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Algebraic approach to LQ-optimal control of spatially distributed systems: 2-D case ⋆

P. Augusta * **Z. Hurák** **

* *Institute of Information Theory and Automation, Czech Academy of Sciences, Pod Vodárenskou věží 4, 182 08 Praha 8, Czech Republic, e-mail: augusta@utia.cas.cz*
** *Department of Control Engineering, Czech Technical University in Prague, Technická 2, 166 27 Praha 6, Czech Republic, e-mail: hurak@fel.cvut.cz*

Abstract: In this paper we develop new results on control systems design for spatially distributed linear systems using an n-D systems approach. The basic ideas are explained using as an example heat conduction in a rod where the measurements and control action applied are based on an array of sensors and heaters. The first part of the analysis given shows how the process dynamics for this case can be approximately described by a 2-D transfer-function, i.e. a fraction of two bivariate polynomials. This is followed by stability analysis and tests. Finally, a simple algorithm for design of LQ controller is proposed.

1. INTRODUCTION

The control of distributed-parameter systems has always been a very active topic with applications in many areas. Also the last few years, in particular, have seen developments in associated technology which have a direct impact on actual implementation. In particular, there have been massive developments in the design and implementation of very high quality sensors and actuators and also the associated costs are dropping. As a result, it is now feasible to consider using arrays of sensors and actuators in the control of spatially distributed systems.

Suppose now that a particular application requires hundreds of control inputs and measured outputs in order for progress with control design and (eventual) implementation to be made. In such cases it is still feasible to design and implement a lumped (centralized) control scheme. Some of the more challenging and relevant applications now emerging will, however, require thousands of actuators and sensors and for such cases the central approach is clearly (at very best) inefficient. Hence there is a need for a major revision of the control paradigm for distributed systems and the only feasible approach appears to be the design and implementation of distributed (decentralized) controllers with a regular (and dense) mesh of sensors and actuators. Figs. 1 and 2 respectively illustrate the centralized and distributed control approaches (and the basic difference between them) for temperature regulation in a rod.

Results on some new trends in the design of spatially distributed controllers can be found, for example, in Bamieh et al. [2002], D'Andrea and Dullerud [2003], Stewart et al. [2003], Stein and Gorinevsky [2005], Langbort and D'Andrea [2003], Cichy et al. [2005] and Rogers et al. [2007]. A more applications focus can be found in Kulkarni et al. [2002] where the subject area is adaptive optics. In this case, suitably developed spatially distributed control is required if the objectives set down are to be achieved to the maximum possible extent, e.g. precise control of the shape of a deformable mirror by employing a huge area of piezoelectric actuators.

One possible approach to the problem of the control of systems described by linear partial differential equations using a mesh of sensors and actuators is first formulated as that of control systems design for a discrete approximation (in both time and spatial position) of the dynamics. Under some simplifying assumptions, the dynamics can then be described by a state-space model of n-D systems theory. Relevant background literature on this latter area can be found, for example, in Youla and Gnavi [1979], Šebek et al. [1988], Šebek [1988], Šebek [1994]. This paper gives further results on this approach for 2-D systems where one variable represents time and the second a spatial coordinate.

The systems considered here belong to the class of so-called non-causal (also termed spatially non-causal) systems. A physical example is a heat conducting rod with an array of temperature sensors and actuators in the form of heaters as shown schematically in Fig. 2. This case is used here to illustrate the results developed as a necessary preliminary for eventual application to more complex but physically relevant examples such as deformable mirrors for astronomical telescopes (see, for example, Augusta and Hurák [2006]).

The paper is organized as follows. Section 2 deals with stability of distributed parameter systems. Section 3 proposes an algorithm for controller design. A numerical example is shown in Section 4. Remarks and discussion conclude the paper.

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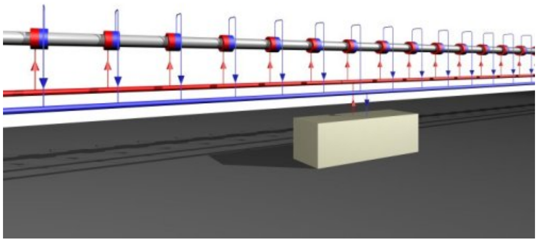


Fig. 1. Centralized control of a distributed parameter system: a rod with an array of heaters and temperature sensors and one controller.

