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Robust Stability/Performance Analysis for Polytopic Systems via Multiple Slack Variable Approach

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Abstract: This paper investigates robust stability, H_2 performance, and H_∞ performance analysis for polytopic systems, i.e. Linear Time-Invariant Parameter-Dependent (LTIPD) systems in which the parameters lie in the unit simplex. Our results are derived via multiple "slack variable" approach, which has previously been proposed for the non-negativity check of polynomial functions, using Polynomially Parameter-Dependent Lyapunov Functions (PPDLFs). Our derived conditions are only sufficient conditions for our addressed problems; however, they encompass existing methods via single slack variable approach. Numerical examples are included to demonstrate the effectiveness of our methods.

1. INTRODUCTION

Exact robust stability analysis and performance analysis have been recognized as very important topics for control theory and much research for them has been conducted in this decade. The first approach for these problems uses quadratic stability; that is, parameter-independent Lyapunov functions are used, e.g. [Boyd *et al.*, 1994]; however, the obtained result is very conservative. To reduce conservatism, parametrically affine or quadratically Parameter-Dependent Lyapunov Functions (PDLFs) are used, e.g. [Feron *et al.*, 1996, Trofino and de Souza, 2001]. Although they successfully reduce conservatism, the results are still conservative. At the next step, Polynomially PDLFs (PPDLFs) are used to further reduce conservatism, and the existing methods using PPDLFs are roughly categorized into four groups: Sums-Of-Squares (SOS) approach [Chesi *et al.*, 2005b,a], KYP-lemma approach [Bliman, 2004], Slack Variable (SV) approach ([Ebihara *et al.*, 2005, Peaucelle *et al.*, 2006, Sato and Peaucelle, 2006a] and references therein), which has been extended to parameter-dependent SV approach [Oliveira and Peres, 2005a,b,c, Oishi, 2006, Oliveira *et al.*, 2006], and quadratic separator approach [Iwasaki and Shibata, 2001]. Some of these methods, i.e. KYP-lemma approach and parameter-dependent SV approach [Oliveira and Peres, 2005b,c, Oishi, 2006], give the necessary and sufficient conditions for their addressed problems. Although SV approach using parameter-independent SVs has not been proved to give the exact analysis even if the parameter-dependency of PPDLFs is sufficiently increased, numerical examples demonstrate that SV approach using parameter-independent SVs is not so conservative [Sato and Peaucelle, 2006a,b].

In this manuscript, we propose new formulations for robust stability analysis and performance analysis for polytopic systems. Our methods are based on parameter-independent SV approach using multiple SVs (we call it multiple SV approach in short), which has been proposed for the non-negativity check of polynomial functions and has been proved to encompass SV approach using a single SV [Sato and Peaucelle, 2007a,b]. Consequently, our new methods are no more conservative than analysis methods via single SV approach. We show the effectiveness of our methods with numerical examples.

Hereafter, $\langle X \rangle$ is a shorthand notation of $X + X^T$, $0_{n,m}$, I_n and $\mathbf{0}$ respectively denote an $n \times m$-dimensional zero matrix, an n-dimensional identity matrix and an appropriately dimensioned zero matrix, $\mathcal{R}^{n\times m}$ and $\mathcal{S}^n$ respectively denote sets of $n \times m$ dimensional real matrices and $n \times n$ dimensional symmetric real matrices, $\mathsf{diag}(X_1, \cdots, X_k)$ denotes a block diagonal matrix composed of $X_1, \cdots$ and X_k, $\otimes$ denotes the Kronecker product, $X^\perp \in \mathcal{R}^{(n-r)\times n}$ denotes a matrix satisfying $X^\perp X = \mathbf{0}$ and $X^\perp X^{\perp T} > 0$, where $X \in \mathcal{R}^{n\times p}$ and $\mathrm{rank}(X) = r$, and the notation $\mathsf{Tr}_n(X)$ denotes $\begin{bmatrix} \mathrm{Tr}(X_{11}) & \cdots & \mathrm{Tr}(X_{1i}) \\ \vdots & \ddots & \vdots \\ \mathrm{Tr}(X_{i1}) & \cdots & \mathrm{Tr}(X_{ii}) \end{bmatrix}$ for $X = \begin{bmatrix} X_{11} & \cdots & X_{1i} \\ \vdots & \ddots & \vdots \\ X_{i1} & \cdots & X_{ii} \end{bmatrix}$, where $X_{kl} \in \mathcal{R}^{n\times n}$ $(k, l = 1, \cdots, i)$.

2. PRELIMINARIES

In this section, we make some preliminaries for our proposed methods.

