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Lyapunov Function Design using Quantization of Markov Process

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Abstract: In this paper, we propose a Lyapunov function design method using a difference approximation scheme and quantization of the Markov process. First, we approximate a Lyapunov equation by a Schrödinger-like equation. Second, we obtain sufficient conditions for a function to be a Lyapunov function. Then, we provide a Lyapunov function design procedure.

1. INTRODUCTION

For nonlinear control systems, the existence of control Lyapunov functions assures that the origin is asymptotically stabilizable. In the previous works, e.g., [3, 12], some stabilizing controllers were proposed under the assumption that control Lyapunov functions are given. However, it is very hard to construct control Lyapunov functions.

For nonlinear systems without inputs or closed-loop systems, the existence of Lyapunov functions implies that the origin is asymptotically stable. While Lyapunov function design problems are studied by Krasovskii [5, 11], Schultz [5], Zubov [11], Vannelli-Vidyasagar [14], and so on, the problems are not fully resolved yet. Lyapunov functions are useful even for controller design as follows.

If Lyapunov functions are constructed for closed-loop systems with ad-hoc controllers, we can obtain better controllers (such as inverse optimal controllers) by employing these Lyapunov functions as control Lyapunov functions.

In this paper, we propose a Lyapunov function design method using a difference approximation scheme and quantization of the Markov process. In Section 2, we introduce some definitions and previous results. In Section 3, we describe a problem. In Section 4, we approximate a Lyapunov equation by a Schrödinger-like equation. Moreover, we describe the general solution of the approximate Lyapunov equation by using eigenvalues and eigenfunctions of the Hamiltonian operator. In Section 5, we obtain sufficient conditions for a function to be an approximate Lyapunov function. In Section 6, we provide an approximate Lyapunov function design procedure. In Section 7, we confirm the effectiveness of the proposed method by an example. Section 8 concludes this paper.

Note that this paper is different from researches for control problems of quantum systems; e.g., Rosenbrock [10].

2. PRELIMINARY DISCUSSION

We introduce definitions and show previous works.

Let $\mathbb{R}^+$ be the set of positive real numbers, $\mathbb{N}^+$ be the set of positive integer numbers, and $\mathbb{N}_0$ be the set of non-negative integer numbers.

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ and $x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n$. We consider the following ordinary differential equation:

$$\dot{x} = f(x) := (f_1(x), f_2(x), \ldots, f_n(x))^T \tag{1}$$

and assume that the origin is locally asymptotically stable.

2.1 Definitions

The origin of the system (1) is locally asymptotically stable if and only if a Lyapunov function exists [5].

Definition 1. (Lyapunov function). Let M be a subspace satisfying $0 \in M \subset \mathbb{R}^n$. A C^1 positive define function $W : M \to \mathbb{R}$ is said to be a Lyapunov function of the system (1) if $\dot{W}(x)$ is negative definite.

Definition 2. (Lyapunov equation). Let $q : \mathbb{R}^n \to \mathbb{R}^+$ and $V : \mathbb{R}^n \times [0, \infty) \to \mathbb{R}$ be a function such that $\dot{V}(x,t) = -q(x)V(x,t)$. Then,

$$\frac{\partial V}{\partial t}(x,t) = -q(x)V(x,t) - \frac{\partial V}{\partial x}(x,t) \cdot f(x) \tag{2}$$

is said to be a Lyapunov equation. □

If $V(x,t)$ is a time-invariant positive definite function satisfying (2), $W(x) := V(x,t)$ is a Lyapunov function of the system (1).

Definition 3. (discrete state space). We define a discrete state space by

$$M_d := \left\{ \delta \sum_{i=1}^{n} \gamma_i e_i,\ \forall \gamma_i \in \mathbb{Z} \right\} = \{x_d, {x_d}', {x_d}'', \ldots\}, \tag{3}$$

where δ is a spatial step and $e_1, e_2, \ldots, e_n$ are orthogonal bases of the state vector. □

Definition 4. (transition probability). We define $p((x_d,t) \to ({x_d}', t'))$ as a transition probability from (x_d, t) to $({x_d}', t')$. □

Definition 5. (transition probability rate). We define a transition probability rate by

$$w(x_d \to {x_d}', t) := \lim_{h \to 0} \frac{p((x_d,t) \to ({x_d}', t+h)) - p((x_d,t) \to ({x_d}', t))}{h}. \tag{4}$$

When the transition probability rate is time-invariant, $w(x_d \to {x_d}')$ denotes $w(x_d \to {x_d}', t)$. □

