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A high/low gain bundle of observers: application to the input estimation of a bioreactor model

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Abstract: This paper proposes a new interval observer design based on a high/low gain concept, developed for the estimation of the input of a bioreactor. In a first step, a high gain bounded error observer is proposed, which is able to estimate the unknown input of the considered system. This observer is useful when an accurate model and noise–free measurements are available. We show that the error of the high gain observer can be dynamically bounded, and then we generate guaranteed bounds on this error through an interval observer. The estimation scheme is then extended to an uncertain framework. Taking advantage of the nature of the interval estimates, we run in parallel various observers using high or low gains values and then we take the best estimates. The method is applied to the input estimation of a simple bioreactor model.

1. INTRODUCTION

The lack of accurate information when dealing with biotechnological processes is a well known problem. For example, system inputs can be uncertain, measurements are often biased and only rough approximate model of the system dynamics is available. These drawbacks have motivated the development of robust methods for the control of these processes and also for the estimation of those variables that cannot be directly measured.

Robust state estimation of uncertain dynamical systems is a consolidated trend of research. Various techniques in this field have been developed, for example: state estimation by means of convex sets, such as ellipsoidal methods (Kurzhanski et al., 1994), interval observers (Gouzé et al., 2000), bounded error estimators using interval analysis (Jaulin, 2002), etc. On the other hand, estimation of unknown inputs appears as a relatively new subject. The objective is to solve the so called inverse estimation problem, that is, going back from measured outputs to the unknown inputs of the system. This problem has been solved with observers that are constructed to simultaneously perform estimates over the state and the inputs. Most of these methods have been formulated for linear models (Kim and Goodall, 2005). Some approaches apply to nonlinear models where nonlinearities are perfectly known, and cancel when writing error dynamics equations. In a nonlinear framework, (Liu and Peng, 2002) consider an *a-priori* modeling of the disturbance estimates according to the available system knowledge. Another approach can be found in (Corless and Tu, 1998), where the disturbance is treated as a nonlinear non autonomous function. This method is characterized by Lyapunov functionals for the stability analysis of the estimations.

In the case of biotechnological processes (in particular in depollution processes), it can often occur that influent concentrations are unknown. (Aubrun et al., 2001) present a method based on a stochastic approach in order to obtain estimates of the influent substrate. In (Theilliol et al., 2003) a method that uses online derivatives of available outputs is provided.

In this paper we propose a new scheme to obtain an interval observer for the input. Our development consists in obtaining guaranteed bounds of the error of a conventional observer, which assures convergence for high values of its feedback gain. In particular, we consider the observer of the input presented in (Mazenc, 2007), which guarantees convergence for a high gain value, and then we derive an interval observer for its error. After considering some technical details, we construct an interval observer through a change of coordinates that guarantees stable interval estimates.

The proposed scheme is then developed and applied considering an uncertain framework, where the conventional observer cannot guarantee any convergence result. Taking advantage of the guaranteed interval estimates, we run various observers in parallel generating a bundle of observers (Bernard and Gouzé, 2004). This approach allows to combine low or high gain observers and then to compare the obtained estimates.

This paper is organized as follows. In section 2 we present an example of a general biotechnological model in dimension two. Properties and hypotheses are also introduced. In section 3 a conventional observer for the estimation of the input in presented. Section 4 is devoted to the formulation of an interval observer for the input. We show some technical drawbacks to be faced when constructing an interval observer, and then we present our contribution considering both perfect knowledge and uncertain frameworks. Finally, section 5 presents the application of the estimation scheme with simulation results.

2. ASSUMPTIONS AND NOTATIONS

Mass balance models of biotechnological processes have been widely used for automatic control applications. We consider a simple model (Monod, 1942) which describes the behavior of the concentrations of a biomass x and a substrate s in a perfectly mixed bioreactor:

$$\begin{cases} \dot{x} = r(s,x) - ux, \quad x(0) = x_0 \\ \dot{s} = u(v-s) - k_1 r(s,x), \quad s(0) = s_0 \end{cases} \tag{1}$$

where x stands for the biomass and s for the substrate. $u(t)$ is the dilution input and k_1 is the a conversion yield coefficient. The biological activity of the system is featured by the reaction rate $r(s,x) \geq 0$, such that $r(s,x)_{s=0} = 0$, for which we assume that there is no explicit model available. $v = v(t) \in \mathcal{C}^1(\mathbb{R})$ corresponds to the unknown concentration of influent substrate.
Consider that the system output can be written as:

$$y = \begin{cases} y_1 = s \\ y_2 = k_2 r(s,x) \end{cases} \tag{2}$$

using equation (2), system (1) can be written as a linear system plus a term related to the uncertain input, similar to the one presented in (Moisan and Bernard, 2006).

