t) + A_d x(t-d(t)) + Bu(t) + B_\omega \omega(t),\ t > 0 \\ z(t) = \begin{bmatrix} Cx(t) + D_\omega \omega(t) \\ C_d x(t-d(t)) \\ Du(t) \end{bmatrix} \\ x(t) = \phi(t), t \in [-h, 0] \end{cases} \tag{1}$$

where $x(t) \in \mathcal{R}^n$ is the state vector, $u(t) \in \mathcal{R}^m$ is the controlled input, $\omega(t) \in \mathcal{L}_2^q[0,\infty)$ is the exogenous disturbance signal and $z(t) \in \mathcal{R}^r$ is the controlled output. A, A_d, B, B_ω, C, D_ω, C_d and D are constant matrices with appropriate dimensions, the time delay, $d(t)$, is a time-varying differential function that satisfies

$$0 \le d(t) \le h \tag{2}$$

and

$$\dot{d}(t) \le \mu \tag{3}$$

where $h > 0$ and μ are constants. The initial condition, $\phi(t)$, is a continuous vector-valued initial function of $t \in [-h, 0]$.

For a given scalar $\gamma > 0$, the performance of the system is defined to be

$$J(\omega) = \int_0^\infty (z^T(t)z(t) - \gamma^2 \omega^T(t)\omega(t))dt. \tag{4}$$

We are interested in finding a state-feedback gain, $K \in \mathcal{R}^{m\times n}$, in the control law

$$u(t) = Kx(t) \tag{5}$$

such that, for any delay $d(t)$ satisfying (2) and (3),

(i) the closed-loop system of (1)

$$\dot{x}(t) = (A + BK)x(t) + A_d x(t - d(t)) + B_\omega \omega(t) \tag{6}$$

is asymptotically stable under the condition $\omega(t) = 0$, $\forall t \ge 0$;

(ii)$J(\omega) < 0$ for all non-zero $\omega(t) \in \mathcal{L}_2^q[0,\infty)$ and a prescribed $\gamma > 0$ under the condition $x(t) = 0$, $\forall t \in [-h, 0]$.

3. BOUNDED REAL LEMMA

In the following, the terms ignored in He *et al.* [2004b], Wu *et al.* [2004a], Xu *et al.* [2006] are retained like those in He *et al.* [2007a,b]. However, $d(t)X$ with $X \ge 0$ is enlarged as hX and $(h - d(t))X$ is enlarged as hX. In fact, $d(t)X + (h - d(t))X$ is exactly equal to hX. In what follows, this characteristic is observed and a new delay-dependent BRL is presented.

Theorem 1. Given scalars $h \ge 0$, μ and $\gamma > 0$, system (1) with $u(t) = 0$ and a time-varying delay $d(t)$ satisfying (2) and (3) is asymptotically stable and satisfies $J(\omega) < 0$ for all non-zero $\omega(t) \in \mathcal{L}_2^q[0,\infty)$ under the condition $x(t) = 0$, $\forall t \in [-h, 0]$ if there exist matrices $P = P^T > 0$, $Q_i = Q_i^T \ge 0$, $i = 1, 2$, N, M, $Z = Z^T > 0$ and $X = X^T \ge 0$, such that the following LMIs hold,

$$\Phi = \begin{bmatrix} \Phi_1 + \Phi_2 + \Phi_2^T + hX & \sqrt{h}\Phi_3^T Z & \Phi_4^T & \Phi_5^T \\ \star & -Z & 0 & 0 \\ \star & \star & -I & 0 \\ \star & \star & \star & -I \end{bmatrix} < 0 \tag{7}$$

$$\Psi_1 = \begin{bmatrix} X & N \\ \star & Z \end{bmatrix} \ge 0 \tag{8}$$

$$\Psi_2 = \begin{bmatrix} X & M \\ \star & Z \end{bmatrix} \ge 0 \tag{9}$$

where

$$\Phi_1 = \begin{bmatrix} PA{+}A^TP{+}\sum_{i=1}^{2} Q_i & PA_d & 0 & PB_\omega \\ \star & -(1-\mu)Q_2 & 0 & 0 \\ \star & \star & -Q_1 & 0 \\ \star & \star & \star & -\gamma^2 I \end{bmatrix}$$

$$\Phi_2 = [N \quad M - N \quad -M \quad 0]$$
$$\Phi_3 = [A \quad A_d \quad 0 \quad B_\omega]$$
$$\Phi_4 = [C \quad 0 \quad 0 \quad D_\omega]$$
$$\Phi_5 = [0 \quad C_d \quad 0 \quad 0].$$

