Proceedings of the 17th World Congress
The International Federation of Automatic Control
Seoul, Korea, July 6-11, 2008

A High Gain Observer based LMI Approach

M. Rodrigues * **H. Hammouri** * **C. Mechmeche** ** **N. Benhadj Braiek** **

* *LAGEP, UMR CNRS 5007, Université Lyon 1 - ESCPE Lyon 1, 69622 Villeurbanne Cedex, France (Tel: +334 724 318 92; e-mail: rodrigues@lagep.univ-lyon1.fr).*
** *Laboratoire d'Etude et Commande Automatique des Processus (LECAP), Ecole Polytechnique de Tunisie, Tunis*

Abstract: This paper deals with the observability analysis and the observer synthesis of a class of nonlinear systems. The observer that we consider has a constant gain and concerns a class of uniformly observable systems. A sufficient condition permitting to design such a constant gain observer is given. The calculation of this gain is based on LMI technics.

Keywords: Nonlinear systems, uniform observability, nonlinear observer, LMI

1. INTRODUCTION

In this paper, we propose sufficient conditions under which a controlled nonlinear system admits an exponential observer. The need to study the observer design problem for nonlinear dynamical systems is, from a control point of view, well understood by now.

The existence of an observer relies on observability properties which are a bit more involved than in the linear case. Many techniques have been developed for designing an observer for nonlinear systems. Among these techniques, a rather natural approach consists in considering systems which can be steered by a change of coordinates into state affine systems up to output injection. From observability point of view, these systems possess similar properties as linear systems and an extended Luenberger observer can be designed. Several authors Krener and Isidori [1983], Krener and Respondek [1985] Xia and Gao [1988], have characterized such nonlinear systems. An extension of these results consists to ask how one can design a constant gain observer of nonlinear systems (so-called an extended Luenberger observer). The idea consists to develop normal forms characterizing the class of nonlinear systems which are observable independently on the inputs (called uniformly observable systems). Based on these normal forms and the high gain technics, the authors in Hammouri [1991], Bornard and Hammouri [1991], Gauthier et al. [1992], Gauthier and Kupka [1994] gave an extended Luenberger. Many extensions of these results to multi-output uniformly observable systems have been proposed (see for instance H. Hammouri [1977], Hammouri and Farza [2003]).

Under a geometric condition (called uniform observability structure), the authors in Hammouri and Farza [2003] gave a normal form which extend those proposed in Hammouri [1991], Gauthier et al. [1992], Gauthier and Kupka [1994]. Based on this normal form, the authors in Hammouri and Farza [2003] propose a constant gain observer. The algorithm permitting to calculate the gain of the observer is based on the existence of some cone of matrices. The computation of this cone is very difficult to check in practical cases. Based on the LMI technics, our aim here consists to replace the above condition by a more simpler. Indeed, using LMI approach in this new observer gain design, gives a sufficient condition which is more easy to be verified and implemented.

The paper is organized as follows. The next section resumes some previous necessary results and gives the problem statement. In Section 2, a high gain observer is designed through LMI technics under a new condition. We end this paper by an illustrative example before concluding.

2. OBSERVER SYNTHESIS

The system that we consider (Hammouri and Farza [2003]) has the following normal form:

$$\begin{cases} \dot{z} = F(u, z) \\ y = Cz \end{cases} \tag{1}$$

where $F(u,z) = \begin{pmatrix} F^1(u,z) \\ \vdots \\ F^q(u,z) \end{pmatrix}$, $z = \begin{pmatrix} z^1 \\ \vdots \\ z^q \end{pmatrix} \in \mathbb{R}^n$; $u \in U$ a compact subset of $\mathbb{R}^m$; $z^i \in \mathcal{R}^{n_i}$; $n_1 \geq n_2 \geq \ldots \geq n_q$; $n_1 + \ldots + n_q = n$. Each function $F^i(u,z)$, $i = 1, \ldots, q-1$ satisfies the following triangular structure:

$$F^i(u,z) = F^i(u, z^1, \ldots, z^{i+1}),\ z^i \in \mathcal{R}^{n_i} \tag{2}$$

with the following rank condition:

$$Rank(\frac{\partial F^i}{\partial z^{i+1}}(u,z)) = n_{i+1} \quad \forall z \in \mathcal{R}^n; \forall u \in U \qquad (3)$$

Nonlinear systems that can be steered by a change of coordinates to the form (1), (2), (3) are those satisfying some geometrical condition (called a U-uniform observability structure Hammouri and Farza [2003]).

