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Closed form Filtering for Linear Fractional Transformation Models ⋆

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Abstract: The nonlinear optimal Bayes filter is computationally intractable and at present there exists no analytical method for the nonlinear filtering problem. The standard approaches for linear approximation in the literature work reasonably well for the class of systems that are only mildly nonlinear. In this paper, we introduce a filtering approach for the most general class of nonlinear models by transforming the state space model to an equivalent representation given by the linear fractional transformation (LFT) model which is nonlinear in the feedback loop only. Based on an approximation localized to the feedback path only, we derive a closed form solution to Bayes recursion for the LFT model. We give simulation results to demonstrate the potential of the proposed filtering approach for applications where conventional methods fail.

1. INTRODUCTION

For linear systems the optimal Bayes filter is the celebrated Kalman filter. However, most real processes do not exhibit linear behavior and evolve in a nonlinear fashion with time. Unfortunately, the nonlinear optimal Bayes filter is computationally intractable and there exists no analytical method to determine the optimal solution. For this reason, nonlinear filtering relies on approximation methods for a solution that lies in the proximity of the exact solution. In the literature there are two standard approaches for linear approximation. The extended Kalman filter (EKF) applies local linearization of the nonlinear mapping around the state estimate. The approximation is based on the assumption that the state estimate lies in the proximity of the global trajectory. In general, such an assumption is weak and breaks down for large state covariance. The unscented Kalman filter (UKF) Julier and Uhlmann [1997, 2004] on the other hand employs the unscented transformation which is based on statistical linear regression technique Lefebvre et al. [2002] to approximate the filtering distribution. While it has been shown that in general UKF performs better than EKF Julier and Uhlmann [2004], both methods are more suited to a class of models that are only mildly nonlinear to give reasonably good approximation.

In recent years sequential Monte Carlo methods have been applied to nonlinear Bayesian filtering problems Doucet et al. [2000, 2001]. These methods approximate the filtering distribution by a set of samples drawn from a proposal distribution and give better approximation than the linearization techniques. However, this is conditional on the convergence of the estimates which is guaranteed only in the limit that the number of samples is infinity. The large number of samples needed in practice makes these methods computationally inefficient. Moreover, these methods suffer from inconsistency due to the unreliability of clustering techniques for extracting the state estimate.

In gain-scheduling control, the linear fractional transformation model (LFT) K. Zhou and Glover [1996] is applied extensively to describe nonlinear plants (see e.g., Apkarian and Gahinet [1995], Apkarian and Adams [1997] and the references therein). The model gives an equivalent representation for any smooth nonlinear mapping in terms of a linear structure and a simple nonlinear feedback connection K. Zhou and Glover [1996], Tuan et al. [2003, 2004]. Unlike the feedback linearization technique Isidori [1989], Khalil [2002] which is applicable in certain situations only, the LFT model is flexible enough to accommodate the most general class of nonlinear models. In particular, for systems involving complicated fractional terms where a transformation using feedback linearization may not be possible, the LFT model gives an exact equivalent representation. In this paper, we exploit the efficiency of the LFT model and introduce Bayesian filtering technique for such models. By localizing the application of the unscented transformation to the feedback loop only we derive a closed form solution to Bayes recursion for estimating the state of a system given a sequence of data. Simulation results demonstrate that the proposed filter can be applied to highly nonlinear problems where conventional techniques do not work.

The paper is organized as follows: section II gives some background on the nonlinear filtering problem. In section III we review the standard approaches for linear approximation and discuss the LFT model before presenting the main result of this paper, a closed form solution to Bayes recursion for the LFT model. In section IV we demonstrate through simulations the performance of the proposed filter. We conclude with some final comments in section V.

The notations in the paper are standard. In particular, $x|y$ denotes a random variable x restricted by a realization of the conditioning random variable y, $x \sim \mathcal{N}(x; \bar{x}, R_x)$

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denotes a random variable with normal or Gaussian distribution with expected value or mean $\bar{x}$ and covariance R_x, while $\mathcal{N}(\cdot;\bar{x},R_x)$ is its probability density function.