Definition 6. (master equation formalism). Let w_1 and w_2 be time-invariant transition probability rates and $V : M_d \times [0, \infty) \to \mathbb{R}$. Then, the equation

$$\frac{\partial}{\partial t} V(x_d, t) = \sum_{x_d' \in M} \{ w_1(x_d \to x_d') V(x_d', t) - w_2(x_d' \to x_d) V(x_d, t) \} \tag{5}$$

is said to be a "master equation." □

Definition 7. (bra-ket notation). Let $\psi_1, \psi_2 : M_d \to \mathbb{C}^n$. The bracket $\langle \psi_1 | \psi_2 \rangle$ denotes the inner product of $\psi_1(x_d)$ and $\psi_2(x_d)$, and

$$\langle \psi_1 | \mathcal{O} | \psi_2 \rangle := \sum_{x_d' \in M_d} \psi_1^*(x_d') (\mathcal{O} \psi_2)(x_d'), \tag{6}$$

where $\psi_1^*(x_d')$ is the complex conjugate of $\psi_1(x_d')$, and $\mathcal{O}$ is an operator. A vector $\langle \cdots |$ is called a bra vector, and $| \cdots \rangle$ is called a ket vector. □

When we write $\langle x_d |$ or $| x_d \rangle$, x_d denotes $\delta_{x_d, x_d'}$, i.e., $\langle x_d | \psi \rangle = \psi(x_d)$.

For a function $V : M_d \times [0, \infty) \to \mathbb{R}$, we define

$$| \bar{V}(t) \rangle := \sum_{x_d \in M_d} V(x_d, t) | x_d \rangle. \tag{7}$$

Then,

$$V(x_d, t) = \langle x_d | \bar{V}(t) \rangle \tag{8}$$

is satisfied.

Definition 8. (Schrödinger-like equation). An operator $\mathcal{H}$ is called a Hamiltonian operator if

$$\frac{\partial}{\partial t} | \bar{V}(t) \rangle = -\mathcal{H} | \bar{V}(t) \rangle. \tag{9}$$

The equation (9) is called a Schrödinger-like equation. □

The Schrödinger-like equation (9) can be expressed by a matrix equation because (9) is linear.

Definition 9. (Hamiltonian matrix). A matrix $\mathcal{H}_m$ is called a Hamiltonian matrix if

$$\mathcal{H}_m := \begin{bmatrix} \langle x_d | \mathcal{H} | x_d \rangle & \langle x_d | \mathcal{H} | x_d' \rangle & \cdots \\ \langle x_d' | \mathcal{H} | x_d \rangle & \langle x_d' | \mathcal{H} | x_d' \rangle & \\ \vdots & & \ddots \end{bmatrix}. \tag{10}$$

□

Lemma 1. The Schrödinger-like equation (9) is equivalent to the following matrix equation:

$$\begin{bmatrix} \langle x_d | \mathcal{H} | \bar{V}(t) \rangle \\ \langle x_d' | \mathcal{H} | \bar{V}(t) \rangle \\ \vdots \end{bmatrix} = -\mathcal{H}_m \cdot \begin{bmatrix} \langle x_d | \bar{V}(t) \rangle \\ \langle x_d' | \bar{V}(t) \rangle \\ \vdots \end{bmatrix}. \tag{11}$$

♦

Proof . By (7)-(9),

$$\begin{aligned} \langle x_d | \frac{\partial}{\partial t} | \bar{V}(t) \rangle &= -\langle x_d | \mathcal{H} | \bar{V}(t) \rangle \\ &= -\langle x_d | \mathcal{H} \sum_{x_d' \in M_d} \langle x_d' | \bar{V}(t) \rangle | x_d' \rangle \\ &= -\sum_{x_d' \in M_d} \langle x_d | \mathcal{H} | x_d' \rangle \langle x_d' | \bar{V}(t) \rangle. \end{aligned} \tag{12}$$

Therefore,

$$\begin{bmatrix} \langle x_d | \mathcal{H} | \bar{V}(t) \rangle \\ \langle x_d' | \mathcal{H} | \bar{V}(t) \rangle \\ \vdots \end{bmatrix} = -\begin{bmatrix} \langle x_d | \mathcal{H} | x_d \rangle & \langle x_d | \mathcal{H} | x_d' \rangle & \cdots \\ \langle x_d' | \mathcal{H} | x_d \rangle & \langle x_d' | \mathcal{H} | x_d' \rangle & \\ \vdots & & \ddots \end{bmatrix} \cdot \begin{bmatrix} \langle x_d | \bar{V}(t) \rangle \\ \langle x_d' | \bar{V}(t) \rangle \\ \vdots \end{bmatrix}. \tag{13}$$

We obtain (11) by (10) and (13). □

Remark 1. Eigenvalues of a Hamiltonian matrix $\mathcal{H}_m$ coincide with eigenvalues of a Hamiltonian operator $\mathcal{H}$. Eigenvectors of $\mathcal{H}_m$ consist of values of eigenfunctions of $\mathcal{H}$: i.e.,

$$\phi_j^v = (\phi_j(x_d), \phi_j(x_d'), \phi_j(x_d''), \ldots)^T, \quad j \in \mathbb{N}_0, \tag{14}$$

where ϕ_j^v denotes an eigenvector of $\mathcal{H}_m$ and ϕ_j denotes an eigenfunction of $\mathcal{H}$. ♦