2. STABILITY ANALYSIS OF SYSTEMS WITH DISTRIBUTED PARAMETERS

A linear shift-invariant system is bounded input bounded output (BIBO) stable if, and only if, its impulse response is absolutely summable. Also for systems described by transfer functions on approach to determining if this property holds for a given example is based on *root maps.*

Root map, see Dudgeon and Mersereau [1984], is 2-D graph consisting of n parts. Each part shows the loci of the roots of $a[z_1, \ldots, z_{i-1},\ z_{i+1}, \ldots z_n][z_i]$ as the parameters $z_1, \ldots, z_{i-1},\ z_{i+1}, \ldots z_n$ traverse the unit circle $z_k = e^{j\,\omega_k}$ for $-\pi \leq \omega_k \leq \pi,\ k \neq i,$ for $i = 1, 2, \ldots,\ n.$

Systems with distributed parameters are a special class of n-D systems. The following result expresses BIBO stability in terms of the root map just for systems with distributed parameters.

Lemma 1. An n-D spatially distributed system with transfer function

$$P = \frac{b(z, z_1, z_1^{-1}, \ldots, z_n, z_n^{-1})}{a(z, z_1, z_1^{-1}, \ldots, z_n, z_n^{-1})}, \tag{1}$$

where z and z_l, $l = 1, \ldots n$ correspond to time and space respectively is BIBO stable if, and only if, its root map generated by $a[z_1, z_1^{-1}, \ldots, z_n, z_n^{-1}][z]$ lies inside the unit circle in the z plane.

It is clear from the above lemma that only one root map is interesting when finding out stability of systems with distributed parameters. For proof see Appendix A.

3. LQ CONTROL DESIGN VIA POLYNOMIAL METHODS

In this section a simple procedure for design of LQ controller will be proposed. In theory of systems with lumped parameters to design LQ controller means for a given transfer function $P = \frac{b}{a}$ of a linear system to find controller $C = \frac{y}{x}$ optimal in sense of minimizing quadratic criterion

$$\sum_{k=0}^{\infty} Q\,u^2(k) + R\,y^2(k), \tag{2}$$

where $u(k)$ and $y(k)$ denote input and output of plant respectively, see Fig. 3. In algebraic approach design of such a controller leads to procedure consisting of two steps. The first one is to solve polynomial equation

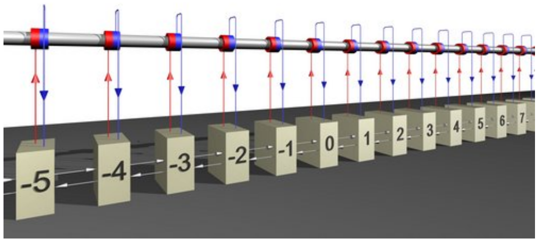


Fig. 2. Distributed control of a distributed parameter system: a rod with an array of heaters and temperature sensors and a distributed controller (an array of controllers).

$$a^\star Q\,a + b^\star R\,b = g^\star g \tag{3}$$

for polynomial g, where star ($\star$) denotes complex conjugate of polynomial. The above equation is often called *polynomial spectral factorization.* The second step is to solve equation

$$a\,x + b\,y = g \tag{4}$$

for polynomials x and y and taking the solution with $\deg y(z) < \deg a(z)$.

In a heuristic way, we will base our algorithm for LQ controller design on the same steps as in the case with lumped parameter systems. Here 1-D polynomials become in n-D ones. In what follows it is shown how n-D polynomial spectral factorization and equations can be solved.

3.1 n-D two-sided polynomial spectral factorization

We are given a real n-D two-sided polynomial
$f = \varphi(z_1, z_1^{-1}, \ldots z_n, z_n^{-1})(z) \cdot \varphi(z_1, z_1^{-1}, \ldots z_n, z_n^{-1})(z^{-1}).$
Spectral factorization of f is defined as a polynomial g if

$$g(z_1, z_1^{-1}, \ldots z_n, z_n^{-1})(z) \cdot g(z_1, z_1^{-1}, \ldots z_n, z_n^{-1})(z^{-1}) = f$$

and $g(z_1, z_1^{-1}, \ldots z_n, z_n^{-1})(z) \neq 0$ for $|z| > 1$.