Property 1. For any positive initial condition, trajectories of system (1) remain positive and bounded for any positive time. We denote:

$$x(t) \leq \overline{x}, \quad s(t) \leq \overline{s}, \quad \forall t \geq 0 \tag{3}$$

Proof. Positivity of variables becomes trivial because $r(s,x)_{s=0} = 0$. Now consider $z = s + k_1 x$. It is clear that:

$$\min\{z(0), z_{in}\} \leq z \leq \max\{z(0), z_{in}\}$$

which proves the boundedness of the state vector. □

The following hypotheses are needed for the statement of our result.

Hypothesis 1. There exists a constant $\gamma \in \mathbb{R}_+$ such that:

$$\int_0^t u(\sigma) d\sigma \geq \gamma t$$

Hypothesis 2. There exists a known constant $\beta \in \mathbb{R}_+$ such that:

$$|\dot{v}(t)| \leq \beta$$

Hypothesis 3. There exist known intervals containing the initial condition of the state of system (1) and the unknown initial condition of the input v:

$$x_0 \in [x_0^-, x_0^+], \quad s_0 \in [s_0^-, s_0^+], \quad v_0 \in [v_0^-, v_0^+]$$

Our objective is to obtain an interval estimate of the unknown input $v(t)$, using the available information provided in equation (2).

In a first step, a conventional observer with convergence guaranteed for high gain values is considered. Then, on the basis of a bounded error estimation scheme, an interval observer for the input is constructed.

3. HIGH GAIN OBSERVER FOR THE INPUT

Let us consider the following input observer candidate for system (1):

$$\begin{aligned} \dot{\hat{s}} &= u(\hat{v} - y_1) - \frac{k_1}{k_2} y_2 + u(\theta + \theta^2)(y_1 - \hat{s}) \\ \dot{\hat{v}} &= u\theta^3 (y_1 - \hat{s}) \end{aligned} \tag{4}$$

where $\theta \in \mathbb{R}$ is a tunable gain (for the sake of simplicity, θ will be considered constant). System (4) is initialised considering:

$$\hat{s}_0 = s_0 \quad \text{and} \quad \hat{v}_0 \in [v_0^-, v_0^+] \tag{5}$$

Equation (4) corresponds to a slightly modified equation of the input observer proposed by (Mazenc, 2007), following a scheme similar to the observer introduced in (Gauthier et al., 1992).

Let us denote $e_s = \hat{s} - s$ and $e_v = \hat{v} - v$. The error dynamics are then expressed by the system:

$$\begin{aligned} \dot{e}_s &= -u(\theta + \theta^2) e_s + u e_v \\ \dot{e}_v &= -u\theta^3 e_s - \dot{v}(t) \end{aligned} \tag{6}$$

Proposition 1. If Hypotheses 1 and 2 are verified, then for a gain $\theta > 1$ there exists a bound $\mathbb{B}(\theta) \in \mathbb{R}_+$ such that $|e_v(t)| \leq \mathbb{B}(\theta)$, with:

$$\mathbb{B}(\theta) = \frac{2\beta}{\theta - 1} + \frac{\theta}{\theta - 1}(v_0^+ - v_0^-) e^{-\gamma\theta t}, \quad \forall t \geq 0 \tag{7}$$

Proof. See the proof in Appendix A. □
See also (Mazenc, 2007) for more details.

From equation (7), it is clearly seen that for a high gain θ, the bound $\mathbb{B}(\theta)$ can be made as small as desired with a convergence rate arbitrary fast. Note that if $\beta = 0$ (the case where the input is a constant) then asymptotic exponential convergence is reached.