Remark 2. The Hamiltonian operator of the Schrödinger-like equation (9) generally has complex eigenvalues and eigenfunctions because $\mathcal{H}$ is non-Hermite [1, 9]. The fact is different from the case of quantum mechanics [13]. ♦

2.2 Quantization of Markov Process ([9], p.157)

We can immediately extend the results in [9] as the followings:

Lemma 2. If a Hamiltonian operator $\mathcal{H}$ satisfies

$$\langle x_d | \mathcal{H} = \sum_{x_d' \in M_d} \{ -w_1(x_d' \to x_d) \langle x_d' | + w_2(x_d \to x_d') \langle x_d | \}, \tag{15}$$

the master equation (5) is equivalent to the Schrödinger-like equation (9). ♦

Lemma 3. Let $\mathcal{H}$ be a Hamiltonian operator in Lemma 2. Then, the elements of Hamiltonian matrix $\mathcal{H}_m$ are obtained as the followings:

$$\langle x_d | \mathcal{H} | x_d'' \rangle = -w_1(x_d \to x_d'') \quad \text{if} \quad x_d'' \neq x_d, \tag{16}$$

$$\langle x_d | \mathcal{H} | x_d \rangle = -w_1(x_d \to x_d) + \sum_{x_d' \in M_d} w_2(x_d' \to x_d). \tag{17}$$

♦

If $w_1 = w_2$, Lemma 2 and Lemma 3 consist with Rajewsky's results [9]. The transformation from the master equation (5) to the Schrödinger-like equation (9) is called quantization of Markov process.

2.3 Difference Approximation ([2], IX.3, [6], 177/178)

We show the approximation of the Lyapunov equation (2) introduced by Kushner [6], Fleming and Soner [2]. Although their original method is constructed for optimal control of a stochastic system, we consider a deterministic system without inputs.

We discretize the state space $\mathbb{R}^n$ into M_d and define the approximation of the partial derivative of $V(x, t)$ in x as

$$\frac{\partial V}{\partial x_i}(x, t) \approx \begin{cases} V_i^+(x_d, t) & \text{if} \quad f_i(x_d) \geq 0 \\ V_i^-(x_d, t) & \text{if} \quad f_i(x_d) < 0 \end{cases}, \tag{18}$$

where

$$V_i^+(x_d, t) := \frac{V(x_d + \delta e_i, t) - V(x_d, t)}{\delta}, \tag{19}$$

$$V_i^-(x_d, t) := \frac{V(x_d, t) - V(x_d - \delta e_i, t)}{\delta}. \tag{20}$$

Moreover, we define the approximation of the partial derivative of $V(x, t)$ in t as

$$\frac{\partial V}{\partial t}(x, t) \approx V_t(x_d, t) := \frac{V(x_d, t) - V(x_d, t - h)}{h}. \tag{21}$$

According to [2, 6], discretization of system (1) with the spatial step δ and the time step h yields transition probabilities of (1) as follows:

$$p((x_d,t)\to(x_d+\delta e_i,t+h)) = \frac{h}{\delta}\max(f_i(x_d),0), \quad (22)$$

$$p((x_d,t)\to(x_d-\delta e_i,t+h)) = \frac{h}{\delta}\max(-f_i(x_d),0), \quad (23)$$

$$p((x_d,t)\to(x_d,t+h)) = h - \frac{h}{\delta}\sum_{i=1}^{n}|f_i(x_d)|, \quad (24)$$

$$p((x_d,t)\to(x_d',t+h)) = 0 \text{ if } x_d' \neq x_d, x_d+\delta e_i, x_d-\delta e_i. \quad (25)$$

Lemma 4. If we use the approximations (18), (21), and transition probabilities (22)-(25), the Lyapunov equation (2) is approximated by

$$V_t(x_d,t) = -q(x_d)V(x_d,t) - \frac{1}{h}\sum_{x_d'\in M_d} p((x_d,t)\to(x_d',t+h))V(x_d',t). \quad (26)$$

♦

Equations (22)-(25) and Lemma 4 are modified for a deterministic system without inputs. If we replace $q(x)V(x,t)$ by $L:\mathbb{R}^n\times U\times[0,\infty)\to\mathbb{R}$, Lemma 4 coincides with the original result in [2, 6], where U is an input space.

3. PROBLEM FORMULATION

The purpose of this paper is to propose a Lyapunov function design method for the system (1).

In Section 4, we approximate a Lyapunov equation as follows. First, we discretize the state space and derive a difference approximation of a Lyapunov equation by the method in Subsection 2.3. Then, we show that the approximate Lyapunov equation is equivalent to a Schrödinger-like equation by the scheme in Subsection 2.2. Moreover, we describe the general solution of the approximate Lyapunov equation by using eigenvalues and eigenfunctions of the Hamiltonian operator.

In Section 5, we obtain sufficient conditions for a function to be an approximate Lyapunov function.