2.1 Definitions

In this paper, we consider the following Linear Time-Invariant Parameter-Dependent (LTIPD) system.

$$G(\zeta) : \begin{cases} \dot{x} = A(\zeta)x + B(\zeta)w \\ z = C(\zeta)x + D(\zeta)w \end{cases}, \quad (1)$$

where $x \in \mathcal{R}^n$ is the state vector, $w \in \mathcal{R}^{n_w}$ is the disturbance input vector, and $z \in \mathcal{R}^{n_z}$ is the performance output vector. Matrices $A(\zeta)$, $B(\zeta)$, etc. are continuously parameter-dependent matrices of appropriate dimensions and assumed to have the following representations.

$$\begin{bmatrix} A(\zeta) & B(\zeta) \\ C(\zeta) & C(\zeta) \end{bmatrix} = \sum_{i=1}^{N} \zeta_i \begin{bmatrix} A_i & B_i \\ C_i & D_i \end{bmatrix}, \qquad (2)$$

where the parameter vector $\zeta = [\zeta_1 \ \cdots \ \zeta_N]^T$ are supposed to lie in the unit simplex:

$$\Omega = \left\{ \zeta : \sum_{i=1}^{N} \zeta_i = 1, \ \zeta_i \geq 0 \ (i = 1, \cdots, N) \right\}, \qquad (3)$$

and matrices A_i, B_i, C_i and D_i are constant matrices of compatible dimensions.

For the LTIPD system (1), we will look for a quadratic PPDLF which is dependent of all ζ_i $(i = 1, \cdots, N)$ with their power series up to m_i $(i = 1, \cdots N)$ and all their by-products. To define such PPDLFs, we give some definitions in the following. For each parameter ζ_i $(i = 1, \cdots, N)$, define the vector of its power series ranging from zero to m_i as

$$\zeta_i^{[m_i]} = [1 \ \zeta_i \ \cdots \ \zeta_i^{m_i}]^T \in \mathcal{R}^{\sigma_i},$$

where $\sigma_i = m_i + 1$. Define the vector of all monomials obtained as products of all ζ_i^j elements with $i = 1, \cdots, N$, $j = 0, \cdots, m_i$:

$$\bar{\zeta} = \zeta_1^{[m_1]} \otimes \cdots \otimes \zeta_N^{[m_N]} \in \mathcal{R}^{\pi^{[1,N]}},$$

where $\pi^{[q,r]} = \Pi_{i=q}^{r} \sigma_i$. By definition, let $\pi^{[1,0]} = 1$ and $\pi^{[N+1,N]} = 1$. With these definitions, we define PPDLF as $x^T P(\zeta) x$ with the following $P(\zeta)$:

$$P(\zeta) = \bar{\zeta}_n^T \bar{P} \bar{\zeta}_n, \ \bar{P} \in \mathcal{S}^{n\pi^{[1,N]}}, \qquad (4)$$

where $\bar{\zeta}_n = \bar{\zeta} \otimes I_n$.

Remark 1. We can set arbitrary-order PPDLFs with appropriately defined $\bar{P}$. For example, when $N = 2$, setting $m_1 = m_2 = 1$ and $\bar{P} = \begin{bmatrix} \bar{P}_0 & \bar{P}_1 & \bar{P}_2 \\ \bar{P}_1^T & \mathbf{0} & \mathbf{0} \\ \bar{P}_2^T & \mathbf{0} & \mathbf{0} \end{bmatrix}$ with $\bar{P}_0 \in \mathcal{S}^n$, $\bar{P}_1 \in \mathcal{R}^{n \times n}$ and $\bar{P}_2 \in \mathcal{R}^{n \times n}$ gives a parametrically affine Lyapunov function.

Two additional notations are now defined. Let the vector

$$e = [1 \ \underbrace{0 \ \cdots \ 0}_{\sum_{i=1}^{N}(m_i \times \pi^{[i+1,N]})} \ ]^T \in \mathcal{R}^{\pi^{[1,N]}}$$

and let $e_n = e \otimes I_n$. This notation works as an index for getting parameter-independent elements, e.g. $e_n^T \bar{\zeta}_n = I_n$. Let as well affine parameter-dependent matrices

$$\eta(\zeta_i) = \begin{bmatrix} \zeta_i I_{\sigma_i - 1} \\ \mathbf{0} \end{bmatrix} - \begin{bmatrix} \mathbf{0} \\ I_{\sigma_i - 1} \end{bmatrix} \in \mathcal{R}^{\sigma_i \times (\sigma_i - 1)}, \qquad (5)$$

which satisfy $\eta(\zeta_i)^{\perp} = \zeta_i^{[m_i]^T}$ $(i = 1, \cdots, N)$. At last, define

$$\begin{aligned} \Xi(\zeta_i) &= I_{\pi^{[1,i-1]}} \otimes (\eta(\zeta_i) \otimes I_{\pi^{[i+1,N]}}) \in \mathcal{R}^{\pi^{[1,N]} \times \pi^{[1,N]-[i]}}, \\ \Xi_n(\zeta_i) &= \Xi(\zeta_i) \otimes I_n, \\ \Xi_{n_w}(\zeta_i) &= \Xi(\zeta_i) \otimes I_{n_w}, \end{aligned}$$

where $\pi^{[1,N]-[i]}$ denotes $\pi^{[1,i-1]} (\sigma_i - 1) \pi^{[i+1,N]}$. They respectively prove to satisfy the following relations.