2. BACKGROUND

Consider the state-space model given by

$$x_{k+1} = f(x_k) + B_k w_k, \tag{1}$$
$$z_k = g(x_k) + D_k v_k, \tag{2}$$

where $f(x_k)$ and $g(x_k)$ denote arbitrary mappings (not necessarily linear), $x_k \in \mathbb{R}^n$ denotes the state of the system at time k, $z_k \in \mathbb{R}^m$ denotes the measurement at time k, $B_k \in \mathbb{R}^{n\times p}$ and $D_k \in \mathbb{R}^{m\times q}$. The process noise $w_k \sim (w_k; 0, Q_k)$ and the measurement noise $v_k \sim \mathcal{N}(v_k; 0, R_k)$ are mutually uncorrelated and independent of the state x_k.

From Bayesian filtering viewpoint, the problem is to estimate the state x_k of the system at time k given a sequence of measurements $Z_k = \{z_0, z_1, \ldots, z_k\} \subset \mathbb{R}^m$ at time k. Under linear assumption on the mappings $f(x_k)$ and $g(x_k)$, the state-space model (1)-(2) can be rewritten as

$$x_{k+1} = A_k x_k + B_k w_k, \tag{3}$$
$$z_k = C_k x_k + D_k v_k, \tag{4}$$

where $A_k \in \mathbb{R}^{n\times n}$ and $C_k \in \mathbb{R}^{m\times n}$. For x_k with Gaussian distribution, linear transformations involving x_k give Gaussian random variables. Moreover, for two random variables $x \sim \mathcal{N}(x;\bar{x},R_x)$ and $y \sim \mathcal{N}(y;\bar{y},R_y)$ related linearly, the distribution of the random variable x conditional on y is given by the standard result

$$x|y \sim \mathcal{N}\big(\cdot;\bar{x} + R_{yx}^T R_y^{-1}(y-\bar{y}), R_x - R_{yx}^T R_y^{-1} R_{yx}\big). \tag{5}$$

Given an initial estimate of the state $x_0 \sim \mathcal{N}(x_0;\bar{x}_0,R_{x,0})$, the distribution of z_0 in (4) conditional on x_0 is $z_0|x_0 \sim \mathcal{N}(\cdot; C_0\bar{x}_0, C_0 R_{x,0} C_0^T + D_0 R_0 D_0^T)$ and the cross-covariance of z_0 and x_0 is $C_0 R_{x,0}$. Let $\bar{z}_0$ and $R_{z,0}$ denote the mean and covariance of $z_0|x_0$ respectively and $R_{zx,0}$ denote the cross-covariance of the two random variables, then on arrival of data z_0 the distribution of x_0 conditional on z_0 follows from the standard result (5),

$$x_0|z_0 \sim \mathcal{N}\big(\cdot;\bar{x}_0 + R_{zx,0}^T R_{z,0}^{-1}(z_0-\bar{z}_0), R_{x,0} - R_{zx,0}^T R_{z,0}^{-1} R_{zx,0}\big). \tag{6}$$

Using the notation $\bar{x}_{0|0}$ and $R_{x,0|0}$ to denote the first two moments of $x_0|z_0$, the prediction of the state x_1 in (3) conditional on data z_0 is given by $x_1|z_0 \sim \mathcal{N}(\cdot; A_0\bar{x}_{0|0}, A_0 R_{x,0|0} A_0^T + B_0 Q_0 B_0^T)$. Using this estimate the distribution of $z_1|x_1$ and the cross-covariance of z_1 and x_1 can be determined as above. On arrival of data z_1, the estimate of $x_1|Z_1$ ($x_1|z_0$ also conditional on z_1) is given by (5). The steps above outline the optimal Bayes filter recursions under linear assumption on $f(x_k)$ and $g(x_k)$ known as Kalman filter prediction and update steps to estimate the conditional random variable $x_k|Z_k$ at each time step.