Remark 1. The bound provided by equation (7) is valid for well known stoichiometric coefficients k_1 and k_2, and noise free measurements y_1 and y_2.

4. AN INTERVAL OBSERVER FOR THE INPUT

Interval observers are based on positive differential systems (Smith, 1995) and offer a way to deal with uncertainty in the system, when known bounds of the uncertain terms are available (Gouzé et al., 2000).
Let us note that, as there is no model available for the unknown input $v(t)$, proposing an interval observer is not straightforward.

We develop an interval observer for the input, assuming known bounds on the initial conditions.

Remark 2. The operator $\leq$ applied between vectors or matrices should be understood as a set of inequalities applied component by component.

The idea is to dynamically compute guaranteed upper and lower bounds for the error system (6), that is, if:

$$\begin{aligned} &\begin{bmatrix} e_s^-(0) \\ e_v^-(0) \end{bmatrix} \leq \begin{bmatrix} e_s(0) \\ e_v(0) \end{bmatrix} \leq \begin{bmatrix} e_s^+(0) \\ e_v^+(0) \end{bmatrix}, \quad \text{then} \\ &\begin{bmatrix} e_s^-(t) \\ e_v^-(t) \end{bmatrix} \leq \begin{bmatrix} e_s(t) \\ e_v(t) \end{bmatrix} \leq \begin{bmatrix} e_s^+(t) \\ e_v^+(t) \end{bmatrix}, \quad \forall t \geq 0 \end{aligned} \tag{8}$$

and then, to obtain the bounds on the input considering:

$$\hat{v}(t) - e_v^+(t) \leq v(t) \leq \hat{v}(t) - e_v^-(t) \tag{9}$$

We propose an interval observer considering a similar framework of the observers proposed in (Moisan and Bernard, 2006). The following concepts are required:

Definition 1. A matrix $\mathcal{K}$ is said to be cooperative if all its off–diagonal elements are nonnegative: $k_{ij} \geq 0$, $\forall i \neq j$.

Property 2. Consider a dynamical system of the form

$$\dot{z} = \mathcal{K}z + c$$

with $z \in \mathbb{R}^n$, $\mathcal{K} \in \mathbb{R}^{n\times n}$ a cooperative matrix and $c \in \mathbb{R}^n_+$, then:

$$\forall z_0 \geq 0 \Rightarrow z(t) \geq 0$$

This means that $z(t)$ remains nonnegative if $z_0 \geq 0$.

Proof. See (Moisan and Bernard, 2006) for a demonstration. □

Now consider the error dynamics given by equation (6). Denoting $e = [e_s \;\; e_v]^T$, it can be rewritten in the following form:

$$\begin{cases} \dot{e}(t) \;=\; uAe(t) + b(t) \\ \quad z \;=\; Ce \end{cases} \tag{10}$$

with:

$$A = \begin{bmatrix} -(\theta+\theta^2) & 1 \\ -\theta^3 & 0 \end{bmatrix}, \quad b(t) = \begin{bmatrix} 0 \\ -\dot{v}(t) \end{bmatrix}, \quad C = [1 \;\; 0] \tag{11}$$

It is clear that, from Hypothesis 2, the term $b(t)$ is bounded by:

$$b^-(t) = \begin{bmatrix} 0 \\ -\beta \end{bmatrix} \quad \text{and} \quad b^+(t) = \begin{bmatrix} 0 \\ \beta \end{bmatrix} \tag{12}$$

It is not difficult to verify that the interval observer structure introduced in (Moisan and Bernard, 2006), given by the pair of systems:

$$\begin{cases} \dot{e}^+(t) \;=\; Ae^+(t) + b^+ + \Gamma(Ce^+(t) - z) \\ \dot{e}^-(t) \;=\; Ae^-(t) + b^- - \Gamma(Ce^-(t) - z) \end{cases} \tag{13}$$

where $\Gamma = [\gamma_1, \ldots, \gamma_n]^T$ does not provide at the same time stability and cooperativity. Indeed, construct the differential comparison $e^\star = [e^+ - e, e - e^-]^T$:

$$\dot{e}^\star(t) \;=\; \left[\begin{array}{c|c} A+\Gamma C & 0 \\ \hline 0 & A+\Gamma C \end{array}\right] e^\star(t) + \begin{bmatrix} b^+(t) - b(t) \\ b(t) - b^-(t) \end{bmatrix} \tag{14}$$

where, considering equation (10), it is possible to check that matrix $A + \Gamma C$ can be written as:

$$A + \Gamma C = \begin{bmatrix} -(\theta+\theta^2)+\gamma_1 & 1 \\ -\theta^3+\gamma_2 & 0 \end{bmatrix} \tag{15}$$

which is cooperative if $\gamma_2 \geq \theta^3$, however the resulting matrix is not stable.