Proof. Choose a Lyapunov-Krasovskii functional candidate to be

$$\begin{aligned} V(x_t) = {} & x^T(t)Px(t) + \int_{t-h}^{t} x^T(s)Q_1 x(s)ds \\ & + \int_{t-d(t)}^{t} x^T(s)Q_2 x(s)ds \\ & + \int_{-h}^{0}\int_{t+\theta}^{t} \dot{x}^T(s)Z\dot{x}(s)dsd\theta \end{aligned} \tag{10}$$

where $P = P^T > 0$, $Q_i = Q_i^T \ge 0$, $i = 1, 2$ and $Z = Z^T > 0$, are to be determined.

Calculating the derivative of $V(x_t)$ along the solutions of system (1) yields

$$\begin{aligned} \dot{V}(x_t) \le {} & 2x^T(t)P\dot{x}(t) + x^T(t)(Q_1 + Q_2)x(t) \\ & -x^T(t-h)Q_1x(t-h) \\ & -(1-\mu)x^T(t-d(t))Q_2x(t-d(t)) \\ & +h\dot{x}^T(t)Z\dot{x}(t) - \int_{t-h}^{t} \dot{x}^T(s)Z\dot{x}(s)ds. \end{aligned} \tag{11}$$

From the Leibniz-Newton formula, the following equations are true for any matrices N and M, with appropriate dimensions,

$$0 = 2\zeta^T(t)N\left[x(t) - x(t-d(t)) - \int_{t-d(t)}^{t} \dot{x}(s)ds\right] \quad (12)$$

$$0 = 2\zeta^T(t)M\left[x(t-d(t)) - x(t-h) - \int_{t-h}^{t-d(t)} \dot{x}(s)ds\right] \quad (13)$$

where

$$\zeta(t)(t) = [x^T(t) \;\; x^T(t-d(t)) \;\; x^T(t-h) \;\; \omega^T(t)]^T.$$

On the other hand, for any appropriately dimensioned matrix $X = X^T \geq 0$, the following equality is true:

$$\begin{aligned} 0 &= \int_{t-h}^{t} \zeta^T(t)X\zeta(t)ds - \int_{t-h}^{t} \zeta^T(t)X\zeta(t)ds \\ &= h\zeta^T(t)X\zeta(t) - \int_{t-h}^{t-d(t)} \zeta^T(t)X\zeta(t)ds \\ &\quad - \int_{t-d(t)}^{t} \zeta^T(t)X\zeta(t)ds \end{aligned} \quad (14)$$

In addition, the following equations are also true:

$$\begin{aligned} &-\int_{t-h}^{t} \dot{x}^T(s)Z\dot{x}(s)ds \\ &= -\int_{t-d(t)}^{t} \dot{x}^T(s)Z\dot{x}(s)ds - \int_{t-h}^{t-d(t)} \dot{x}^T(s)Z\dot{x}(s)ds \end{aligned} \quad (15)$$

Adding the right sides of (12)-(14) into $\dot{V}(x_t)$ and using (15) yield

$$\begin{aligned} &\dot{V}(x_t) + z^T(t)z(t) - \gamma^2\omega^T(t)\omega(t) \\ &\leq \zeta^T(t)\left[\Phi_1+\Phi_2+\Phi_2^T+hX+h\Phi_3^TZ\Phi_3+\Phi_4^T\Phi_4+\Phi_5^T\Phi_5\right]\zeta(t) \\ &- \int_{t-d(t)}^{t} \xi^T(t,s)\Psi_1\xi(t,s)ds - \int_{t-h}^{t-d(t)} \xi^T(t,s)\Psi_2\xi(t,s)ds \end{aligned} \quad (16)$$

where

$$\xi(t,s) = [\zeta^T(t) \;\; \dot{x}^T(s)]^T.$$

Thus, if $\Psi_i \geq 0$, $i = 1,2$, and $\Phi_1 + \Phi_2 + \Phi_2^T + hX + h\Phi_3^TZ\Phi_3 + \Phi_4^T\Phi_4 + \Phi_5^T < 0$, which is equivalent to (7) by Schur complements, $\dot{V}(x_t) + z^T(t)z(t) - \gamma^2\omega^T(t)\omega(t) < 0$, which ensures $J(\omega) < 0$.