In Section 6, we propose an approximate Lyapunov function design method as follows. First, we extract finite elements from the discrete state space and describe the Hamiltonian matrix by a block tridiagonal matrix. Then, we provide an approximate Lyapunov function design procedure.

In Section 7, we confirm the effectiveness of the proposed method by an example. Section 8 concludes this paper.

While the Lyapunov equation (2) is different from a general case, the fact does not put any restriction for the system (1) because the following theorem is held:

Theorem 1. There exists a Lyapunov function of (1) in M if and only if there exists a Lyapunov function of (1) in M satisfying (2). ♦

Proof . Let t_0 be an initial time, and x_0 be an initial state. If $W(x)$ is a Lyapunov function of (1) in M,

$$W'(x) := e^{-\int_{t_0}^{t} q(x(s))ds}W(x_0), \quad \forall x_0\in M \quad (27)$$

is a positive definite function defined in M. Moreover,

$$\frac{d}{dt}W'(x) = -q(x)W'(x),\ x\in M. \quad (28)$$

Therefore, $W'(x)$ is a Lyapunov function satisfying (2). The converse is trivial. □

4. APPROXIMATION OF LYAPUNOV EQUATION

We suggest a new approximation of the Lyapunov equation (2) by using methods in Subsections 2.2 and 2.3.

4.1 Approximate Lyapunov Equation

We discretize the state space $\mathbb{R}^n$ into M_d and consider the transition probabilities (22)-(25) and transition probability rate (4). Then, we obtain

$$w(x_d\to x_d+\delta e_i) = \frac{1}{\delta}\max(f_i(x_d),0), \quad (29)$$

$$w(x_d\to x_d-\delta e_i) = \frac{1}{\delta}\max(-f_i(x_d),0), \quad (30)$$

$$w(x_d\to x_d) = -\frac{1}{\delta}\sum_{i=1}^{n}|f_i(x_d)|, \quad (31)$$

$$w(x_d\to x_d') = 0 \quad \text{if} \quad x_d'\neq x_d, x_d+\delta e_i, x_d-\delta e_i. \quad (32)$$

When $h\to 0$, (26) becomes

$$\frac{\partial}{\partial t}V(x_d,t) = -\sum_{x_d'\in M_d} w(x_d\to x_d')V(x_d',t) - q(x_d)V(x_d,t) \quad (33)$$

with transition probability rates (29)-(32).

Definition 10. (approximate Lyapunov equation). The equation (33) is said to be an approximate Lyapunov equation. □

We define

$$w^\sharp(x_d'\to x_d) := w(x_d'\to x_d) \quad \text{if} \quad x_d'\neq x_d, \quad (34)$$

$$w^\sharp(x_d\to x_d) := q(x_d) - \sum_{x_d'\neq x_d} w^\sharp(x_d'\to x_d) \quad (35)$$

with (29)-(32) and consider a Hamiltonian operator $\mathcal{H}$ defined by

$$\langle x_d|\mathcal{H} = \sum_{x_d'\in M_d}\{w(x_d'\to x_d)\langle x_d'| + w^\sharp(x_d\to x_d')\langle x_d|\}. \quad (36)$$

Then, we obtain the following Lemma:

Lemma 5. The approximate Lyapunov equation (33) with (29)-(32) is equivalent to the Schrödinger-like equation (9) with $\mathcal{H}$ defined by (36). ♦

Proof . By (34) and (35), we obtain

$$q(x_d) = \sum_{x_d'\in M_d} w^\sharp(x_d'\to x_d). \quad (37)$$

By (33) and (37),

$$\frac{\partial}{\partial t}V(x_d,t) = -\sum_{x_d'\in M_d}\{w(x_d\to x_d')V(x_d',t) + w^\sharp(x_d'\to x_d)V(x_d,t)\}. \quad (38)$$

Equation (38) coincides with the Schrödinger-like equation (9) by Lemma 2 with $w_1 = -w$ and $w_2 = w^\sharp$. □

4.2 Solution of Approximate Lyapunov Equation

We show that the general solution of the approximate Lyapunov equation (33) with (29)-(32) can be represented by using eigenvalues and eigenfunctions of a Hamiltonian operator. Let

$$E_j := E_{jR} + \mathbf{i}E_{jI},\ E_j^* := E_{jR} - \mathbf{i}E_{jI} \tag{39}$$

be eigenvalues of $\mathcal{H}$ with (36) and

$$\begin{aligned} \phi_j(x_d) &:= \phi_{jR}(x_d) + \mathbf{i}\phi_{jI}(x_d) \\ \phi_j^*(x_d) &:= \phi_{jR}(x_d) - \mathbf{i}\phi_{jI}(x_d) \end{aligned} \tag{40}$$

be eigenfunctions corresponding to E_j, E_j^*, where $j \in \mathbb{N}_0$, $E_{jR}, E_{jI} \in \mathbb{R}$, $\phi_{jR}, \phi_{jI} : \mathbb{R}^n \to \mathbb{R}$, $\mathbf{i}$ is the imaginary unit, and $|E_0| \le |E_1| \le \cdots$.