Definition 2.1. A constant gain exponential observer for system (1) is a dynamical system of the form:

$$\dot{\hat{z}} = F(u,\hat{z}) + G(C\hat{z} - y) \qquad (4)$$

where G is a constant matrix such that: $\|\hat{z}(t) - z(t)\| \leq \lambda e^{-\mu t}\|\hat{z}(0) - z(0)\|$ where $\lambda > 0$ and $\mu > 0$ are constants which do not depend on the input $u \in L_\infty(\mathbb{R}^+, U)$ nor on $\hat{z}(0), z(0)$. ■

2.1 The existing results

In Hammouri and Farza [2003], the authors gave some sufficient conditions under which a constant gain observer was designed for system (1). These conditions may be formulated as follows:

H1) **Global Lipschitz condition**:

$\exists c > 0; \forall u \in U; \forall z, z' \in \mathbb{R}^n$,

$\|F(u,z) - F(u,z')\| \leq c\|z - z'\|$. ■

H2) **The Cone condition**:

$\forall k, 1 \leq k \leq q-1$, there exists $n_k \times n_{k+1}$ constant matrix $S_{k,k+1}$ such that

$$\frac{\partial F^k}{\partial z^{k+1}}(u,z) \in \mathcal{C}(n_k, n_{k+1}; -1; S_{k,k+1}); \forall (u,z) \in U \times \mathcal{R}^n$$

where $\mathcal{C}(n_k, n_{k+1}; -1)$ is the cone defined by $\{M \in \mathcal{M}(n_k, n_{k+1}; \mathbb{R});\quad \text{s.t. } M^TS + S^TM < \alpha I_{k+1}\}$.
$\mathcal{M}(n_k, n_{k+1}; \mathbb{R})$ is the space of $n_k \times n_{k+1}$ real matrices and I_{k+1} is the $(k+1)\times(k+1)$ identity matrix. ■

Set $Z = (Z_1, \ldots, Z_{q-1}) \in \mathbb{R}^n \times \ldots \times \mathbb{R}^n$. Under the above hypotheses, the authors in Hammouri and Farza [2003], show that there exists a symmetric positive definite matrix P and constants $\rho > 0$ and $\eta > 0$ such that for every $(u, Z_1, \ldots, Z_{q-1}) \in U \times \mathbb{R}^n \times \ldots \times \mathbb{R}^n$, we have:

$PA(u,Z) + A^T(u,Z)P - \rho C^TC \leq -\eta I$, where

$$A(u,Z) = \begin{pmatrix} 0 & A_1(u,Z_1) & 0 & \ldots & & 0 \\ \vdots & 0 & A_2(u,Z_2) & 0 & \ldots & \vdots \\ \vdots & \ldots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ldots & \ldots & \ddots & \ddots & 0 \\ \vdots & \ldots & \ldots & \ddots & \ddots & A_{q-1}(u,Z_{q-1}) \\ 0 & \ldots & \ldots & \ldots & \ldots & 0 \end{pmatrix}$$

with $A_k(u,Z_k) = \dfrac{\partial F^k}{\partial Z_k^{k+1}}(u,Z_k)$.

Using this construction, the authors show that an exponential observer for system (1) takes the form:

$$\dot{\hat{z}} = F(u,\hat{z}) + \Delta_\theta P^{-1}C^T(C\hat{z} - y) \qquad (5)$$

where $\Delta_\theta = \begin{pmatrix} \theta I_{n_1} & 0 & \ldots & 0 \\ 0 & \theta^2 I_{n_2} & 0 & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \ldots & 0 & \theta^q I_{n_q} \end{pmatrix}$, I_{n_k} is the $n_k \times n_k$ identity matrix, $k = 1, \ldots, q$ and θ .

The drawback of the above observer's gain construction lies into the fact that the observer's gain depends on the matrices $S_{k,k+1}$ and that their construction is a very difficult task.

In what follows, we will give a more simpler construction based on the LMI technics. As in the control problems, this new construction by LMI opens a new field of investigation where the observer gain may take into account of the parameter uncertainty for example. We present this new observer gain design in the following section.