On the other hand, (7)-(9) imply that the following LMIs (17)-(19) hold, which guarantee $\dot{V}(x_t) < -\varepsilon\|x(t)\|^2$ for a sufficiently small $\varepsilon > 0$ such that system (1) with $u(t) = 0$ and $\omega(t) = 0$ is asymptotically stable,

$$\begin{bmatrix} \hat{\Phi}_1 + \hat{\Phi}_2 + \hat{\Phi}_2^T + h\hat{X} & \sqrt{h}\hat{\Phi}_3^TZ \\ \star & -Z \end{bmatrix} < 0 \quad (17)$$

$$\hat{\Psi}_1 = \begin{bmatrix} \hat{X} & \hat{N} \\ \star & Z \end{bmatrix} \geq 0 \quad (18)$$

$$\hat{\Psi}_2 = \begin{bmatrix} \hat{X} & \hat{M} \\ \star & Z \end{bmatrix} \geq 0 \quad (19)$$

where

$$\hat{\Phi}_1 = \begin{bmatrix} PA + A^TP + \sum_{i=1}^{2} Q_i & PA_d & 0 \\ \star & -(1-\mu)Q_2 & 0 \\ \star & \star & -Q_1 \end{bmatrix}$$

$$\hat{\Phi}_2 = \begin{bmatrix} \hat{N} & \hat{M} - \hat{N} & -\hat{M} \end{bmatrix}$$

$$\hat{\Phi}_3 = [A \;\; A_d \;\; 0]$$

and $P = P^T > 0$, $Q_i = Q_i^T \geq 0$, $i = 1,2$, $\hat{N}$, $\hat{M}$, $Z = Z^T > 0$ and $\hat{X} = \hat{X}^T \geq 0$, are decision variables. This completes the proof.

Remark 2. In many cases, the information on the derivative of delay is unknown. Regarding this circumstance, a rate-independent bounded real lemma for a delay only satisfying (2) can be derived by choosing $Q_2 = 0$ in Theorem 1. In what follows, the corresponding results for the derivative of delay being unknown can be derived following the similar line.

From the proof procedure of Theorem 1, we have the result regarding the stability of system (1) with $u(t) = 0$ and $\omega(t) = 0$.

Corollary 3. Given scalars $h \geq 0$ and μ, the system (1) with $u(t) = 0$, $\omega(t) = 0$ and a time-varying delay $d(t)$ satisfying (2) and (3) is asymptotically stable if there exist matrices $P = P^T > 0$, $Q_i = Q_i^T \geq 0$, $i = 1,2$, $\hat{N}$, $\hat{M}$, $Z = Z^T > 0$ and $\hat{X} = \hat{X}^T \geq 0$, such that LMIs (17)-(19) are feasible.

Remark 4. In fact, if $Q_2 = Q$, $Q_1 = \varepsilon I$, with $\varepsilon > 0$ being sufficient small scalars, $\hat{M} = 0$, and $\hat{N} = \begin{bmatrix} Y^T & T^T & 0 \end{bmatrix}^T$, $X = \begin{bmatrix} X_{11} & X_{12} & 0 \\ \star & X_{22} & 0 \\ \star & \star & 0 \end{bmatrix} \geq 0$, Corollary 3 yields Theorem 2 in Wu *et al.* [2004a].

4. STATE FEEDBACK H_∞ CONTROLLER DESIGN

In this section, Theorem 1 is extended to design an H_∞ controller for system (1) under control law (5).