It is well-known fact that, unlike 1-D case, n-D polynomial factorization problem cannot be solved by applying the Fundamental Theorem of Algebra. In general, it is simply not possible to find polynomial spectral factor of finite form, however, there exist a factorization that has infinite number of terms, see Šebek [1994] for details.

Spectral factorization of multivariate polynomials are discussed in a couple of publications. Algorithm for spectral factorization of 2-D polynomial via computing complex cepstrum was proposed by Dudgeon and Mersereau [1984], but there is no extension to n-D case. Bose and Shi [1988] extended Wilson spectral factorization to 2-D case. Gao et al. [2004] deal with approximate factorization of multivariate polynomials

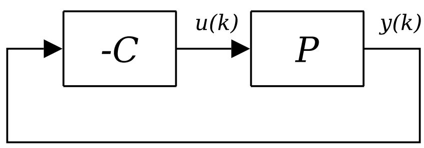


Fig. 3. Closed-loop system

Inspired by Gao et al. [2004] we will use a simple algorithm for approximate spectral factorization of multivariate polynomial based on minimizing polynomial 2-norm. Let $f(z, z_1, \ldots, z_n)$ and $g(z, z_1, \ldots, z_n)$ denote a polynomial to be factorized and spectral factor, respectively. The algorithm can be described by two following steps.

(1) Choose a structure of spectral factor g.
(2) Find coefficients of g such that
  - g is stable in sense of Lemma 1,
  - $\|f - g^\star g\|_2$ is minimized.

Note that possibility to choose a structure of spectral factor should be useful for spatially distributed systems with two (and more) plan coordinates, where the choice has to correspond to actuators/sensors placement (and grid used in spatial discretization, see, for instance, Augusta and Hurák [2006] for details).

As an example, consider

$$f = \left(z - 0.35\left(z_1 + z_1^{-1}\right) - 0.3\right) \cdot \left(z^{-1} - 0.35\left(z_1 + z_1^{-1}\right) - 0.3\right) + 1.$$

Choose spectral factor

$$A\,z + B\left(z_1 + z_1^{-1}\right) + C$$

with $A, B, C \in \mathbb{R}$ satisfying the following condition to spectral factor be stable.

$$C + \frac{B}{2} \leq A, \quad C - \frac{B}{2} \leq A.$$

Square of norm $\|f - g^\star g\|_2$ is

$$6 - 4.7\,A^2 + 2.8\,A\,B + 1.2\,A\,C - 7.3\,B^2 - 1.7\,B\,C - 4.7\,C^2 + A^4 + 8\,A^2\,B^2 + 4\,A^2\,C^2 + 4\,B^4 + 12\,B^2\,C^2 + C^4.$$

Its minimizing with the above constraint can be done using GloptiPoly by Henrion and Garulli [2005], Henrion et al. [2007], which gives

$$A = 1.5, \quad B = -0.2, \quad C = -0.23.$$

Approximate spectral factor of f is

$$g = 1.5\,z - 0.2\left(z_1 + z_1^{-1}\right) - 0.23.$$

Due to using of approximation of spectral factor and getting finite series (polynomials), the resulting controller is practically always suboptimal one.

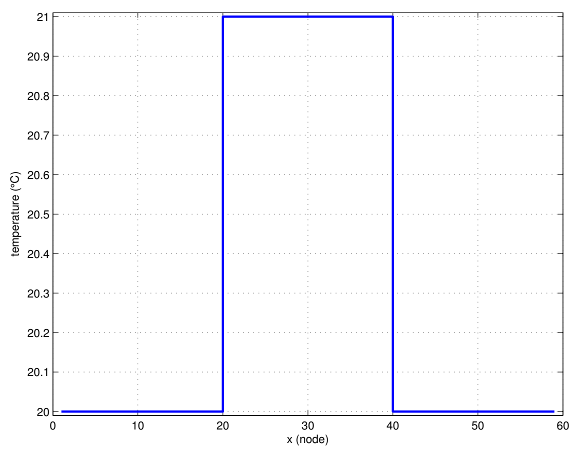


Fig. 4. Initial condition

3.2 Linear equations with n-D two-sided polynomials

Let us consider polynomials a, b, g to be elements of ring $\mathcal{R}[z_1, z_1^{-1}, \ldots z_n, z_n^{-1}][z]$. The polynomial equation (4) can now be solved by algorithms for polynomial equations introduced by Kučera [1979]. See Hunt [1993], Šebek [1994] for details.