$$\begin{aligned} \Xi(\zeta_i)^{\perp} &= I_{\pi^{[1,i-1]}} \otimes \left(\zeta_i^{[m_i]^T} \otimes I_{\pi^{[i+1,N]}} \right) \\ \Xi_n(\zeta_i)^{\perp} &= I_{\pi^{[1,i-1]}} \otimes \left(\zeta_i^{[m_i]^T} \otimes I_{n\pi^{[i+1,N]}} \right) \\ \Xi_{n_w}(\zeta_i)^{\perp} &= I_{\pi^{[1,i-1]}} \otimes \left(\zeta_i^{[m_i]^T} \otimes I_{n_w\pi^{[i+1,N]}} \right) \end{aligned}$$

2.2 Basic Lemmas

Well-known stability, H_2 performance, and H_∞ performance analysis conditions for the LTIPD system (1) are now recalled. In these lemmas, Lyapunov functions are set as $x^T P(\zeta) x$.

Lemma 2. (Stability). The system (1) is robustly stable for all admissible ζ if and only if there exists $P(\zeta) > 0$ that satisfies (6) for all admissible ζ.

$$\langle P(\zeta) A(\zeta) \rangle < 0 \qquad (6)$$

Lemma 3. (H_2 performance). The system (1) is robustly stable and its H_2 performance is bounded by γ_2 for all admissible ζ if and only if $D(\zeta) = \mathbf{0}$ and there exist $P(\zeta) > 0$ and $N(\zeta) > 0$ that satisfy (7), (8) and (9) for all admissible ζ.

$$\langle P(\zeta) A(\zeta) \rangle + C(\zeta)^T C(\zeta) < 0 \qquad (7)$$

$$N(\zeta) - B(\zeta)^T P(\zeta) B(\zeta) > 0 \qquad (8)$$

$$\gamma_2^2 > \mathrm{Tr}\,(N(\zeta)) \qquad (9)$$

Lemma 4. (H_∞ performance). The system (1) is robustly stable and its H_∞ performance is bounded by γ_∞ for all admissible ζ if and only if there exists $P(\zeta) > 0$ that satisfies (10) for all admissible ζ.

$$\begin{bmatrix} \langle P(\zeta) A(\zeta) \rangle & P(\zeta) B(\zeta) & C(\zeta)^T \\ B(\zeta)^T P(\zeta) & -\gamma_\infty^2 I_{n_w} & D(\zeta)^T \\ C(\zeta) & D(\zeta) & -I_{n_z} \end{bmatrix} < 0 \qquad (10)$$

In Lemmas 2, 3 and 4, the positivity of $P(\zeta)$ is redundant if the system (1) for one admissible value of the uncertainty is stable [Feron *et al.*, 1996, Peaucelle *et al.*, 2006], which can be assumed without loss of generality. Thus, we do not consider the positivity condition of $P(\zeta)$ hereafter.

3. MAIN RESULTS

In this section, we first show our proposed methods for robust stability analysis and performance analysis for the LTIPD system (1) via multiple SV approach. Next, we show that the methods encompass the previously proposed counterpart methods via single SV approach, which have been proposed in [Sato and Peaucelle, 2006a].

3.1 Proposed Methods

In this subsection, we show our proposed methods for robust stability analysis using Lemma 2, robust H_2 performance analysis using Lemma 3, and robust H_∞ performance analysis using Lemma 4 for the LTIPD system (1) using $P(\zeta)$ defined in (4).

We first show our result for robust stability analysis.

Theorem 5. (Stability Analysis). If there exist a symmetric matrix $\bar{P} \in \mathcal{S}^{n\pi^{[1,N]}}$, and matrices $M_i \in \mathcal{R}^{n\pi^{[1,N]-[i]}\times n\pi^{[1,N]}}$ $(i = 1, \cdots, N)$ such that (11) for all the vertices of Ω, then the LTIPD system (1) is robustly stable.

$$\left\langle \bar{P}\left(I_{\pi^{[1,N]}} \otimes A(\zeta)\right)\right\rangle + \textstyle\sum_{i=1}^{N} \left\langle \Xi_n(\zeta_i) M_i \right\rangle < 0 \tag{11}$$

Proof. Note that inequality (11) is affine with respect to ζ. Therefore, if it holds for all the vertices of Ω, then it also holds for all ζ in Ω. Noting that $(I_{\pi^{[1,N]}} \otimes A(\zeta))\,\bar{\zeta}_n = \bar{\zeta}_n A(\zeta)$, multiplying (11) from the left and the right by $\bar{\zeta}_n^T$ and $\bar{\zeta}_n$ respectively leads to $\langle \bar{\zeta}_n^T \bar{P} \bar{\zeta}_n A(\zeta)\rangle < 0$, which is equivalent to (6). This completes the proof.