Theorem 5. Given scalars $h \geq 0$, μ and $\gamma > 0$, closed-loop system (6) with a time-varying delay $d(t)$ satisfying (2) and (3) is asymptotically stable and satisfies $J(\omega) < 0$ for all non-zero $\omega(t) \in \mathcal{L}_2^q[0,\infty)$ under the condition $x(t) = 0$, $\forall t \in [-h, 0]$ if there exist matrices $L = L^T > 0$, $R_i = R_i^T \geq 0$, $i = 1,2$, $Y = Y^T$, $W = W^T > 0$, S, T and V, such that the following matrix inequalities hold,

$$\begin{bmatrix} \Xi_1 + \Xi_2 + \Xi_2^T + hY & \sqrt{h}\Xi_3^T & \Xi_4^T & \Xi_5^T & \Xi_6^T \\ \star & -W & 0 & 0 & 0 \\ \star & \star & -I & 0 & 0 \\ \star & \star & \star & -I & 0 \\ \star & \star & \star & \star & -I \end{bmatrix} < 0 \quad (20)$$

$$\begin{bmatrix} Y & S \\ \star & LW^{-1}L \end{bmatrix} \geq 0 \quad (21)$$

$$\begin{bmatrix} Y & T \\ \star & LW^{-1}L \end{bmatrix} \geq 0 \quad (22)$$

where

$$\Xi_1 = \begin{bmatrix} \Xi_{11} & A_d L & 0 & B_\omega \\ \star & -(1-\mu)R_2 & 0 & 0 \\ \star & \star & -R_1 & 0 \\ \star & \star & \star & -\gamma^2 I \end{bmatrix}$$

$$\Xi_{11} = AL + LA^T + BV + V^T B^T + \sum_{i=1}^{2} R_i$$

$$\begin{aligned} \Xi_2 &= [S \;\; T-S \;\; -T \;\; 0] \\ \Xi_3 &= [AL+BV \;\; A_d L \;\; 0 \;\; B_\omega] \\ \Xi_4 &= [CL \;\; 0 \;\; 0 \;\; D_\omega] \\ \Xi_5 &= [0 \;\; C_d L \;\; 0 \;\; 0] \\ \Xi_6 &= [DV \;\; 0 \;\; 0 \;\; 0]. \end{aligned}$$

Moreover, a stabilizing H_∞ controller is given by $u(t) = VL^{-1}x(t)$.

Proof. From Theorem 1, it is clear that closed-loop system (6) with a time-varying delay $d(t)$ satisfying (2) and (3) is asymptotically stable and satisfies $J(\omega) < 0$ for all non-zero $\omega(t) \in \mathcal{L}_2^q[0,\infty)$ under the condition $x(t) = 0, \ \forall t \in [-h, 0]$ if there exist matrices $P = P^T > 0$, $Q_i = Q_i^T \geq 0, \ i = 1, 2, \ N, \ M, \ Z = Z^T > 0$ and $X = X^T \geq 0$, such that matrix inequalities (8)-(9) and (23) hold

$$\tilde{\Phi} = \begin{bmatrix} \tilde{\Phi}_1 + \Phi_2 + \Phi_2^T + hX & \sqrt{h}\tilde{\Phi}_3^T Z & \Phi_4^T & \Phi_5^T & \tilde{\Phi}_6^T \\ \star & -Z & 0 & 0 & 0 \\ \star & \star & -I & 0 & 0 \\ \star & \star & \star & -I & 0 \\ \star & \star & \star & \star & -I \end{bmatrix} < 0 \quad (23)$$

where

$$\tilde{\Phi}_1 = \begin{bmatrix} \tilde{\Phi}_{11} & PA_d & 0 & PB_\omega \\ \star & -(1-\mu)Q_2 & 0 & 0 \\ \star & \star & -Q_1 & 0 \\ \star & \star & \star & -\gamma^2 I \end{bmatrix}$$

$$\tilde{\Phi}_{11} = P(A+BK) + (A+BK)^T P + \sum_{i=1}^{2} Q_i$$

$$\begin{aligned} \tilde{\Phi}_3 &= [A+BK \;\; A_d \;\; 0 \;\; B_\omega] \\ \tilde{\Phi}_6 &= [DK \;\; 0 \;\; 0 \;\; 0] \end{aligned}$$

and the other parameters are defined in Theorem 1. Define

$$\begin{aligned} \Pi &= \mathrm{diag}\{P^{-1}, \ P^{-1}, \ P^{-1}, \ I\} \\ \Theta &= \mathrm{diag}\{\Pi, \ Z^{-1}, \ I, \ I, \ I\}. \end{aligned}$$