3. HIGH GAIN OBSERVER BASED LMI TECHNICS

As in Hammouri and Farza [2003], the assumption **H1**) is maintained. However, assumption **H2**) will be reformulated as follows:

$A(u,Z)$ is the matrix defined above, from **H1**), the set of matrices $\mathcal{E} = \{A(u,Z); (u,z) \in U \times \mathbb{R}^n\}$ is a bounded subset of $\mathcal{M}(n,n;\mathbb{R})$. ■

H3) **Polytopic condition**:

$\mathcal{E}$ is contained in a polytopic convex set $\mathcal{P} = Co\{M_1, \ldots, M_l\}$ of $\mathcal{M}(n,n;\mathbb{R})$, where the M_i's are the vertices of $\mathcal{P}$ for which there exist a $n \times n$ symmetric positive definite matrix P and a $n \times p$ matrix W such that:

$$\text{for } 1 \leq i \leq l, \quad (M_i + KC)^TP + P(M_i + KC) < 0 \quad (6)$$

This inequality (6) can be linearized as follows

$$\text{for } 1 \leq i \leq l, \quad M_i^TP + PM_i + C^TW^T + WC < 0 \quad (7)$$

where $W = PK$. ■

Remark *3.1.* Since the polytopic set $\mathcal{P}$ is a compact convex set, it follows that there exists $\alpha > 0$ such that:

$$\text{for } 1 \le i \le l, \quad M_i^T P + P M_i + C^T W^T + WC \le -\alpha I \quad (8)$$

for every $M \in \mathcal{P}$.

Theorem *3.1.* Under hypotheses **H1**) and **H3**), an exponential observer for system (1) takes the form:

$$\dot{\hat{z}} = F(u,\hat{z}) + \Delta_\theta P^{-1} W (C\hat{z} - y) \quad (9)$$

where Δ_θ is the diagonal matrix defined above. The gain of the observer is $K = P^{-1}W$. ■

Proof *3.1.* We will show that there exists θ_0, such that $\forall \theta \ge \theta_0$, the error $e(t) = \hat{z}(t) - z(t)$ exponentially converges to 0,where $z(t)$ is the unknown state and $\hat{z}(t)$ is the its estimate given by (9).

Set $\widetilde{F}^i(u,z,\hat{z}) = F^i(u,\hat{z}^1,\ldots,\hat{z}^i,z^{i+1})$ for $1 \le i \le q-1$; $\widetilde{F}^q(u,z,\hat{z}) = F^q(u,\hat{z})$ and, $\widetilde{F}(u,z,\hat{z}) = \begin{pmatrix} \widetilde{F}^1(u,z,\hat{z}) \\ \vdots \\ \widetilde{F}^q(u,z,\hat{z}) \end{pmatrix}$, we obtain:

$$\begin{aligned} \dot{e} &= (F(u,\hat{z}) - \widetilde{F}(u,z,\hat{z})) + (\widetilde{F}(u,z,\hat{z}) - F(u,z)) \\ &\quad + \Delta_\theta P^{-1} W (C\hat{z} - y) \end{aligned} \quad (10)$$

To show that $e(t)$ exponentially converges to 0, it suffices to show that $\epsilon(t) = \Delta_\theta^{-1} e(t)$ exponentially converges to 0.

From (10), we deduce:

$$\begin{aligned} \dot{\epsilon} &= \Delta_\theta^{-1}(F(u,\hat{z}) - \widetilde{F}(u,z,\hat{z})) + \Delta_\theta^{-1}(\widetilde{F}(u,z,\hat{z}) - F(u,z)) \\ &\quad + P^{-1}W(C\hat{z} - y) \end{aligned} \quad (11)$$

Using the expression of $\widetilde{F}(u,z,\hat{z})$ and the fact that $F(u,z)$ has a triangular structure, the main value theorem yields to:
$F(u,\hat{z}) - \widetilde{F}(u,z,\hat{z}) = A(u,Z)e$,

where $Z = (Z_1,\ldots,Z_{q-1})$ and $A(u,Z)$ is the upper diagonal matrix given above. Moreover the k th bloc of $A(u,Z)$ takes of the form $A_k(u,Z_k) = \dfrac{\partial F^k}{\partial z^{k+1}}(u, Z_k^1, \ldots, Z_k^{k+1})$,

where $Z_k^i(t) = z^i(t) + \Lambda^i(t) e^i(t)$ and Λ is a diagonal matrix hose coefficients are in $[0,1]$.