In general, where $f(x_k)$ and $g(x_k)$ are not restricted to linear mappings, transformations involving x_k do not give Gaussian random variables. Moreover, for a random variable y related nonlinearly to x with Gaussian distribution, the standard result (5) does not hold. With no tractable method to estimate the conditional random variable $x_k|Z_k$, nonlinear filtering involves an approximation for $y = f(x_k)$ to propagate the moments of the state to the next time step given an estimate of the state at time k. Similarly, an approximation for $y = g(x_k)$ given the distribution of the predicted state, gives the required estimate for $x_k|Z_k$ on applying the standard result (5). In the literature the following two approaches for linear approximation are standard. The extended Kalman filter (EKF) uses the first-order Taylor series approximation of $f(x_k)$ and $g(x_k)$ around $\bar{x}_k$. The approximation neglects higher order terms of the series under the assumption that $\bar{x}_k$ lies in the neighborhood of the global trajectory. This poses a potential problem if either $R_{x,k}$ is large or $f(x_k)$ and $g(x_k)$ depart from linear behavior. A better approximation given by the unscented Kalman filter (UKF) Julier and Uhlmann [1997, 2004], Lefebvre et al. [2002] applies the unscented transformation to approximate the filtering distribution. This approach is based on the assumption that it is easier to approximate the distribution of a random variable than an arbitrary nonlinear mapping. Despite better performance, UKF gives reasonably good approximation for mildly nonlinear systems only.

The efficiency and ease of implementation of the LFT model have been demonstrated in Tuan et al. [2004] by applying it to a highly nonlinear control problem. This gives the motivation for introducing Bayesian filtering for the LFT model to address the nonlinear filtering problem for a general class of models which to our knowledge has not been considered. The simple nonlinear structure in the feedback loop only compounded by the highly uncorrelated feedback connection give better approximation than conventional methods for highly nonlinear models.

3. FILTERING FOR THE LFT MODEL

3.1 The linear fractional transformation (LFT) model

Given a random variable $x = [\,x(1), x(2), \ldots, x(n)\,]^T$ with mean $\bar{x} = [\,\bar{x}(1), \bar{x}(2), \ldots, \bar{x}(n)\,]^T$ and covariance R_x and the observation $y = g(x)$ where $g(x) = [\,g_1(x), g_2(x), \ldots, g_m(x)\,]^T$ is a smooth nonlinear mapping, the local linearization of $g(x)$ around $\bar{x}$ gives

$$g(x) \approx \left.\frac{\partial g(x)}{\partial x}\right|_{x=\bar{x}} (x-\bar{x}) + g(\bar{x}). \tag{7}$$

Based on this approximation EKF estimates y given x with mean $g(\bar{x})$ and covariance AR_xA^T and the cross-covariance of the two random variables as AR_x where A denotes the derivative of $g(x)$ w.r.t. x evaluated at $\bar{x}$. The estimate of $x|y$ is then given by applying the standard result (5).

The observation y can be expressed in terms of the conditional expectation of y given x as

$$g(x) = R_{yx}R_x^{-1}(x-\bar{x}) + g(\bar{x}) + e, \tag{8}$$

where R_{yx} denotes the cross-covariance of y and x. The error $e = y - \mathbb{E}\{y|x\}$ is independent of x and $\mathbb{E}\{\cdot\}$ denotes the expectation operator. UKF applies the so called unscented transformation which is based on the statistical linear regression of $g(x)$ around $\bar{x}$,

$$g(x) \approx (Ax + b) + e, \tag{9}$$

where $A = R_{yx}R_x^{-1}$ and $b = g(\bar{x}) - A\bar{x}$. The procedure for the unscented transformation is as follows: p regression points x_i, $i = 1, \ldots, p$ are selected around $\bar{x}$ such that

$$\bar{x} = \frac{1}{p}\sum_{i=1}^{p} x_i, \quad R_x = \frac{1}{p}\sum_{i=1}^{p}(x_i - \bar{x})(x_i - \bar{x})^T. \tag{10}$$