Pre- and post-multiply $\tilde{\Phi}$ in (23) by Θ and Θ, respectively, and pre- and post-multiply $\Psi_i, \ i = 1, 2$ in (8)-(9) by $\mathrm{diag}\{\Pi, \ L\}$ and $\mathrm{diag}\{\Pi, \ L\}$, respectively, and make the following changes to the variables,

$$\begin{aligned} &L := P^{-1}, \ V := KL, \ S := \Pi NL, \ T := \Pi ML \\ &R_i := LQ_iL, \ i = 1, 2, Y := \Pi X \Pi, \ W := Z^{-1}. \end{aligned}$$

So, (20)-(22) can be derived using Schur complement. This completes the proof.

It is noted that the conditions in Theorem 5 are no longer LMI ones due to the term $LW^{-1}L$ in (21)-(22). An appropriate state feedback controller gain matrix K cannot be found using a convex optimization algorithm. However, as mentioned in Moon *et al.* [2001], the cone complementary linearisation (CCL) algorithm proposed in Ghaoui *et al.* [1997] can be employed to solve this non-convex problem.

Define new variables U such that $LW^{-1}L \geq U$, and replace conditions (21), (22) with

$$\begin{bmatrix} Y & S \\ \star & U \end{bmatrix} \geq 0 \quad (24)$$

$$\begin{bmatrix} Y & T \\ \star & U \end{bmatrix} \geq 0 \quad (25)$$

and

$$LW^{-1}L \geq U. \quad (26)$$

(26) is equivalent to $L^{-1}WL^{-1} \leq U^{-1}$, which is expressed as

$$\begin{bmatrix} U^{-1} & L^{-1} \\ L^{-1} & W^{-1} \end{bmatrix} \geq 0 \quad (27)$$

using Schur complements. Thus, by introducing new variables $P, \ H, \ Z$, the original conditions (21)-(22) are represented as (24)-(25) and

$$\begin{bmatrix} H & P \\ P & Z \end{bmatrix} \geq 0, \ P = L^{-1}, \ H = U^{-1}, \ Z = W^{-1}. \quad (28)$$

Then, this non-convex problem is converted to the following LMI-based nonlinear minimization problem:

Minimize $\mathrm{tr}\{LP + UH + WZ\}$
subject to (20), (24), (25) and

$$\begin{bmatrix} H & P \\ P & Z \end{bmatrix} \geq 0, \ \begin{bmatrix} L & I \\ I & P \end{bmatrix} \geq 0, \quad (29)$$

$$\begin{bmatrix} U & I \\ I & H \end{bmatrix} \geq 0, \ \begin{bmatrix} W & I \\ I & Z \end{bmatrix} \geq 0. \quad (30)$$

Then, the minimum H_∞ performance γ_{min} can be found for a given $h \geq 0$ through the following algorithm.

Algorithm 1. Step 1. Choose a sufficiently large initial $\gamma > 0$ such that there exists a feasible solution to (20), (24), (25), (29)and (30). Set $\gamma_{min} = \gamma$.

Step 2. Find a feasible set $(P_0, L_0, R_{10}, R_{20}, S_0, T_0, \ Y_0, \ Z_0, \ W_0, \ U_0, \ H_0, \ V_0)$ satisfying (20), (24), (25), (29) and (30). Set $k = 0$.

Step 3. Solve the following LMI problem for the variables $(P, L, R_1, R_2, S, \ T, \ Y, \ Z, W, \ U, \ H, \ V)$

Minimize $tr\{LP_k + L_kP + UH_k + U_kH + WZ_k + W_kZ\}$

subject to (20), (24), (25), (29) and (30).

Set $P_{k+1} = P$, $L_{k+1} = L$, $U_{k+1} = U$, $H_{k+1} = H$, $W_{k+1} = W$, $Z_{k+1} = Z$.

Step 4. If LMIs (8), (9) and (23) are feasible with a given K derived in Step 3 for the variables $P, Q_1, Q_2, N, M, \ X, \ Z$, then set $\gamma_{min} = \gamma$ and return to Step 2 after decreasing γ to some extent. If LMIs (8), (9) and (23) are infeasible within a specified number of iterations, then exit. Otherwise, set $k = k + 1$ and go to Step 3.