To overcome this problem we propose a change of coordinates of system (6) in which stable estimates can be obtained.

4.1 Change of coordinates

It appears that matrix A is stable, with real and strictly negative eigenvalues given by $\lambda_1 = -\theta$ and $\lambda_2 = -\theta^2$. Therefore, it admits the diagonalization:

$$A = P\Delta P^{-1} \tag{16}$$

where:

$$\Delta = \begin{bmatrix} -\theta & 0 \\ 0 & -\theta^2 \end{bmatrix}, P^{-1} = \frac{1}{\theta-1}\begin{bmatrix} -1 & 1/\theta \\ \theta^2 & -1 \end{bmatrix} \tag{17}$$

Consider the change of variables:

$$\zeta = P^{-1}e \tag{18}$$

then, in the new coordinates system (10) is expressed by:

$$\dot{\zeta}(t) \;=\; u\Delta\zeta(t) + P^{-1}b(t) \tag{19}$$

Taking advantage of the diagonal (and thus cooperative) structure of matrix Δ, plus its stability for a gain $\theta \in \mathbb{R}_+ - \{0, 1\}$, an interval observer can be obtained in the ζ-coordinates, considering the bounds on the term $b(t)$ provided by Hypothesis (2).

4.2 An interval observer with perfect knowledge

Consider the following system:

$$\begin{aligned} \begin{bmatrix} \dot{\zeta}^+(t) \\ \dot{\zeta}^-(t) \end{bmatrix} &= u\left[\begin{array}{c|c} \Delta(\theta) & 0 \\ \hline 0 & \Delta(\theta) \end{array}\right]\begin{bmatrix} \zeta^+(t) \\ \zeta^-(t) \end{bmatrix} + \mathcal{M}(\theta)\begin{bmatrix} b^+ \\ b^- \end{bmatrix} \\ \begin{bmatrix} e^+(t) \\ e^-(t) \end{bmatrix} &= \mathcal{G}(\theta)\begin{bmatrix} \zeta^+(t) \\ \zeta^-(t) \end{bmatrix} \end{aligned} \tag{20}$$

where $\mathcal{M}(\theta)$ and $\mathcal{G}(\theta)$ are linear transformations that fulfil the following properties:

Property 3. For $e \in [\underline{e}, \overline{e}] \subset \mathbb{R}^n$ and $z = P^{-1}e$:

$$\mathcal{M}(\theta)\begin{bmatrix} \overline{e} \\ \underline{e} \end{bmatrix} = \begin{bmatrix} \overline{z} \\ \underline{z} \end{bmatrix} \quad \text{such that} \quad \underline{z} \leq z \leq \overline{z}$$

Property 4. For $\zeta \in [\underline{\zeta}, \overline{\zeta}] \subset \mathbb{R}^n$ and $\varepsilon = P\zeta$:

$$\mathcal{G}(\theta)\begin{bmatrix} \overline{\zeta} \\ \underline{\zeta} \end{bmatrix} = \begin{bmatrix} \overline{\varepsilon} \\ \underline{\varepsilon} \end{bmatrix} \quad \text{such that} \quad \underline{\varepsilon} \leq \varepsilon \leq \overline{\varepsilon}$$

See Appendix B for more details about $\mathcal{M}(\theta)$ and $\mathcal{G}(\theta)$.

Proposition 2. Given ζ_0^-, ζ_0^+ such that $\zeta_0 \in [\zeta_0^-, \zeta_0^+]$ and a gain $\theta \in \mathbb{R}_+ - \{0, 1\}$, then system (20) is an interval observer of system (10), leading to bounds on the input $v(t)$ by equation (9).