With $R_x > 0$, Cholesky decomposition of $R_x = \sum_i^n q_i q_i^T$. A choice of the regression points satisfying (10) is 2κ points x_0, n points x_i and n points x_{n+i} with

$$x_0 = \bar{x}, \quad x_i = \bar{x} + \sqrt{p/2}\, q_i, \quad x_{n+i} = \bar{x} - \sqrt{p/2}\, q_i, \tag{11}$$

where κ denotes degree of freedom in the selection of the regression points Julier and Uhlmann [1997]. Let $y_i = g(x_i)$, $i = 1, 2, \dots, p$, then the mean and covariance of y and the cross-covariance with x are approximated by the distribution of the regression points y_i and x_i, $i = 1, \dots, p$ as

$$\bar{y} = \frac{1}{p}\sum_{i=1}^{p} y_i, \quad R_y = \frac{1}{p}\sum_{i=1}^{p}(y_i - \bar{y})(y_i - \bar{y})^T, \tag{12}$$

$$R_{yx} = \frac{1}{p}\sum_{i=1}^{p}(y_i - \bar{y})(x_i - \bar{x})^T. \tag{13}$$

The conditional expectation and covariance of $x|y$ are then given by substituting (12) and (13) in (5). An empirical study of the performance of EKF and UKF has shown that in general the latter approach gives better approximation Julier and Uhlmann [1997, 2004], LaViola [2003].

From control theory it is known that the observation $y = g(x)$ admits the LFT model K. Zhou and Glover [1996], Tuan et al. [2003, 2004],

$$\begin{bmatrix} y \\ y_\Delta \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x \\ w_\Delta \end{bmatrix}, \tag{14}$$

$$w_\Delta = \Delta(x) y_\Delta, \tag{15}$$

where $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{m\times p_\Delta}$, $C \in \mathbb{R}^{p_\Delta \times n}$ and $D \in \mathbb{R}^{p_\Delta \times p_\Delta}$. The auxiliary variables $w_\Delta \in \mathbb{R}^{p_\Delta}$ and $y_\Delta \in \mathbb{R}^{p_\Delta}$ are related via the feedback connection $\Delta(x)$ which has a simple structure of the form $\Delta(x) = \sum_{j=1}^n \Delta_j x(j)$. The elimination of the auxiliary variables in (14)-(15) gives the standard LFT model K. Zhou and Glover [1996],

$$y = \big(A + B\Delta(x)\big(I - D\Delta(x)\big)^{-1} C\big)x, \tag{16}$$

where $\Delta(x)$ enters the relation in a highly nonlinear fashion. Using either (7) or (9) to linearize y in the standard LFT form gives an approximation that is equivalent to EKF or UKF respectively. Instead, the representation (14)-(15) is nonlinear in the feedback loop only. Given the first two moments of x, the distribution of w_Δ in (15) and the cross-covariance with x can be approximated using either (7) or (9). From (14), y is a linear transformation of $[\,x,\ w_\Delta\,]^T$, applying (5) gives the estimate of $x|y$.

Suppose $y = f(x) + B_1 w$ where $B_1 \in \mathbb{R}^{m\times p}$ and $w \sim \mathcal{N}(w; 0, R_w)$ is independent of x. Then y admits the LFT model

$$y = Ax + B_1 w + B_2 w_\Delta, \tag{17}$$

$$y_\Delta = Cx + D w_\Delta, \tag{18}$$

$$w_\Delta = \Delta(x) y_\Delta, \tag{19}$$

with $B_2 \in \mathbb{R}^{m\times p_\Delta}$. Let $w_{\Delta i} = \Delta(x_i) y_{\Delta i}$, $i = 1, \dots, p$ denote the transformed regression points with $y_{\Delta i} \approx Cx_i + D\bar{w}_\Delta$ where $\bar{w}_\Delta = \mathbb{E}\{w_\Delta\}$. Then, the first two moments of w_Δ are determined from the distribution of the points $w_{\Delta i}$ as (12),

$$\bar{w}_\Delta = \Big(I - \Delta(\bar{x})D\Big)^{-1} \frac{1}{p}\sum_i \Delta(x_i) C x_i, \tag{20}$$

$$R_\Delta = \frac{1}{p}\sum_{i=1}^{p}(w_{\Delta i} - \bar{w}_\Delta)(w_{\Delta i} - \bar{w}_\Delta)^T. \tag{21}$$

where $\Delta(\bar{x}) = 1/p\sum_i \Delta(x_i)$. The cross-covariance of w_Δ and x is computed from the distribution of $w_{\Delta i}$ and x_i as (13),

$$R_{\Delta x} = \frac{1}{p}\sum_{i=1}^{p}(w_{\Delta,i} - \bar{w}_\Delta)(x_i - \bar{x})^T. \tag{22}$$