We next show our result for robust H_2 performance analysis.

Theorem 6. (H_2 Performance Analysis). If there exist a symmetric matrix $\bar{P} \in \mathcal{S}^{n\pi^{[1,N]}}$, a parametrically affine symmetric matrix $\bar{N}(\zeta) \in \mathcal{S}^{n_w\pi^{[1,N]}}$, matrices $F_i \in \mathcal{R}^{n\pi^{[1,N]-[i]}\times\left(n\pi^{[1,N]}+n_z\right)}$ $(i = 1, \cdots, N)$, $M_i \in \mathcal{R}^{\left(n_w\pi^{[1,N]-[i]}+n\pi^{[1,N]-[i]}\right)\times\left(n_w\pi^{[1,N]}+n\pi^{[1,N]}\right)}$ $(i = 1, \cdots, N)$ and $H_i \in \mathcal{R}^{\pi^{[1,N]-[i]}\times\pi^{[1,N]}}$ $(i = 1, \cdots, N)$, and a positive number γ_2 such that (12), (13) and (14) for all the vertices of Ω, then the LTIPD system (1) is robustly stable and an upper bound of its H_2 performance is given by γ_2.

$$\begin{bmatrix} \left\langle \bar{P}\left(I_{\pi^{[1,N]}} \otimes A(\zeta)\right)\right\rangle & e \otimes C(\zeta)^T \\ e^T \otimes C(\zeta) & -I_{n_z} \end{bmatrix} + \textstyle\sum_{i=1}^{N} \left\langle \begin{bmatrix} \Xi_n(\zeta_i) \\ \mathbf{0} \end{bmatrix} F_i \right\rangle < 0 \tag{12}$$

$$\begin{bmatrix} \bar{N}(\zeta) & \left(I_{\pi^{[1,N]}} \otimes B(\zeta)^T\right)\bar{P} \\ \bar{P}\left(I_{\pi^{[1,N]}} \otimes B(\zeta)\right) & \bar{P} \end{bmatrix} + \textstyle\sum_{i=1}^{N} \left\langle \begin{bmatrix} \Xi_{n_w}(\zeta_i) & \mathbf{0} \\ \mathbf{0} & \Xi_n(\zeta_i) \end{bmatrix} M_i \right\rangle > 0 \tag{13}$$

$$\gamma_2^2 ee^T - \mathsf{Tr}_{n_w}\left(\bar{N}(\zeta)\right) + \textstyle\sum_{i=1}^{N} \left\langle \Xi(\zeta_i) H_i \right\rangle > 0 \tag{14}$$

Noting that $\left(e^T \otimes C(\zeta)\right)\bar{\zeta}_n = C(\zeta)$ and $\left(I_{\pi^{[1,N]}} \otimes B(\zeta)\right)\bar{\zeta}_{n_w} = \bar{\zeta}_n B(\zeta)$, the proof of Theorem 6 is straightforward similarly to that of Theorem 5 when choosing $N(\zeta) = \bar{\zeta}_{n_w}^T \bar{N}(\zeta)\bar{\zeta}_{n_w}$ with a parametrically affine symmetric matrix $\bar{N}(\zeta)$. Thus we omit it.

We finally show our result for robust H_∞ performance analysis.

Theorem 7. (H_∞ Performance Analysis). If there exist a symmetric matrix $\bar{P} \in \mathcal{S}^{n\pi^{[1,N]}}$, matrices $M_i \in \mathcal{R}^{\left(n\pi^{[1,N]-[i]}+n_w\pi^{[1,N]-[1]}\right)\times\left(n\pi^{[1,N]}+n_w\pi^{[1,N]}+n_z\right)}$ $(i = 1, \cdots, N)$, and a positive number γ_∞ such that (15) for all the vertices of Ω, then the LTIPD system (1) is robustly stable and an upper bound of its H_∞ performance is given by γ_∞.

$$\begin{bmatrix} \left\langle \bar{P}\left(I_{\pi^{[1,N]}} \otimes A(\zeta)\right)\right\rangle & \bar{P}\left(I_{\pi^{[1,N]}} \otimes B(\zeta)\right) & e \otimes C(\zeta)^T \\ \left(I_{\pi^{[1,N]}} \otimes B(\zeta)^T\right)\bar{P} & -\gamma_\infty^2\left(ee^T\right) \otimes I_{n_w} & e \otimes D(\zeta)^T \\ e^T \otimes C(\zeta) & e^T \otimes D(\zeta) & -I_{n_z} \end{bmatrix} + \textstyle\sum_{i=1}^{N} \left\langle \begin{bmatrix} \Xi_n(\zeta_i) & \mathbf{0} \\ \mathbf{0} & \Xi_{n_w}(\zeta_i) \\ \mathbf{0} & \mathbf{0} \end{bmatrix} M_i \right\rangle < 0 \tag{15}$$

Noting that $\left(e^T \otimes D(\zeta)\right)\bar{\zeta}_{n_w} = D(\zeta)$, the proof of Theorem 7 is straightforward similarly to those of Theorems 5 and 6. Thus we omit it.