Proof. We need to prove that the error generated by comparing systems (19) and (20) is a positive system. The error dynamics are expressed by the equation:

$$\begin{bmatrix} \dot{e}_\zeta^+(t) \\ \dot{e}_\zeta^-(t) \end{bmatrix} \;=\; u\left[\begin{array}{c|c} \Delta(\theta) & 0 \\ \hline 0 & \Delta(\theta) \end{array}\right]\begin{bmatrix} e_\zeta^+(t) \\ e_\zeta^-(t) \end{bmatrix} + \mathcal{B}(\theta, b(t), b^-, b^+) \tag{21}$$

It can be shown that the residual term $\mathcal{B}(.) \in \mathbb{R}^{2n}$ is nonnegative (see Appendix B for details), and then Proposition 2 holds. □

Remark 3. This observer inherits the convergence characteristic of the conventional observer (4).

4.3 Interval observer with uncertainties

Consider now that the stoichiometric coefficients k_1 and k_2 and the measurements are uncertain quantities. The following hypotheses are considered:

Hypothesis 4. The stoichiometric coefficients k_1 and k_2 are unknown but bounded by known positive values.

$$k_1 \in [k_1^-, k_1^+] \quad \text{and} \quad k_2 \in [k_2^-, k_2^+] \tag{22}$$

Hypothesis 5. Online measurements $y_1(t)$ and $y_2(t)$ are perturbed respectively by noises $\delta_s(t)$ and $\delta_q(t)$. We assume that these perturbations are of multiplicative nature:

$$y_1(t) = s(t)(1 + \delta_s(t)) \;\text{ and }\; y_2(t) = k_2(1 + \delta_q(t))r(.) \tag{23}$$

Moreover, these noise signals are bounded such that $|\delta_s(t)| \leq \Delta_s < 1$ and $|\delta_q(t)| \leq \Delta_q < 1$.

Considering Hypotheses 4 and 5, observer equation (4) leads to error dynamics with the same structure as equation (10), except for the vector $b(t)$ which now depends on the uncertain values and on the gain θ. $b(t)$ is then expressed by:

$$b(t) = \begin{bmatrix} us\delta_s(\theta + \theta^2 - 1) - y_2\left(\dfrac{k_1}{k_2(1+\delta_q)} - \dfrac{\tilde{k}_1}{\tilde{k}_2}\right) \\ u\theta^3 s\delta_s - \dot{v}(t) \end{bmatrix} \tag{24}$$

where $\tilde{k}_1 \in [k_1^-, k_1^+]$ and $\tilde{k}_2 \in [k_2^-, k_2^+]$. From the previously introduced hypotheses it is easy to see that $b(t) \in [b^-(t), b^+(t)]$, with:

- for $\theta \in]0, \theta^\star]$:

$$b^\pm(t) = \begin{bmatrix} \mp u\bar{s}\Delta_s(\theta + \theta^2 - 1) + y_2\left(\dfrac{k_1^\pm}{k_2^\mp(1 \mp \Delta_q)} - \dfrac{k_1^\mp}{k_2^\pm}\right) \\ \pm u\theta^3 s\delta_s \pm \beta \end{bmatrix} \tag{25}$$

- for $\theta > \theta^\star$, $\theta \neq 1$:

$$b^\pm(t) = \begin{bmatrix} \pm u\bar{s}\Delta_s(\theta + \theta^2 - 1) + y_2\left(\dfrac{k_1^\pm}{k_2^\mp(1 \mp \Delta_q)} - \dfrac{k_1^\mp}{k_2^\pm}\right) \\ \pm u\theta^3 s\delta_s \pm \beta \end{bmatrix} \tag{26}$$

where $\theta^\star$ is the positive solution of $\theta + \theta^2 - 1 = 0$.

Remark 4. When considering uncertainties, the residual vector $b(t)$ and its bounds depend on the gain θ which may dramatically amplify the uncertainties when using high values of θ.

The observer candidate also keeps the same structure as equation (20), then the following proposition holds.

Proposition 3. Given ζ_0^-, ζ_0^+ such that $\zeta_0 \in [\zeta_0^-, \zeta_0^+]$ and a gain $\theta \in \mathbb{R}_+ - \{0, 1\}$, then system (20) (for bounds $b^-(t)$ and $b^+(t)$ expressed by equation (25) and (26)) is an interval observer of system (10), leading to bounds on the input $v(t)$ by equation (9).