Using the structure of $A(u,Z)$ and the fact that $C = (I_{n_1}, 0, \ldots, 0)$, a simple calculation yields to:

$\Delta_\theta^{-1}(F(u,\hat{z}) - \widetilde{F}(u,z,\hat{z})) = \Delta_\theta^{-1} A(u,Z) e = \theta A(u,Z)\epsilon$ and $WC\Delta_\theta = \theta WC$. Hence (11) becomes:

$$\dot{\epsilon} = \theta(A(u,Z) + P^{-1}WC)\epsilon + \Delta_\theta^{-1}(\widetilde{F}(u,z,\hat{z}) - F(u,z)) \quad (12)$$

To end the proof of the theorem, it suffices to show that $V(t) = \epsilon^T(t) P \epsilon(t)$ exponentially converges to 0.
A simple calculation yields to:

$$\begin{aligned} \dot{V}(t) &= \theta \epsilon^T(t)((A(u,Z) + P^{-1}WC)^T P + P(A(u,Z) \\ &\quad + P^{-1}WC))\epsilon(t) \\ &\quad + 2\epsilon^T(t) P \Delta_\theta^{-1}(\widetilde{F}(u,z,\hat{z}) - F(u,z)) \end{aligned} \quad (13)$$

From hypothesis **H3**) and remark 3.1, we deduce:

$$\dot{V}(t) \le -\alpha\theta \|\epsilon(t)\|^2 + 2\epsilon^T(t) P \Delta_\theta^{-1}(\widetilde{F}(u,z,\hat{z}) - F(u,z)) \quad (14)$$

Using the triangular structure of $F(u,z)$ and Δ_{theta}, we get:

$\Delta_\theta^{-1}(\widetilde{F}(u,z,\hat{z}) - F(u,z)) =$

$$\begin{pmatrix} \theta^{-1}(F^1(u,\widehat{z}^1,z^2) - F^1(u,z^1,z^2)) \\ \vdots \\ \theta^{-k}(F^k(u,\widehat{z}^1,\ldots,\widehat{z}^k,z^{k+1}) - F^1(u,z^1,\ldots,z^k,z^{k+1})) \\ \vdots \\ \theta^{-q}(F^q(u,\widehat{z}^1,\ldots,\widehat{z}^{q-1},z^q) - F^1(u,z^1,\ldots,z^{q-1},z^q)) \end{pmatrix}$$

From assumption **H1**):

$\|F^k(u,\widehat{z}^1,\ldots,\widehat{z}^k,z^{k+1}) - F^1(u,z^1,\ldots,z^k,z^{k+1})\| \le c\sqrt{\|e^1\|^2 + \ldots + \|e^k\|^2}$. Thus, for $\theta \ge 1$, we have
$\theta^{-k}\|F^k(u,\widehat{z}^1,\ldots,\widehat{z}^k,z^{k+1}) - F^1(u,z^1,\ldots,z^k,z^{k+1})\|$
$\le \lambda\sqrt{\|e^1\|^2 + \ldots + \|e^k\|^2}$,

for some constant λ which doesn't depend on θ.

Combining this last fact with (15), we obtain:

$$\dot{V}(t) \le -\alpha\theta\|\epsilon(t)\|^2 + \beta\|\epsilon(t)\|^2 \quad (15)$$

where β is a constant which doesn't depend on θ.
To end the proof, it suffices to take $\theta > \dfrac{\beta}{\alpha}$. □

4. COMMENTS AND EXAMPLES

In this section, we will give some remarks and examples concerning the existence of a constant gain observer.

Remark *4.1.* Assumption **H1**) and conditions (2), (3) are not generally sufficient for the existence of a constant gain observer for system (1).

Indeed, consider the following example:

$$\begin{cases} \dot{z}^1 = u_1 z^3 \\ \dot{z}^2 = u_2 z^3 \\ \dot{z}^3 = 0 \\ y = Cz = \begin{pmatrix} z^1 \\ z^2 \end{pmatrix} \end{cases} \quad (16)$$

where $u = (u_1, u_2)$ belongs to the unit circle $U = \{u \ \ s.t. \ \ \|u\| = 1\}$.

Clearly system (16) takes the form (1) and satisfying (2), (3) and assumption **H1**). Let us show that system (16) doesn't admit a constant gain observer.

Assuming that there exists a constant gain observer of the form:

$$\dot{\hat{z}} = A(u)\hat{z} + K(C\hat{z} - y) \quad (17)$$

where, $A(u) = \begin{pmatrix} 0 & 0 & u_1 \\ 0 & 0 & u_2 \\ 0 & 0 & 0 \end{pmatrix}$, $K = \begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \\ k_{31} & k_{32} \end{pmatrix}$ is a constant matrix and $C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$.

Thus, for every $u \in L^\infty(\mathcal{R}^+, U)$, the error equation:

$$\dot{e} = (A(u) + KC)e \quad (18)$$

is exponentially stable at the origin.