3.2 Inclusion of Single SV Approach

In this subsection, we show that our derived theorems encompass the counterpart theorems using $P(\zeta)$ defined in (4) via single SV approach, which have been proposed in [Sato and Peaucelle, 2006a]. To recall the analysis methods therein, we define the following parametrically affine matrices.

$$\begin{aligned} \eta(\zeta) = & \left[\begin{array}{c} \eta(\zeta_N) \otimes I_{\pi^{[N+1,N]}} \\ 0_{\pi^{[1,N]}-\pi^{[N,N]},(\sigma_N-1)\times\pi^{[N+1,N]}} \end{array} \right| \cdots \\ & \left| \begin{array}{c} \eta(\zeta_2) \otimes I_{\pi^{[3,N]}} \\ 0_{\pi^{[1,N]}-\pi^{[2,N]},(\sigma_2-1)\times\pi^{[3,N]}} \end{array} \right| \eta(\zeta_1) \otimes I_{\pi^{[2,N]}} \Bigg] \\ & \in \mathcal{R}^{\pi^{[1,N]}\times\left(\pi^{[1,N]}-1\right)}, \\ \eta_n(\zeta) = & \eta(\theta) \otimes I_n, \\ \eta_{n_w}(\zeta) = & \eta(\theta) \otimes I_{n_w}, \end{aligned}$$

where $\eta(\zeta_i)$ $(i = 1, \cdots, N)$ have the same definitions as in (5).

We describe the analysis methods via single SV approach.

Lemma 8. [Sato and Peaucelle, 2006a] If there exist a symmetric matrix $\bar{P}$, and a matrix M such that (16) for all the vertices of Ω, then the LTIPD system (1) is robustly stable.

$$\left\langle \bar{P}\left(I_{\pi^{[1,N]}} \otimes A(\zeta)\right) + \eta_n(\zeta) M \right\rangle < 0 \tag{16}$$

For robust H_2 performance analysis, we use the same $N(\theta)$ in Theorem 6, i.e. $N(\theta) = \bar{\theta}_{n_w}^T \bar{N}(\theta)\bar{\theta}_{n_w}$, where $\bar{N}(\theta)$ is a parametrically affine symmetric matrix.

Lemma 9. [Sato and Peaucelle, 2006a] If there exist a symmetric matrix $\bar{P}$, a parametrically affine symmetric matrix $\bar{N}(\zeta)$, matrices F, M and H, and a positive number γ_2 such that (17), (18) and (19) for all the vertices of Ω, then the LTIPD system (1) is robustly stable and an upper bound of its H_2 performance is given by γ_2.

$$\begin{bmatrix} \left\langle \bar{P}\left(I_{\pi^{[1,N]}} \otimes A(\zeta)\right)\right\rangle & e \otimes C(\zeta)^T \\ e^T \otimes C(\zeta) & -I_{n_z} \end{bmatrix} + \left\langle \begin{bmatrix} \eta_n(\zeta) \\ \mathbf{0} \end{bmatrix} F \right\rangle < 0 \tag{17}$$

$$\begin{bmatrix} \bar{N}(\zeta) & \left(I_{\pi^{[1,N]}} \otimes B(\zeta)^T\right)\bar{P} \\ \bar{P}\left(I_{\pi^{[1,N]}} \otimes B(\zeta)\right) & \bar{P} \end{bmatrix} + \left\langle \begin{bmatrix} \eta_{n_w}(\zeta) & \mathbf{0} \\ \mathbf{0} & \eta_n(\zeta) \end{bmatrix} M \right\rangle > 0 \tag{18}$$

$$\gamma_2^2 ee^T - \mathsf{Tr}_{n_w}\left(\bar{N}(\zeta)\right) + \left\langle \eta(\zeta) H \right\rangle > 0 \tag{19}$$

Lemma 10. [Sato and Peaucelle, 2006a] If there exist a symmetric matrix $\bar{P}$, a matrix M, and a positive number γ_∞ such that (20) for all the vertices of Ω, then the LTIPD system (1) is robustly stable and an upper bound of its H_∞ performance is given by γ_∞.

$$\begin{bmatrix} \langle \bar{P}\left(I_{\pi^{[1,N]}} \otimes A(\zeta)\right)\rangle & \bar{P}\left(I_{\pi^{[1,N]}} \otimes B(\zeta)\right) & e \otimes C(\zeta)^T \\ \left(I_{\pi^{[1,N]}} \otimes B(\zeta)^T\right)\bar{P} & -\gamma_\infty^2\left(ee^T\right)\otimes I_{n_w} & e \otimes D(\zeta)^T \\ e^T \otimes C(\zeta) & e^T \otimes D(\zeta) & -I_{n_z} \end{bmatrix} + \left\langle \begin{bmatrix} \eta_n(\zeta) & \mathbf{0} \\ \mathbf{0} & \eta_{n_w}(\zeta) \\ \mathbf{0} & \mathbf{0} \end{bmatrix} M \right\rangle < 0 \quad (20)$$

For Theorem 5 and Lemma 8, we give the following assertion.