4. A NUMERICAL EXAMPLE: HEAT CONDUCTION IN A ROD

Consider a model of heat conduction in a rod, with array of temperature sensors and actuators, schematically sketched in Fig. 2. This system is described by the well-known heat equation

$$\frac{\partial u(x,t)}{\partial t} = \kappa\,\frac{\partial^2 u(x,t)}{\partial x^2} + q(t,x) \tag{5}$$

where u denotes temperature (°C), t and x denote time (s) and a spatial coordinate (m) respectively, κ is a constant $(\mathrm{m^2\,s^{-1}})$ and q is the input heat $(\mathrm{°C\,s^{-1}})$.

Transfer function between input heat and temperature was derived as follows, see Augusta et al. [2007a] for details.

Discretization of (5) using finite difference methods, Strikwerda [1989], approximates partial derivatives by central differences, i.e.

$$\left(\frac{\partial u}{\partial t}\right)_{k,i} = \frac{u_{k+1,i} - u_{k,i}}{T}, \quad \left(\frac{\partial^2 u}{\partial x^2}\right)_{k,i} = \frac{u_{k,i-1} - 2\,u_{k,i} + u_{k,i+1}}{h^2} \tag{6}$$

where $T > 0$ is the sampling (time) period and $h > 0$ denotes the distance between the nodes along the rod. Hence the partial differential equation description of (5) is approximated by the partial difference (recurrence) equation

$$u_{k+1,i} = \frac{T}{h^2} u_{k,i-1} + \left(1 - 2\frac{T}{h^2}\right) u_{k,i} + \frac{T}{h^2} u_{k,i+1} + q_{k,i} \tag{7}$$

where k corresponds to discrete "time" and i to the coordinate of the node and setting $\kappa = 1$ incurs no loss of generality.

The recurrence equation can be more compactly expressed by application of a two-sided z-transform to yield

$$\frac{U(z,z_1)}{Q(z,z_1)} = P(z,z_1) = \frac{b(z,z_1)}{a(z,z_1)} = \frac{1}{z - \frac{T}{h^2}\,z_1^{-1} - 1 + 2\,\frac{T}{h^2} - \frac{T}{h^2}\,z_1}$$