Theorem 2. The general solution of the approximate Lyapunov equation (33) with (29)-(32) is represented by

$$V^\natural(x_d, t) = \sum_{j\in\mathbb{N}_0} \{C_j e^{-E_j t}\phi_j(x_d) + C_j^* e^{-E_j^* t}\phi_j^*(x_d)\}, \tag{41}$$

where C_j is an arbitrary complex constant and C_j^* is the complex conjugate of C_j. ♦

Proof . By the linearity of the Schrödinger-like equation (9), the general solution of (9) with (36) is represented as

$$V^\natural(x_d, t) = \sum_{j\in\mathbb{N}_0} \{C_j e^{-E_j t}\phi_j(x_d) + D_j e^{-E_j^* t}\phi_j^*(x_d)\}, \tag{42}$$

where C_j and D_j are arbitrary complex constants. By Definition 2, Lemma 5 and (42), (41) is the general solution of the approximate Lyapunov equation (33). □

5. SUFFICIENT CONDITIONS FOR LYAPUNOV FUNCTION

We obtain sufficient conditions for an approximate Lyapunov function by using Lemma 5 and Theorem 2.

We define the following time-invariant function:

$$W^\natural(x_d) := V^\natural(x_d, 0) = \sum_{j\in\mathbb{N}_0} \left(C_j\phi_j(x_d) + C_j^*\phi_j^*(x_d)\right). \tag{43}$$

Assumption 1. Let $0 \in M_d^\natural \subset M_d$. There exists a combination $(C_0, C_1, \ldots)$ which makes $W^\natural(x_d)$ positive definite in $M_d^\natural$. Moreover, there exists a real constant $K \in \mathbb{R}$ satisfying

$$\sum_{j\in N^\natural} \left(\left|W_j^\sharp(x_d)\right| + \left|W_j^\flat(x_d)\right|\right) \le KW^\natural(x_d),\ \forall x_d \in M_d^\natural, \tag{44}$$

where

$$N^\natural := \{j | C_j \ne 0\} \subset \mathbb{N}_0, \tag{45}$$

$$W_j^\flat(x_d) := C_j\phi_j(x_d) + C_j^*\phi_j^*(x_d), \tag{46}$$

$$W_j^\sharp(x_d) := -i(C_j\phi_j(x_d) - C_j^*\phi_j^*(x_d)). \tag{47}$$

□

Remark 3. If the state space is bounded, a constant K satisfying (44) exists. ♦

Under the assumption 1, we obtain the following theorem:

Theorem 3. Suppose that (41) is a satisfactory approximate solution of the Lyapunov equation (2). If Assumption 1 is held and

$$K \max_{j\in N^\natural} (|E_{jR}|, |E_{jI}|) < q(x_d),\ \forall x_d \in M_d^\natural, \tag{48}$$

$W^\natural(x_d)$ is an approximate Lyapunov function in $M_d^\natural$. ♦

Proof . Suppose that all conditions in Theorem 3 are satisfied. Let

$$C_j := k_j(\cos\mu_j + \mathbf{i}\sin\mu_j),\ k_j, \mu_j \in \mathbb{R}. \tag{49}$$

By (39)-(41), (45), and (49), we obtain

$$V^\natural(x_d, t) = \sum_{j\in N^\natural} \big[2e^{-E_{jR}t}k_j\{\phi_{jR}(x_d)\cos(\mu_j - E_{jI}t) - \phi_{jI}(x_d)\sin(\mu_j - E_{jI}t)\}\big]. \tag{50}$$

Then, the approximate derivative of (50) becomes

$$\begin{aligned} \dot V^\natural(x, t) \approx \sum_{j\in N^\natural} \big[2e^{-E_{jR}t}k_j\big\{&\cos(\mu_j - E_{jI}t) \\ &\times \left(\phi'_{jR}(x_d)\cdot f(x_d) + E_{jI}\phi_{jI}(x_d) - E_{jR}\phi_{jR}(x_d)\right) \\ &+ \sin(\mu_j - E_{jI}t) \\ &\times \left(-\phi'_{jI}(x_d)\cdot f(x_d) + E_{jI}\phi_{jR}(x_d) + E_{jR}\phi_{jI}\right)\big\}\big], \end{aligned} \tag{51}$$

where $\phi'_{jR}(x_d)$ and $\phi'_{jI}(x_d)$ are the approximations of $\partial\phi_{jR}(x)/\partial x$ and $\partial\phi_{jI}(x)/\partial x$, respectively. On the other hand,

$$\dot V^\natural(x, t) \approx -q(x_d)V^\natural(x_d, t) \tag{52}$$

because $V^\natural(x_d, t)$ is an approximate solution of the Lyapunov equation (2). By (51) and (52), we obtain

$$\phi'_{jR}(x_d)\cdot f(x_d) \approx E_{jR}\phi_{jR}(x_d) - E_{jI}\phi_{jI}(x_d) - q(x_d)\phi_{jR}(x_d), \tag{53}$$

$$\phi'_{jI}(x_d)\cdot f(x_d) \approx E_{jR}\phi_{jI}(x_d) + E_{jI}\phi_{jR}(x_d) - q(x_d)\phi_{jI}(x_d). \tag{54}$$