Proposition 11. If there exist a symmetric matrix $\bar{P}$ and a matrix M such that (16) for all the vertices of Ω, then there always exist matrices M_i $(i = 1, \cdots, N)$ such that (11) for all the vertices of Ω with the same $\bar{P}$.

Proof. Suppose that there exist $\bar{P}$ and M such that (16) for all the vertices of Ω. Let matrix M be partitioned as $M = \left[\hat{M}_N^T \cdots \hat{M}_1^T\right]^T$, where $\hat{M}_i \in \mathcal{R}^{n(\sigma_i-1)\pi^{[i+1,N]} \times n\pi^{[1,N]}}$ $(i = 1, \cdots, N)$. Set $M_1 = \hat{M}_1$ and $M_i = \left[\hat{M}_i^T \ \mathbf{0}\right]^T$ $(i = 2, \cdots, N)$, then $\sum_{i=1}^N \Xi_n(\zeta_i)M_i = \eta_n(\zeta)M$ holds. Therefore, inequality (11) holds with the above defined M_i $(i = 1, \cdots, N)$ and the same $\bar{P}$. This completes the proof.

Similarly to Proposition 11, we give the following assertions for robust performance analysis.

Proposition 12. For a given positive number γ_2, if there exist a symmetric matrix $\bar{P}$, a parametrically affine symmetric matrix $\bar{N}(\zeta)$, matrices F, M and H such that (17), (18) and (19) for all the vertices of Ω, then there always exist matrices F_i $(i = 1, \cdots, N)$, M_i $(i = 1, \cdots, N)$ and H_i $(i = 1, \cdots, N)$ such that (12), (13) and (14) for all the vertices of Ω with the same γ_2, $\bar{N}(\zeta)$ and $\bar{P}$.

Proposition 13. For a given positive number γ_∞, if there exist a symmetric matrix $\bar{P}$ and a matrix M such that (20) for all the vertices of Ω, then there always exist matrices M_i $(i = 1, \cdots, N)$ such that (15) for all the vertices of Ω with the same γ_∞ and $\bar{P}$.

Proofs are straightforward similarly to that of Proposition 11. Thus we omit them.

Propositions 11, 12 and 13 show that the analysis methods via multiple SV approach encompass the counterpart methods via single SV approach.

Remark 14. In [Sato and Peaucelle, 2006a], it has been proved that analysis methods via single SV approach reduce conservatism with the increase of the parameter-dependency of $P(\zeta)$. Similarly to them, Theorems 5, 6 and 7 also reduce conservatism with the increase of the parameter-dependency of $P(\zeta)$.

4. NUMERICAL EXAMPLES

In this section, we illustrate the effectiveness of our methods with numerical examples. The comparison between analysis methods via single SV approach and other SV approach methods using parameter-independent SVs, i.e. the methods in [Ebihara *et al.*, 2005, Peaucelle *et al.*, 2006], has already been shown in [Sato and Peaucelle, 2006a], in which the methods via single SV approach, i.e. Lemmas 8, 9 and 10, give no worse analysis result than those methods. Further, in [Sato and Peaucelle, 2006b] which is an in-house research report prepared for [Sato and Peaucelle, 2006a], numerical examples borrowed from [Chesi *et al.*, 2005b,a] show that Lemmas 8, 9 and 10 give no worse analysis result than SOS approach methods in [Chesi *et al.*, 2005b,a]. Therefore, in the following, we focus on the comparison between multiple SV approach methods and single SV approach methods.

The calculation for LMIs has been conducted using SeDuMi ver. 1.1 [Sturm, 1999] along with the parser YALMIP ver. 3 [Lofberg, 2004].

4.1 Stability Analysis for Randomly Generated Examples in [Oliveira and Peres, 2005c]

We conduct robust stability analysis for the numerical examples in Table 1 in [Oliveira and Peres, 2005c] using Theorem 5 and Lemma 8. Here, 1000 examples, all of which are assured to be robustly stable, are checked for the robust stability. The results are shown in Table 1. For reference, the results using the methods in [Oliveira and Peres, 2005c], which are the asymptotic necessary and sufficient conditions for robust stability analysis using parametrically affine Lyapunov functions, are also shown. As proved in [Ebihara and Hagiwara, 2006], parametrically affine Lyapunov functions do not always give the exact analysis. This fact is also illustrated in Table 1. For reference, we also give the number of decision variables in parentheses. Although Theorem 5 and Lemma 8 are neither proved to be the necessary and sufficient condition for robust stability analysis problem, they give the exact analysis result but just for one case.

In Table 1 in [Oliveira and Peres, 2005c], they have conducted robust stability analysis for numerical examples with setting $n = 2, 3, 4$ and $N = 2, 3, 4$. However, the number of decision variables in Theorem 5 for $n = 4$ and $N = 4$ is 10272, which is prohibitive for its real calculation. Thus, our methods are currently for low-order systems with a few vertices due to its numerical complexity.