Proof. The proof is the same as the one of Proposition 2. □.

4.4 Bundle of observers

Taking advantage of the guaranteed nature of the interval estimates provided by equation (20), we run simultaneously several observers with different fixed values of the gain θ, which provide different behaviors of the estimates. By means of a bundle of observers (Bernard and Gouzé, 2004), we can then compare various interval estimates and take advantage of the good transient behavior of some estimates and the good steady state estimates of others. A bundle of observers let us relax the requirement of high gain values of the observer (4): it is possible to run in parallel observers considering either low or high values for gain θ.

4.5 A simple biomass interval observer

An interval observer for the biomass $x(t)$ can be proposed, that uses the computed bounds of the input. Consider the change of variables $z(t) = k_1x(t) + s(t)$ (Bastin and Dochain, 1990), which eliminates the reaction rate $r(x, s)$ obtaining an observer whose convergence rate is given by u.

$$\begin{cases} \dot{z}^+(t) = u(v^+(t) - z^+(t)), \quad \dot{z}^-(t) = u(v^-(t) - z^-(t)) \\ x^+(t) = (z^+(t) - y_1(t))/k_1, \; x^-(t) = (z^-(t) - y_1(t))/k_1 \end{cases} \tag{27}$$

It is is possible to check that the interval estimates for the biomass depend directly on the width of the interval estimates of the unknown influent concentration.

5. APPLICATION

A simple application of the estimation scheme is presented, inspired from anaerobic digestion processes. In this kind of processes, the output y_2 corresponds to the methane flow rate and is usually online monitored.

Dilution and influent substrate (unknown input) are shown in Fig. 1. Available outputs $y_1(t)$ (substrate) and $y_2(t)$ (methane flow rate) are shown in Fig. 2.

We have run the proposed interval observer considering a perfect knowledge of the system and noise–free measurements. Simulation results for a single interval estimate considering $\theta = 100$ are shown in Fig. 3. Fast convergence of the interval estimates toward the influent substrate unknown value is verified.

A bundle of interval estimates has been run under an uncertain framework. We considered a $\pm 10\%$ uncertainty for the stoichiometric coefficients and a multiplicative noise on the measurements up to a 3% (see Table 1). An

Table 1. System parameters and uncertainty.

parameter	meaning	value	uncertainty
k_1	biomass yield coef.	1.28	[1.15,1.4]
k_2	methane yield coef.	25	[22.5,27.5]
Δ_s	y_1 noise upper bound	-	0.03
Δ_q	y_2 noise upper bound	-	0.03

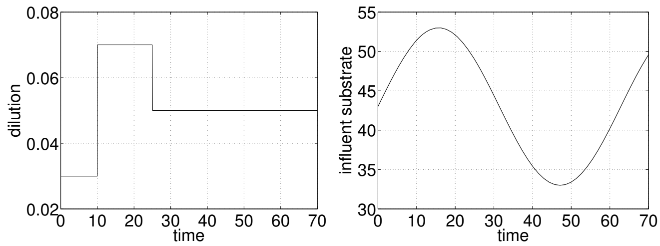

Fig. 1. Dilution and unknown influent substrate.

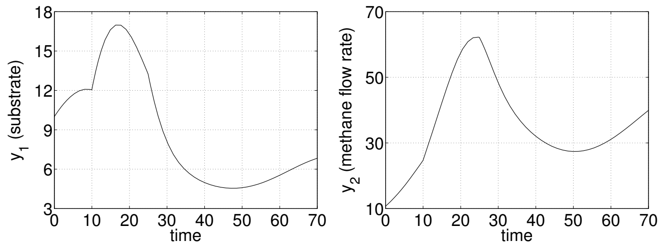

Fig. 2. Online measurements y_1 and y_2.