By (40), (43), (46)-(47), and (53)-(54), the following approximation is derived:

$$\dot W^\natural(x) \approx -q(x_d)W^\natural(x_d) + \sum_{j\in N^\natural}\{E_{jR}W_j^\flat(x_d) - E_{jI}W_j^\sharp(x_d)\}. \tag{55}$$

By Assumption 1,

$$\begin{aligned} &\sum_{j\in N^\natural}\left\{E_{jR}W_j^\flat(x_d) - E_{jI}W_j^\sharp(x_d)\right\} \\ &\le \sum_{j\in N^\natural}(|E_{jR}||W^{\flat_j}(x_d)| + |E_{jI}||W^{\sharp_j}(x_d)|) \\ &\le \max_{j\in N^\natural}(|E_{jR}|, |E_{jI}|)\sum_{j\in N^\natural}(|W_j^\sharp(x_d)| + |W_j^\flat(x_d)|) \\ &\le K\max_{j\in N^\natural}(|E_{jR}|, |E_{jI}|)\,W^\natural(x_d). \end{aligned} \tag{56}$$

By (55) and (56),

$$\dot W^\natural(x) < \left\{-q(x_d) + K\max_{j\in N^\natural}(|E_{jR}|, |E_{jI}|)\right\}W^\natural(x_d). \tag{57}$$

By (48) and (57), $W^\natural(x_d)$ is an approximate Lyapunov function. □

In addition, a simple case is described as follows:

Corollary 1. Suppose that (41) is a satisfactory approximate solution of the Lyapunov equation (2). Let $\alpha \in \mathbb{N}_0$ and $C_\alpha, E_\alpha, \phi_\alpha \in \mathbb{R}$. If $W_\alpha^\natural(x_d) := C_\alpha\phi_\alpha(x_d)$ is positive definite in $M_d^\natural$ and

$$E_\alpha < q(x_d),\ \forall x_d \in M_d^\natural, \tag{58}$$

$W_\alpha^\natural(x_d)$ is an approximate Lyapunov function in $M_d^\natural$. ♦

Proof . Because $V_\alpha^\natural(x_d,t) = \exp(-E_\alpha t)W_\alpha^\natural(x_d)$ is an approximate solution of the Lyapunov equation (2), we obtain

$$\dot V_\alpha^\natural(x,t) \approx -q(x_d)V_\alpha^\natural(x_d,t). \tag{59}$$

Hence, we obtain

$$\dot W_\alpha^\natural(x) \approx (E_\alpha - q(x_d))W_\alpha^\natural(x_d). \tag{60}$$

By (58) and (60), $W_\alpha^\natural(x_d)$ is an approximate Lyapunov function. □

6. LYAPUNOV FUNCTION DESIGN

We propose an approximate Lyapunov function design method.

6.1 Block Tridiagonalization of Hamiltonian Matrix

We take finite elements from the discrete state space M_d and describe the Hamiltonian matrix $\mathcal{H}_m$ by a block tridiagonal matrix.

We obtain the following Lemma:

Lemma 6. If $\mathcal{H}$ is defined by (36), the elements of Hamiltonian matrix $\mathcal{H}_m$ are obtained as follows:

$$\langle x_d|\mathcal{H}|x_d''\rangle = \begin{cases} w(x_d\to x_d'') & \text{if} \quad x_d'' \neq x_d \\ w(x_d\to x_d) + q(x_d) & \text{if} \quad x_d'' = x_d. \end{cases} \tag{61}$$

♦

Proof . By using (37) and Lemma 3 with $w_1 = -w$ and $w_2 = w^\sharp$, (61) is derived. □

We extract

$$M_{df} := \left\{ \pm\delta\sum_{i=1}^n \gamma_i e_i,\ \gamma_i = 0,1,2,\ldots,l' \right\} \tag{62}$$

from M_d, where l' is a constant non-negative integer. The number of lattice points of M_{df} is l^n, where $l := 2l'+1$.

Moreover, we sort $x_d \in M_{df}$ by $z_k \in M_q$ in lexicographic order, where

$$M_q := \{z_1, z_2, \ldots, z_{l^n}\}. \tag{63}$$

For example, when $n=2$ and $l=3$,

$$z_1 = -\delta e_1 - \delta e_2,\ z_2 = -\delta e_1 + 0e_2,\ z_3 = -\delta e_1 + \delta e_2,$$
$$z_4 = +0e_1 - \delta e_2,\ z_5 = +0e_1 + 0e_2,\ z_6 = +0e_1 + \delta e_2,$$
$$z_7 = +\delta e_1 - \delta e_2,\ z_8 = +\delta e_1 + 0e_2,\ z_9 = +\delta e_1 + \delta e_2.$$

Then, the ket vector $|x_d\rangle$ is changed to $|z\rangle$.