4.2 H_2 Performance Analysis for Randomly Generated Examples

Next, we conduct robust H_2 performance analysis for randomly generated numerical examples. In our examples, $A(\zeta)$ matrices are the same as in Table 1; however, other matrices $B(\zeta)$ and $C(\zeta)$, all the elements of which lie in $[-0.5, 0.5]$, are randomly generated. For each generated example, we calculate the upper bound of H_2 performance γ_2 by applying Theorem 6 and Lemma 9, and set the obtained γ_2 as $\gamma_{2_{\min}}$. For comparison, we calculate the maximum value of H_2 performance γ_2 using fine gridding method, and set the obtained γ_2 as $\gamma_{2_{grid}}$. Tables 2 and 3 respectively show the average values of $\gamma_{2_{\min}}/\gamma_{2_{grid}}$ for MIMO systems and SISO systems. We also give the number of decision variables in parentheses. They show that Theorem 6 is less conservative than Lemma 9.

A typical example which clearly illustrates Proposition 12 is shown in the following.

$$\left[\begin{array}{c|c} A_i & B_i \\ \hline C_i & \end{array}\right] = \begin{cases} \left[\begin{array}{ccc|c} -0.65720 & 0.45650 & -0.25776 & 0.28247 \\ 0.94968 & -0.85357 & -0.45973 & 0.14439 \\ -0.41159 & -0.093762 & -1.8121 & 0.25151 \\ \hline 0.21149 & 0.21465 & -0.10629 & \end{array}\right] & (i=1) \\ \left[\begin{array}{ccc|c} -0.86242 & -0.82758 & 0.12356 & -0.28094 \\ 0.14989 & -0.31188 & -0.71235 & -0.45806 \\ 0.31463 & -0.041254 & -0.56785 & -0.19322 \\ \hline 0.40763 & 0.36311 & 0.31868 & \end{array}\right] & (i=2) \\ \left[\begin{array}{ccc|c} -0.52594 & -0.20583 & -0.85546 & 0.23945 \\ 0.011705 & -1.4926 & 0.64233 & 0.10013 \\ -0.91841 & -0.74820 & -1.3360 & 0.35546 \\ \hline -0.41186 & 0.062732 & -0.41894 & \end{array}\right] & (i=3) \end{cases}$$

Theorem 6 and Lemma 9 with $m_i = 1$ $(i = 1, 2, 3)$ respectively give the minimum γ_2 as 2.7340 and 3.0159. Fine gridding method gives the worse H_2 performance as 1.8669 at $\zeta = [1\ 0\ 0]^T$. Theorem 6 gives more exact analysis than Lemma 9; that is, Proposition 12 holds for this example. However, they both give conservative results. To obtain more exact analysis, parameter-dependency of PPDLFs should be increased as suggested in Remark 14; however, we cannot apply Theorem 6 with $m_i = 2$ $(i = 1, 2, 3)$ due to the prohibitive numerical complexity.

4.3 H_∞ Performance Analysis for Randomly Generated Examples

Finally, we conduct robust H_∞ performance analysis for randomly generated numerical examples. In our examples, $A(\zeta)$ matrices are the same as in Table 1; however, other matrices $B(\zeta)$, $C(\zeta)$ and $D(\zeta)$, all the elements of which lie in $[-0.5, 0.5]$, are randomly generated. For each generated example, we calculate the upper bound of H_∞ performance γ_∞ by applying Theorem 7 and Lemma 10, and set the obtained γ_∞ as $\gamma_{\infty_{\min}}$. For comparison, we calculate the maximum value of H_∞ performance γ_∞ using fine gridding method, and set the obtained γ_∞ as $\gamma_{\infty_{grid}}$. Tables 4 and 5 respectively show the average values of $\gamma_{\infty_{\min}}/\gamma_{\infty_{grid}}$ for MIMO systems and SISO systems. We also give the number of decision variables in parentheses.

A typical example which clearly illustrates Proposition 13 is shown in the following.

$$\left[\begin{array}{c|c} A_i & B_i \\ \hline C_i & D_i \end{array}\right] = \begin{cases} \left[\begin{array}{ccc|c} -0.44588 & 0.53058 & 0.46577 & -0.37069 \\ 0.18373 & -0.44727 & 0.37579 & -0.46117 \\ 0.69592 & -0.84477 & -0.69260 & -0.038438 \\ \hline 0.24713 & 0.099555 & -0.45061 & 0.18998 \end{array}\right] & (i=1) \\ \left[\begin{array}{ccc|c} -1.2033 & -0.17933 & -0.78222 & -0.42739 \\ -0.96830 & -0.22693 & 0.95020 & -0.49252 \\ 0.40224 & 0.10063 & -0.52708 & -0.43766 \\ \hline 0.34847 & -0.34778 & 0.18830 & 0.29317 \end{array}\right] & (i=2) \\ \left[\begin{array}{ccc|c} -1.3977 & 0.43537 & -0.043558 & -0.22457 \\ 0.58232 & -0.48923 & 0.82965 & -0.21313 \\ 0.95700 & 0.23948 & -1.3940 & -0.081949 \\ \hline -0.090526 & -0.049513 & -0.13226 & -0.29596 \end{array}\right] & (i=3) \end{cases}$$