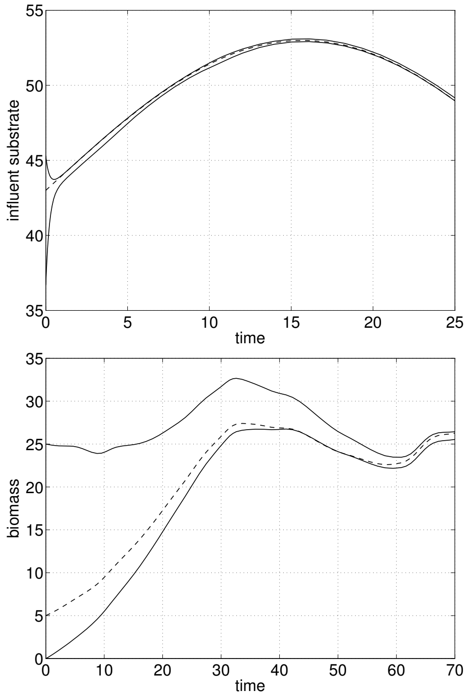

Fig. 3. Interval estimates for the unknown input and biomass under perfect knowledge of the system. Continuous lines: interval estimates, dashed lines: real values.

illustrative example is shown in Fig. 4, where two interval estimates are shown: one for a low gain value $\theta = 1.1$ and other for a higher gain value $\theta = 50$. A more dense bundle of observers, with 30 interval estimates running in parallel, with $\theta \in [0.1, 50]$ is then shown in Fig. 5, giving a more accurate final interval estimate.

6. CONCLUSION

An interval observer for the estimation of the input of a biotechnological process has been proposed. The only condition imposed on the unknown input is the boundedness of its time derivative, which is not restrictive. The interval

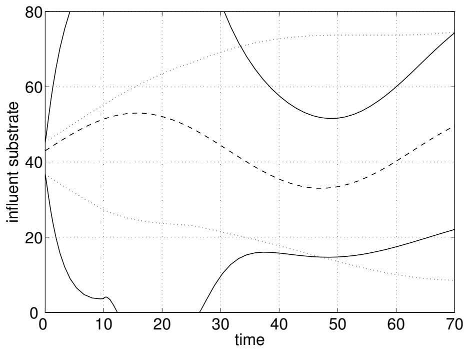

Fig. 4. Two interval observers. Dotted lines: low gain interval estimates, continuous lines: high gain estimates.

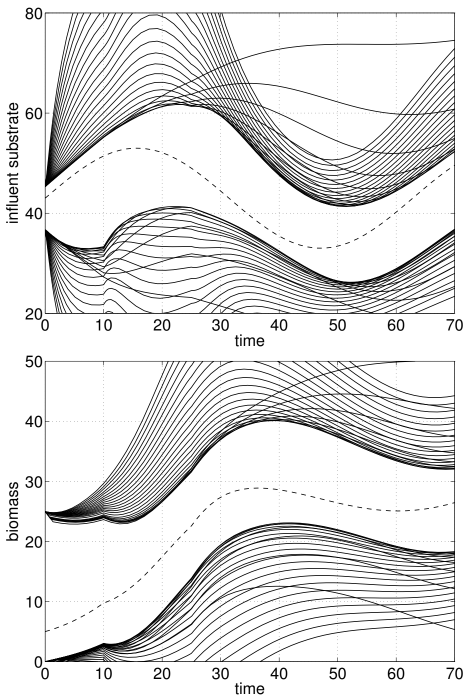

Fig. 5. Interval estimates for the unknown input and biomass under uncertainty. Continuous lines: interval estimates, dashed lines: real values.

estimates are obtained by bounding the error of a conventional observer (Mazenc, 2007), and then extended to an uncertain framework. Interval observers introduce some advantages with respect to classical observation methods. In particular for this development, we relaxed the high gain requirement of the proposed conventional observer, constructing interval estimates with both low and high gain values. Moreover, interval estimates allow us to verify the convergence of observer and assess the estimation accuracy. A simple observer for the biomass has been also tested, that uses the bounds generated for the input. In is worth to remark that the interval estimates obtained for the biomass can be improved using more sophisticated interval observers, as it is shown in (Moisan et al., 2006). As a future work, the application of the method to a real industrial setup is expected.