Let

$$H_{\kappa_1}^{\kappa_2} := -\langle z_{\kappa_1}|\mathcal{H}|z_{\kappa_2}\rangle, \tag{64}$$

$$\alpha_i := (\beta_i - 1)l^i + 1, \quad i = 1,2,\ldots,n, \tag{65}$$

where $\kappa_1, \kappa_2, \beta_1, \beta_2, \ldots, \beta_n \in \mathbb{N}^+$, and define the following diagonal matrix:

$${}_i\mathcal{D}_{\kappa_1}^{\kappa_2} := \begin{bmatrix} H_{\kappa_1}^{\kappa_2} & 0 & 0 & 0 \\ 0 & H_{\kappa_1+1}^{\kappa_2+1} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & H_{\kappa_1+l^i-1}^{\kappa_2+l^i-1} \end{bmatrix}_{l^i\times l^i}. \tag{66}$$

Let us define

$${}_1\mathcal{H}_{\beta_1} := \begin{bmatrix} H_{\alpha_1}^{\alpha_1} & H_{\alpha_1}^{\alpha_1+1} & 0 & \cdots & \cdots \\ H_{\alpha_1+1}^{\alpha_1} & H_{\alpha_1+1}^{\alpha_1+1} & H_{\alpha_1+1}^{\alpha_1+2} & 0 & \cdots \\ 0 & H_{\alpha_1+2}^{\alpha_1+1} & H_{\alpha_1+2}^{\alpha_1+2} & \ddots & \cdots \\ 0 & 0 & \ddots & \ddots & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix}_{l\times l} \tag{67}$$

and

$${}_i\mathcal{H}_{\beta_i} := \begin{bmatrix} {}_{i-1}\mathcal{H}_{\alpha_i} & {}_{i-1}\mathcal{D}_{\alpha_i}^{\alpha_i+1} & 0 & \cdots \\ {}_{i-1}\mathcal{D}_{\alpha_i+1}^{\alpha_i} & {}_{i-1}\mathcal{H}_{\alpha_i+1} & {}_{i-1}\mathcal{D}_{\alpha_i+1}^{\alpha_i+2} & \cdots \\ 0 & {}_{i-1}\mathcal{D}_{\alpha_i+2}^{\alpha_i+1} & {}_{i-1}\mathcal{H}_{\alpha_i+2} & \ddots \\ 0 & 0 & \ddots & \ddots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}_{l^i\times l^i}, \tag{68}$$

where $i = 2,3,\ldots,n$.

Then, we obtain the following Lemma:

Lemma 7. If the state space of the Schrödinger-like equation (9) with (36) is changed from M_d to M_q, the Hamiltonian matrix $\mathcal{H}_m$ becomes the block tridiagonal matrix ${}_n\mathcal{H}_1$. ♦

Proof . If $n=1$,

$$\mathcal{H}_m = \begin{bmatrix} H_1^1 & H_1^2 & 0 & \cdots & \cdots \\ H_2^1 & H_2^2 & H_2^3 & 0 & \cdots \\ 0 & H_3^2 & H_3^3 & \ddots & \cdots \\ 0 & 0 & \ddots & \ddots & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix}_{l\times l} = {}_1\mathcal{H}_1 \tag{69}$$

by the transition probability rates (29)-(32) and Lemma 6. If $n=2$,

$$\mathcal{H}_m = \begin{bmatrix} {}_1\mathcal{H}_1 & {}_1\mathcal{D}_1^{l+1} & 0 & \cdots \\ {}_1\mathcal{D}_{l+1}^1 & {}_1\mathcal{H}_2 & {}_1\mathcal{D}_{l+1}^{2l+1} & \cdots \\ 0 & {}_1\mathcal{D}_{2l+1}^{l+1} & {}_1\mathcal{H}_3 & \ddots \\ 0 & 0 & \ddots & \ddots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}_{l^2\times l^2} = {}_2\mathcal{H}_1. \tag{70}$$

Moreover, if $n=3$,

$$\mathcal{H}_m = \begin{bmatrix} {}_2\mathcal{H}_1 & {}_2\mathcal{D}_1^{l^2+1} & 0 & \cdots \\ {}_2\mathcal{D}_{l^2+1}^1 & {}_2\mathcal{H}_2 & {}_2\mathcal{D}_{l^2+1}^{2l^2+1} & \cdots \\ 0 & {}_2\mathcal{D}_{2l^2+1}^{l^2+1} & {}_2\mathcal{H}_3 & \ddots \\ 0 & 0 & \ddots & \ddots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}_{l^3\times l^3} = {}_3\mathcal{H}_1. \tag{71}$$