Remark 6. It is noted that the stopping conditions for iteration at the beginning of step 4 in Moon *et al.* [2001] and Gao & Wang [2003] are very strict. The gain matrix K and the other decision variables such as L, R_1, R_2, Y, W, S, T obtained in the previous step must satisfy matrix inequalities (20)-(22), which are equivalent to that the matrices

Table 1. Controller gains and No. of iterations using Algorithm 1 for $\gamma = 0.1287$ in Example 7.

h	Feedback gain obtained by Algorithm 1	No. of Iterations		
		Algorithm 1	Lee *et al.* [2004]	Xu *et al.* [2006]
1.1	[-0.1718 -32.0748]	2	19	16
1.2	[-0.1228 -33.6992]	2	32	22
1.25	[-0.0905 -35.0062]	2	86	29
1.40	[0.0009 -19.0760]	7	–	–

P, Q_1, Q_2, X, Z, N, M obtained in the previous step must satisfy (8), (9) and (23) with the given K. In fact, once K is derived, the conditions in Theorem 5 are reduced to LMIs for P, Q_1, Q_2, X, Z, N, M. Thus, in Algorithm 1, the stopping conditions for iteration are modified to verify whether LMIs (8), (9) and (23) are feasible, which may provide more freedom to select the variables such as P, Q_1, Q_2, X, Z, N, M.

5. NUMERICAL EXAMPLES

In this section, two numerical examples are used to show the benefits of the proposed method.

Example 7. Consider system (1) with

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},\ A_d = \begin{bmatrix} -1 & -1 \\ 0 & -0.9 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ B_\omega = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
$$C = [\,0\ 1\,],\ D_\omega = [\,0\,],\ C_d = [\,0\ 0\,],\ D = [\,0.1\,]$$

For constant delay $\mu = 0$ and $\gamma = 0.1287$, it is reported in Gao & Wang [2003], Lee *et al.* [2004] and Xu *et al.* [2006] that the system can be stabilized for $0 \le h \le 1.25$, $0 \le h \le 1.25$ and $0 \le h \le 1.38$, respectively. However, the system can be stabilized for $0 \le h \le 1.40$ using Algorithm 1. Moreover, the number of iterations for some given h and controller gains are listed in Table 1.

As for time-varying delay, the obtained H_∞ performance γ_{min} of the closed-loop system for some given h is listed in Table 2. It is noted that our methods provide improved results over those in Xu *et al.* [2006] because not only an improved bounded real lemma has been established but also a new algorithm has been presented.

Table 2. Obtained γ_{min} for $h = 1$ and various μ in Example 7.

μ	Xu *et al.* [2006]	Algorithm 1
$\mu = 0.5$	0.117	0.111
unknown μ	–	0.118

Example 8. Consider the stability of system (1) with $u(t) = 0$ and $\omega(t) = 0$ and

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix},\ A_d = \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix}.$$

The computed upper bounds, h, which guarantee the stability of system (1) with $u(t) = 0$ and $\omega(t) = 0$ for various μ, are listed in Table 3. It is clear that our results have improvement over those in Fridman & Shaked [2003], Han [2004], He *et al.* [2004b], Wu *et al.* [2004a], Jiang & Han [2005], He *et al.* [2007a] and He *et al.* [2007b].

Table 3. Allowable upper bounds of h for various μ in Example 8.

μ	0.5	0.9	unknown μ
Fridman & Shaked [2003] Han [2004] He *et al.* [2004b] Wu *et al.* [2004a]	0.99	0.56	0.56
Jiang & Han [2005]	–	–	0.67
He *et al.* [2007a,b]	1.08	0.77	0.77
Corollary 3	1.26	1.06	1.06

6. CONCLUSIONS

A new delay-dependent BRL has been established without ignoring any terms in the derivative of Lyapunov-Krasovskii functional by considering the relationship among the time-varying delay, its upper bound and their difference. Based on the derived BRL, the H_∞ controller has been designed using a new CCL algorithm with a modified stopping condition which is less strict than the existing ones. Two numerical examples have verified the less conservativeness.

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