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Appendix A. PROOF OF PROPOSITION 1

Consider the time parametrization:

$$\tau = \int_0^t u(\sigma)d\sigma \geq \gamma t, \quad \gamma,\ t \geq 0 \tag{A.1}$$

then system (10) can be rewritten as:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\begin{bmatrix} e_s \\ e_v \end{bmatrix} = A \begin{bmatrix} e_s \\ e_v \end{bmatrix} + \begin{bmatrix} 0 \\ -\psi(\tau) \end{bmatrix} \tag{A.2}$$

where

$$\psi(\tau) = \frac{\frac{\mathrm{d}v}{\mathrm{d}t}(\tau)}{u(\tau)} \in \left[\frac{-\beta}{\gamma}, \frac{\beta}{\gamma}\right] \tag{A.3}$$

now, considering the change of variables $\zeta_1 = \theta e_s - e_v$ and $\zeta_2 = -\theta^2 e_s + e_v$, it follows that:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\zeta_1 = -\theta\zeta_1 + \psi(\tau), \quad \frac{\mathrm{d}}{\mathrm{d}\tau}\zeta_2 = -\theta^2\zeta_2 - \psi(\tau) \tag{A.4}$$

Bounds for the term $\psi(\tau)$ in equation (A.3), let us write:

$$|\zeta_i(\tau)| \leq \frac{\beta}{\gamma\theta^i} + |\zeta_i(0)|e^{-\theta^i\tau}, \quad i = 1, 2 \tag{A.5}$$

Now we can come back to the original coordinates:

$$|\hat{v}(\tau) - v(\tau)| \leq \frac{(\theta + \theta^{-1})\beta/\gamma + \theta^2(|\zeta_1(0)| + |\zeta_2(0)|)e^{-\theta\tau}}{\theta^2 - \theta} \tag{A.6}$$

and finally considering equation A.1:

$$|\hat{v}(t) - v(t)| \leq \frac{2\beta}{\theta - 1} + \frac{\theta}{\theta - 1}(v_0^+ - v_0^-)e^{-\gamma\theta t}, \quad \forall t \geq 0 \tag{A.7}$$

Then Proposition 1 holds. □

Appendix B. PROOF OF PROPOSITION 2

For the proof of Proposition 2 we need to specify transformations $\mathcal{M}$ and $\mathcal{G}$, and then verify that $\mathcal{B}$ is a nonnegative vector.

Transformations $\mathcal{M}$ and $\mathcal{G}$ provide a link between e–coordinates and ζ–coordinates. Considering the change of variables $\zeta = P^{-1}e$ used in section 4 to obtain stable estimates, we specify $\mathcal{M}$ as follows:

$$\mathcal{M} = \begin{bmatrix} P_+^{-1} & P_-^{-1} \\ P_-^{-1} & P_+^{-1} \end{bmatrix} \tag{B.1}$$

Denote $P^{-1} = \{\tilde{p}_{ij}\}$, $\forall\ i, j = 1, \ldots, n$. Matrices P_-^{-1} and $P_+^{-1} \in \mathbb{R}^{n\times n}$ are defined as:

$$P_+^{-1} = \begin{cases} \tilde{p}_{ij} & \text{if } \tilde{p}_{ij} \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad \text{and } P_-^{-1} = P^{-1} - P_+^{-1} \tag{B.2}$$

that is, the positive and negative parts of matrix P^{-1} have been separated.

Vector $\mathcal{B}$ can be written as:

$$\mathcal{B} = \begin{bmatrix} P_+^{-1}b^+ + P_-^{-1}b^- - P^{-1}b \\ P^{-1}b - P_-^{-1}b^+ - P_+^{-1}b^- \end{bmatrix} \tag{B.3}$$

Considering $b(t) \in [b^-, b^+] = [b(t) - \epsilon^-, b(t) + \epsilon^+]$ with $\epsilon^-, \epsilon^+ \in \mathbb{R}_+^n$, it follows that:

$$\mathcal{B} = \begin{bmatrix} P_+^{-1}\epsilon^+ - P_-^{-1}\epsilon^- \\ P_+^{-1}\epsilon^- - P_-^{-1}\epsilon^+ \end{bmatrix} \tag{B.4}$$

then, from equation (B.2) it is easy to see that $\mathcal{B} \geq 0$. Matrix $\mathcal{G}$ is obtained in an analogous way (considering matrix P), assuring the bounding in the original coordinates. □

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