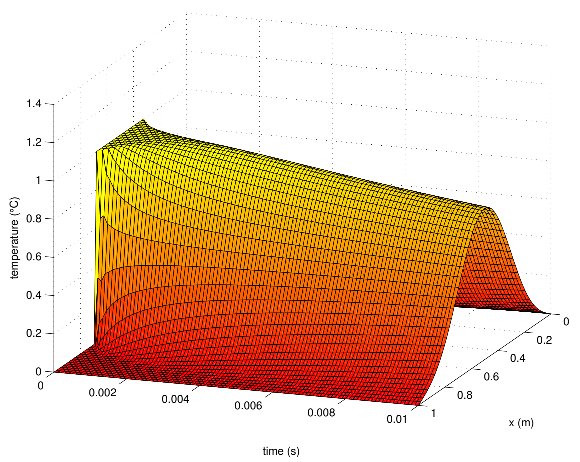
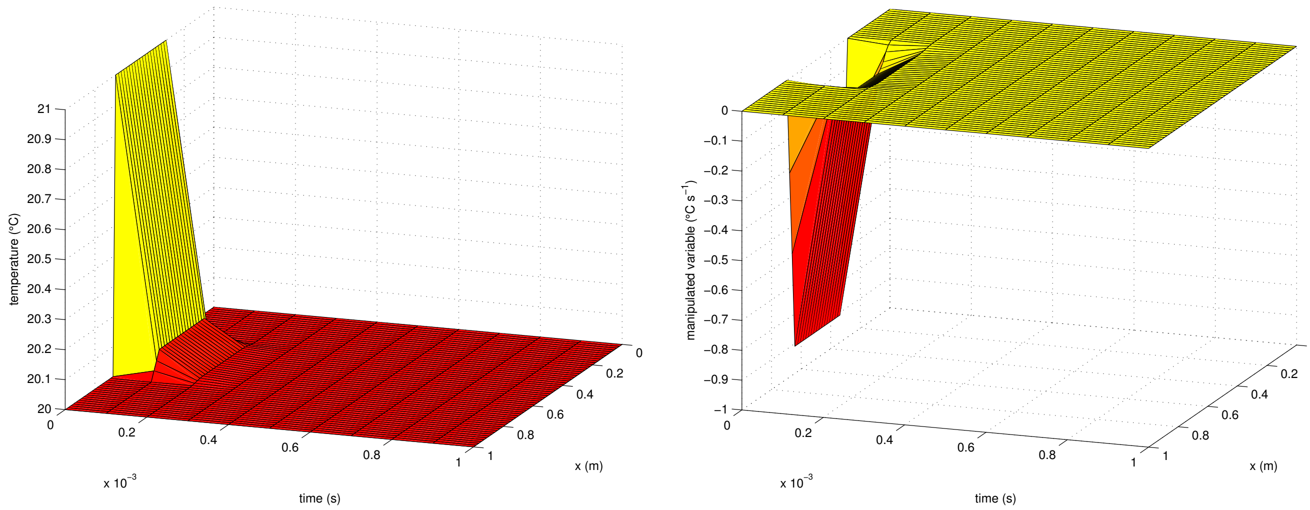
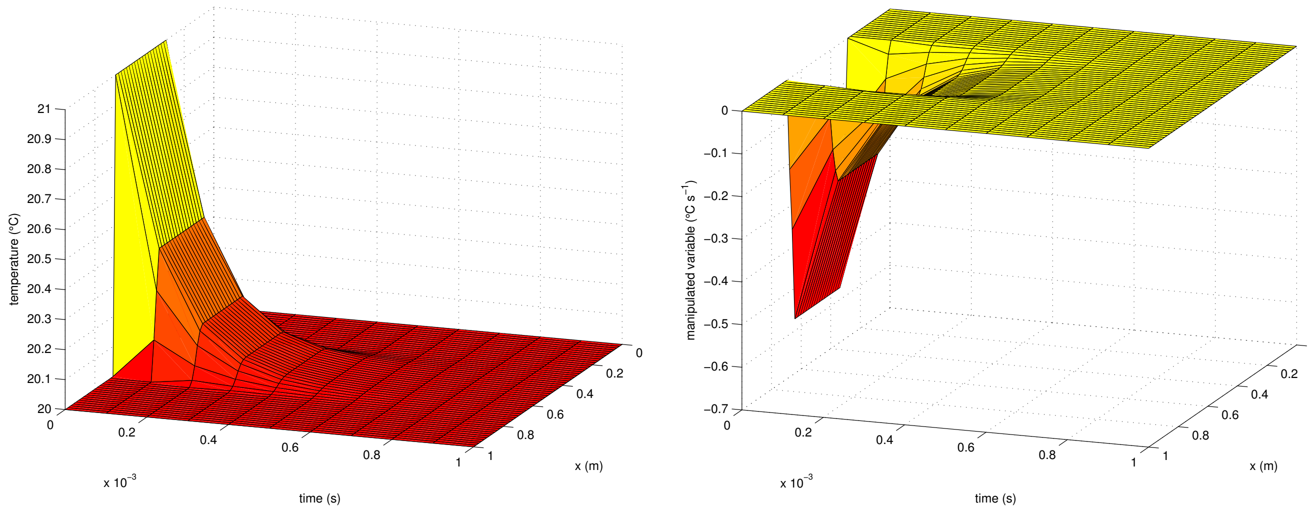