In particular, the error equations associated to inputs $u(t) = (1, 0)$ and $u(t) = (-1, 0)$ are exponentially stable. This implies that:

$\begin{pmatrix} k_{11} & k_{12} & 1 \\ k_{21} & k_{22} & 0 \\ k_{31} & k_{32} & 0 \end{pmatrix}$ and $\begin{pmatrix} k_{11} & k_{12} & -1 \\ k_{21} & k_{22} & 0 \\ k_{31} & k_{32} & 0 \end{pmatrix}$ are both Hurwitz matrices.

A simple calculation shows that this yields to the following contradiction: $k_{21}k_{32} - k_{31}k_{22} < 0$ and $k_{21}k_{32} - k_{31}k_{22} > 0$.

In what follows, we will give an example which illustrate our LMI method. Let consider the following system with $u(t) \in [0, 1]$:

$$\begin{cases} \dot{z}^1 = A_1(u) z^2 \\ \dot{z}^2 = A_2(u) z^3 \\ \dot{z}^3 = 0 \\ y = Cz = z^1 \end{cases} \quad (19)$$

with the following matrices

$$\begin{cases} \dot{z} = A(u) z \\ y = Cz = z^1 \end{cases} \quad (20)$$

where,

$$A(u) = \begin{pmatrix} 0 & 0 & 1 & u & 0 \\ 0 & 0 & -u & 1 & 0 \\ 0 & 0 & 0 & 0 & 1-u \\ 0 & 0 & 0 & 0 & u \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, C = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} \quad (21)$$

The vertices of the polytope are $A(0)$ and $A(1)$.

We solve LMI from equation (7), with $K = P^{-1}W$, and we obtain:

$$P = \begin{pmatrix} 33.54 & 2.71 & -12.74 & -4.78 & -19.24 \\ 2.71 & 28.51 & 4.40 & -13.68 & -8.13 \\ -12.74 & 4.40 & 27.78 & 6.01 & -14.95 \\ -4.78 & -13.68 & 6.01 & 35.32 & -16.14 \\ -19.24 & -8.13 & -14.95 & -16.14 & 78.52 \end{pmatrix} \quad (22)$$

$$W = \begin{pmatrix} 25.50 & 509.18 \\ -509.77 & 25.82 \\ 31.14 & -8.07 \\ 17.81 & 28.16 \\ -6.26 & -1.52 \end{pmatrix} \quad (23)$$

$$K = \begin{pmatrix} 1.59 & 30.89 \\ -25.86 & 3.30 \\ 6.63 & 18.53 \\ -12.06 & 9.18 \\ -3.58 & 13.3155 \end{pmatrix} \quad (24)$$

5. CONCLUSION

In this paper, an observer synthesis of a class of nonlinear systems is presented. The paper focusses on the design of a high gain observer by LMI technics which allow more easier solutions to be verified. A short example illustrates the developed technic by the design of a constant gain. In future works and as in the control problems, this new observer gain design may take into account of the parameter uncertainty which should be more easy by the use of LMI.

REFERENCES

G. Bornard and H. Hammouri. A high gain observer for a class of uniformly observable systems. *Proc. 30th IEEE Conference on Decision and Control, Brighton*, 122:176–192, 1991.

J.P. Gauthier, H. Hammouri, and S. Othman. A simple observer for nonlinear systems - application to bioreactors. *IEEE Trans. on Aut. Control*, 37:875–880, 1992.

J.P. Gauthier and I.A.K. Kupka. Observability and observers for nonlinear systems. *SIAM J. Control. Optim.*, 32:975–994, 1994.

F. Armanet H. Hammouri, B. Targui. High gain observer based on a triangular structure. *IEEE Trans. on Aut. Control*, 22:728–740, 1977.

H. Hammouri. *Sur la théorie de l'observation et des observateurs des systèmes non linéaires.* Thèse d'état, Institut National Polytechnique de Grenoble, 1991.

H. Hammouri and M. Farza. Nonlinear observer for locally uniformly observable systems. *ESAIM Control, Optimisation and Calculus of Variations*, 9:353–370, March 2003.

A.J. Krener and A. Isidori. Linearization by output injection and nonlinear observers. *Syst. and Control Letters*, 3:47–52, 1983.

A.J. Krener and W. Respondek. Nonlinear observers with linealizable error dynamics. *SIAM J. Control. Optim.*, 23:197–216, 1985.

X.H. Xia and W.B. Gao. Nonlinear observer design by observer canonical forms. *Int. J. Control*, 47:1081–1100, 1988.