In this way, we obtain

$$\mathcal{H}_m = \begin{bmatrix} {}_{n-1}\mathcal{H}_1 & {}_{n-1}\mathcal{D}_1^{l^{n-1}+1} & 0 & \cdots \\ {}_{n-1}\mathcal{D}_{l^{n-1}+1}^1 & {}_{n-1}\mathcal{H}_2 & {}_{n-1}\mathcal{D}_{l^{n-1}+1}^{2l^{n-1}+1} & \cdots \\ 0 & {}_{n-1}\mathcal{D}_{2l^{n-1}+1}^{l^{n-1}+1} & {}_{n-1}\mathcal{H}_3 & \ddots \\ 0 & 0 & \ddots & \ddots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}_{l^n\times l^n} = {}_n\mathcal{H}_1 \tag{72}$$

for $n > 2$. □

The previous works; e.g., Press, et al. [8] proposed some algorithms for solving the eigenvalue problems of sparse matrices.

6.2 Lyapunov Function Design Procedure

We propose the following approximate Lyapunov function design procedure:

- **[Step1**] Discretize the state space $\mathbb{R}^n$ into the finite elements M_q.
- **[Step2**] Determine a positive function $q(x)$ and calculate the Hamiltonian matrix $\mathcal{H}_m$ by Lemma 6.
- **[Step3**] Solve the eigenvalue problem for the Hamiltonian matrix $\mathcal{H}_m$.
- **[Step4**] Choose constants $C_0, C_1, \ldots$ such that $W^\natural(x_d)$ becomes a positive definite function satisfying the inequality (48) or (58).

By Step1-4, we can obtain $W^\natural(x_d)$, which is an approximate Lyapunov function of system (1).

The classical previous works, e.g., Schultz [5], Zubov [11], Vannelli-Vidyasagar [14] discussed a Lyapunov function design problem in a small enough neighborhood of the origin. Moreover, Krasovskii [5, 11] discussed the problem restricting the form of a Lyapunov function. Hence, the system may be limited.

The previous works with finite-difference approximations, e.g., Kushner [6] are under the assumption that the boundary satisfies the Neumann condition $\partial V/\partial x = 0$. Hence, if the boundary condition is not held, the iteratively calculated values may diverge.

Our method is available in a domain that the system is asymptotically stable. And, the restriction of a Lyapunov function does not put any restriction for the system because Theorem 1 is held. Moreover, a divergence problem does not cause because our method does not contain any iteratively calculation.

7. EXAMPLE

For example, we consider a two-dimensional system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 - x_2 - x_2^3, \end{aligned} \tag{73}$$

where $x = (x_1, x_2)^T \in M := \{(x_1, x_2)^T | -1.0 \leq x_1 \leq 1.0,\ -1.0 \leq x_2 \leq 1.0\}$.

For the system (73), we obtain an approximate Lyapunov function by using the proposed method in Subsection 6.2.

Let $\delta = 0.1$ and $q(x) = 1.0$. Then, Hamiltonian matrix ${}_2\mathcal{H}_1$ is obtained by (70) and (29)-(32). We solve the eigen equation of ${}_2\mathcal{H}_1$ and choose $C_0, C_1, \ldots$ such that $W^\natural(x_d)$ becomes an approximate Lyapunov function by Theorem 3.

Figures 1 and 2 show the approximate Lyapunov function $W^\natural(x_d)$ and the derivative $dW^\natural(x_d)/dt$ for system (73), where $dW^\natural/dt$ is obtained by multi-linear approximation. The region Ω denotes the largest connected level set of $W^\natural(x_d)$ and $\partial\Omega$ denotes the boundary of Ω.

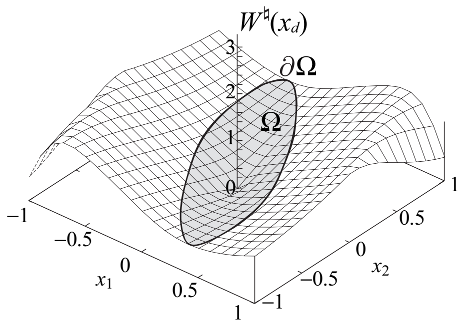

Fig. 1. Approximate Lyapunov function $W^\natural(x_d)$.

8. CONCLUSIONS

We have proposed a Lyapunov function design method as follows. First, we have approximated a Lyapunov equation by a Schrödinger-like equation. Second, we have obtained sufficient conditions for a function to be an approximate Lyapunov function. Then, we have provided an approximate Lyapunov function design procedure.

Moreover, we have confirmed the effectiveness of our method by an example.

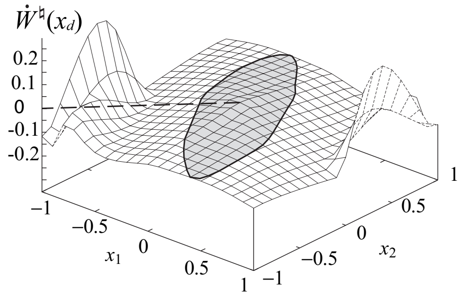

Fig. 2. Approximate $dW^\natural(x_d)/dt$.

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