Fig. 5. Uncontrolled system



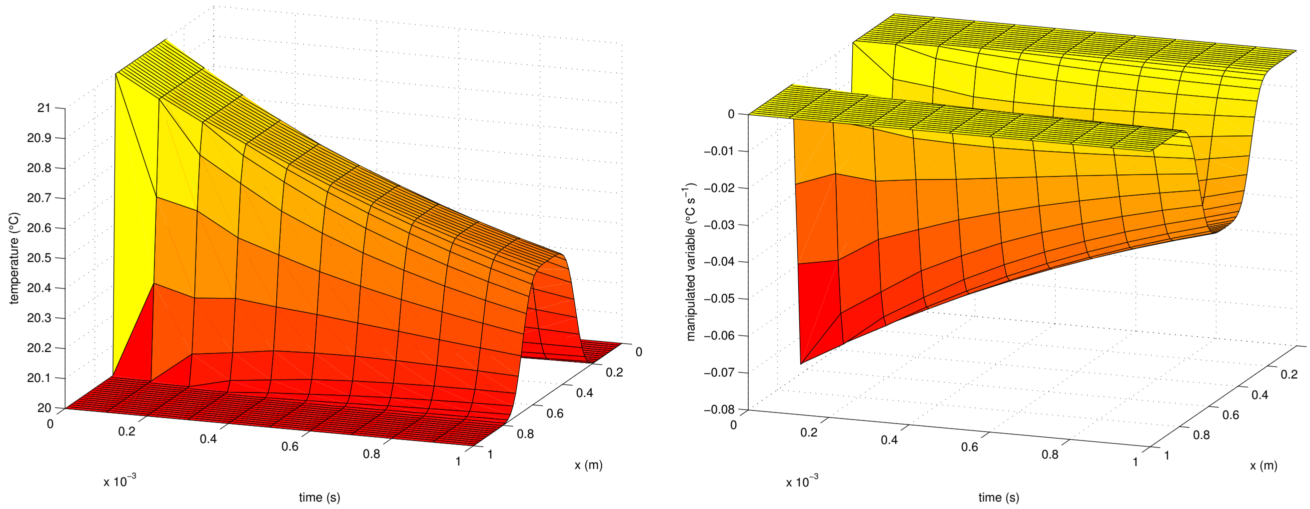
(a) (b)

Fig. 6. LQ control, $Q = 1$, $R = 10$: (a) output of plant, (b) manipulated variable



(a) (b)

Fig. 7. LQ control, $Q = 1$, $R = 1$: (a) output of plant, (b) manipulated variable



(a) (b)

Fig. 8. LQ control, $Q = 10$, $R = 1$: (a) output of plant, (b) manipulated variable, (c) output of plant and manipulated variable at the middle of the rod

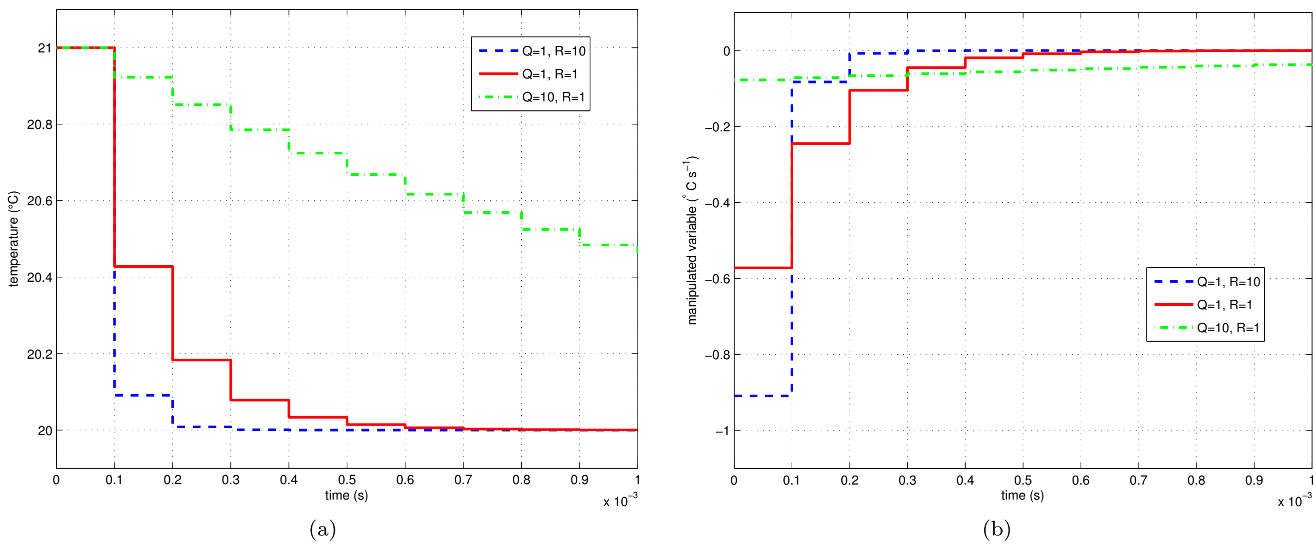


Fig. 9. LQ control: (a) output of plant and (b) manipulated variable at the middle of the rod

where z and z_1 correspond to time and space respectively. The above transfer function is of form (1), so theory described here can be used.