Table 1. Number of systems assured to be robustly stable via Theorem 5 (Thm 5), Lemma 8 (Lem 8), Theorem 3 (Thm 3 [OP2005c]) and Theorem 5 (Thm 5 [OP2005c]) in [Oliveira and Peres, 2005c]

n	N	Thm 5 ($m_i = 1$)	Lem 8 ($m_i = 1$)	Thm 3 [OP2005c]	Thm 5 [OP2005c]
2	2	1000(100)	1000(84)	1000(6)	1000(22)
	3	1000(520)	1000(360)	1000(9)	1000(33)
	4	1000(2576)	1000(1488)	1000(12)	1000(44)
3	2	1000(222)	1000(186)	958(12)	977(48)
	3	1000(1164)	1000(804)	936(18)	963(72)
4	2	1000(392)	1000(328)	952(20)	978(84)
	3	999(2064)	999(1424)	917(30)	953(126)

Table 2. $\gamma_{2_{\min}}/\gamma_{2_{grid}}$ for randomly generated MIMO systems via Theorem 6 (Thm 6) and Lemma 9 (Lem 9)

n	N	$n_w(= n_z)$	Thm 6 ($m_i = 1$)	Lemma 9 ($m_i = 1$)
2	2	2	1.038(461)	1.038(373)
	3	2	1.065(2609)	1.075(1749)
3	2	3	1.013(1007)	1.013(814)
4	2	4	1.011(1769)	1.011(1429)

Table 3. $\gamma_{2_{\min}}/\gamma_{2_{grid}}$ for randomly generated SISO systems via Theorem 6 (Thm 6) and Lemma 9 (Lem 9)

n	N	$n_w(= n_z)$	Thm 6 ($m_i = 1$)	Lemma 9 ($m_i = 1$)
2	2	1	1.053(289)	1.053(231)
	3	1	1.095(1613)	1.112(1043)
3	2	1	1.038(527)	1.040(420)
	3	1	1.119(2941)	1.155(1886)
4	2	1	1.027(845)	1.032(673)

Table 4. $\gamma_{\infty_{\min}}/\gamma_{\infty_{grid}}$ for randomly generated MIMO systems via Theorem 7 (Thm 7) and Lemma 10 (Lem 10)

n	N	$n_w(= n_z)$	Thm 7 ($m_i = 1$)	Lemma 10 ($m_i = 1$)
2	2	2	1.001(325)	1.002(253)
	3	2	1.011(1769)	1.022(1089)
3	2	3	1.028(727)	1.123(565)
4	2	4	1.864(1289)	4.613(1001)

Theorem 7 and Lemma 10 with $m_i = 1$ $(i = 1, 2, 3)$ respectively give the minimum γ_∞ as 735.76 and 2608.5. Fine gridding method gives the worse H_∞ performance as 468.22 at $\zeta = [0\ 1\ 0]^T$. Theorem 7 gives more exact analysis than Lemma 10; that is, Proposition 13 holds for this example. However, they both give conservative results. To obtain more exact analysis, parameter-dependency of PPDLFs should be increased as suggested in Remark 14; however, we cannot apply Theorem 7 with $m_i = 2$ $(i = 1, 2, 3)$ due to the prohibitive numerical complexity.

5. CONCLUSIONS

We propose new methods for robust stability, H_2 performance, and H_∞ performance analysis for linear time-invariant uncertain systems, whose state-space matrices are polytopically parameter-dependent, using polynomi-

Table 5. $\gamma_{\infty_{\min}}/\gamma_{\infty_{grid}}$ for randomly generated SISO systems via Theorem 7 (Thm 7) and Lemma 10 (Lem 10)

n	N	$n_w(=n_z)$	Thm 7 ($m_i=1$)	Lemma 10 ($m_i=1$)
2	2	1	1.001(193)	1.003(154)
	3	1	1.010(1037)	1.011(662)
3	2	1	1.014(351)	1.049(283)
	3	1	1.028(1885)	1.069(1225)
4	2	1	1.021(557)	1.103(452)

ally parameter-dependent Lyapunov functions. The proposed methods are derived via multiple slack variable approach, which encompasses single slack variable approach. Our methods consequently encompass the previously proposed methods via single slack variable approach. Although our methods are only sufficient conditions for our addressed problems, randomly generated examples demonstrate the effectiveness of proposed methods for both robust stability and performance analysis. However, the numerical complexity of our methods is prohibitive for high-order systems with many vertices. This drawback will be further investigated in the future.

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