As a numerical example consider the case of a rod of length 1 m. Let the number of nodes be $n = 59$ with distance between two successive ones $h = 1/n$, and take the sampling period as $T = 0.1$ ms.

Fig. 4 and Fig. 5 gives the initial condition and the response of the uncontrolled system respectively.

Figs. 6, 7, 8 show responses to initial condition of closed-loop system with LQ controller for weights $Q = 1, R = 10$, $Q = 1, R = 1$ and $Q = 10, R = 1$, respectively. Fig. 9 shows the same responses at the middle of the rod.

5. CONCLUSIONS

The paper has considered the stability and control of systems whose dynamics can be approximated by so-called spatially non-causal 2-D systems described by fraction of two multivariate polynomials. In the case of stability, the approach used is based on writing the required condition in terms of a one-variable polynomial with coefficients which are defined in terms of the other variable. The new results obtained consist of design of controller using of computation of approximate spectral factorization of 2-D two-sided polynomial and solving linear equations with 2-D two-sided polynomials.

The responses shown in 6, 7, 8, 9 are acceptable. The closed-loop system behaviour is conformable with that obtained in LQ control theory of systems with lumped parameters.

The results introduced here are highly promising but are clearly still at an early stage in terms of an overall solution. They show that the 2-D/n-D systems can have a role to play in this area, especially we believe in the production of tractable solutions to problems which will provide vital insight into the problems encountered with very large scale examples.

Appendix A. PROOF OF LEMMA 1

A proof of Lemma 1, where von Neumann's theory of stability was used, was introduced by Augusta et al. [2007b]. Here we offer a proof based on properties of $\mathcal{Z}$-transform.

Let $P(z, z_1, z_1^{-1}, \ldots, z_n, z_n^{-1}) = \mathcal{Z}\{g_{k,k_1,\ldots,k_n}\}$ ($\mathcal{Z}$ denotes the $\mathcal{Z}$-transform). Bounded input signal satisfies a condition $|f_{k,\cdot}| < M < +\infty$ for all $k \geq 0$. Response of plant is given by convolution

$$h_{k,k_1,\ldots,k_n} = \sum_{i=0}^{+\infty} \sum_{i_1,\ldots,i_n=-\infty}^{+\infty} f_{i,i_1,\ldots,i_n}\, g_{k-i,k_1-i_1,\ldots,k_n-i_n}.$$

A plant to be stable has to satisfy

$$|h_{k,\ldots,k_n}| = \left| \sum_{i=0}^{+\infty} \sum_{i_1,\ldots,i_n=-\infty}^{+\infty} g_{i,i_1,\ldots,i_n}\, f_{k-i,k_1-i_1,\ldots,k_n-i_n} \right| \leq M \sum_{i=0}^{+\infty} \sum_{i_1,\ldots,i_n=-\infty}^{+\infty} |g_{i,i_1,\ldots,i_n}| < +\infty.$$

Hence, the necessary and sufficient condition is

$$\sum_{k=0}^{+\infty} \sum_{k_1,\ldots,k_n=-\infty}^{+\infty} |g_{k,k_1,\ldots,k_n}| < +\infty. \tag{A.1}$$

The sequence $\{g_{k,k_1,\ldots,k_n}\}$ is the impulse response of a plant, so $\{g_{k,k_1,\ldots,k_n}\} = \mathcal{Z}^{-1}\{P(z, z_1, z_1^{-1}, \ldots, z_n, z_n^{-1})\}$ holds. If the point $(z_0, z_{0\,1}, \ldots, z_{0\,n})$ lies within the region of convergence of P, then the points $(z, z_1, \ldots, z_n)$ satisfying $|z_1| = |z_{0\,1}|, \ldots, |z_n| = |z_{0\,n}|$ and $|z| \geq |z_0|$ lie within the region of convergence too (see also Dudgeon and Mersereau [1984]). To satisfy (A.1), P has to exist for $|z| \geq |z_0| = 1$. It means that a plant to be stable has to have no roots in region $|z| \geq 1$. So, the boundary of the region of convergence is a function only of $|z_1|, \ldots |z_n|$ and all its values have to be less than 1. Finally, this property can be verify by the root map generated by $a[z_1, \ldots, z_n][z]$. This concludes the proof.

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