From (17), the random variable y is Gaussian with distribution $\mathcal{N}(\cdot; \bar{y}, R_y)$ where

$$\bar{y} = A\bar{x} + B_2 \bar{w}_\Delta, \tag{23}$$

$$R_y = AR_xA^T + B_1R_wB_1^T + B_2R_\Delta B_2^T + AR_{\Delta x}^T B_2^T + B_2 R_{\Delta x} A^T, \tag{24}$$

and the cross-covariance of y and x is given by

$$R_{yx} = AR_x + B_2 R_{\Delta x}. \tag{25}$$

The estimate of $x|y$ then follows immediately from (5). Based on the results (20)-(22), in the next subsection we give a closed form solution to Bayes recursion for the LFT representation of the state space model (1)-(2).

3.2 Closed form Bayes recursion

The state space model (1)-(2) admits the following equivalent representation which is nonlinear in the feedback loop only

$$x_{k+1} = A_k x_k + B_{1,k} w_k + B_{2,k} w_{\Delta k}, \tag{26}$$

$$z_k = C_{1,k} x_k + D_{11,k} v_k + D_{12,k} w_{\Delta k}, \tag{27}$$

$$z_{\Delta k} = C_{2,k} x_k + D_{22,k} w_{\Delta k}, \tag{28}$$

$$w_{\Delta k} = \Delta(x_k) z_{\Delta k}, \tag{29}$$

where $A_k \in \mathbb{R}^{n\times n}$, $B_{1,k} \in \mathbb{R}^{n\times p}$, $B_{2,k} \in \mathbb{R}^{n\times p_\Delta}$, $C_{1,k} \in \mathbb{R}^{m\times n}$, $D_{11,k} \in \mathbb{R}^{m\times q}$, $D_{12,k} \in \mathbb{R}^{m\times p_\Delta}$, $C_{2,k} \in \mathbb{R}^{p_\Delta\times n}$, and $D_{22,k} \in \mathbb{R}^{p_\Delta\times p_\Delta}$. $w_{\Delta k}$ and $z_{\Delta k}$ denote auxiliary variables independent of the noise processes w_k and v_k. Under the standard assumption that w_k and v_k are mutually uncorrelated and independent of x_k, we propose the following.

Proposition 1. Suppose that at time $k-1$ the state x_{k-1} conditional on the data Z_{k-1} is Gaussian with distribution

$$x_{k-1}|Z_{k-1} \sim \mathcal{N}(\cdot; m_{k-1}, P_{k-1}). \tag{30}$$

Then, the state propagated to the next time step is also Gaussian

$$x_k|Z_{k-1} \sim \mathcal{N}(\cdot; m_{k|k-1}, P_{k|k-1}), \tag{31}$$

with the moments

$$m_{k|k-1} = A_{k-1}m_{k-1} + B_{2,k-1}\bar{w}_{\Delta k-1}, \tag{32}$$

$$P_{k|k-1} = A_{k-1}P_{k-1}A_{k-1}^T + B_{1,k-1}Q_{k-1}B_{1,k-1}^T + B_{2,k-1}R_{\Delta k-1}B_{2,k-1}^T + A_{k-1}R_{\Delta x,k-1}^T B_{2,k-1}^T + B_{2,k-1}R_{\Delta x,k-1}A_{k-1}^T. \tag{33}$$

Proof. Given the estimate of $x_{k-1}|Z_{k-1}$ with mean m_{k-1} and covariance P_{k-1}, the distribution of $w_{\Delta k-1}$ is determined from (20)-(21) and the cross-covariance of $w_{\Delta k-1}$ and x_{k-1} is given by (22). From (26), x_k given Z_{k-1}

is a sum of Gaussian random variables with mean and covariance (32)-(33). □

Proposition 2. Suppose that the predicted state x_k conditional on the data Z_{k-1} is Gaussian

$$x_k|Z_{k-1} \sim \mathcal{N}(\cdot; m_{k|k-1}, P_{k|k-1}). \tag{34}$$

Then, on arrival of data z_k the estimate of $x_k|Z_{k-1}$ also conditional on z_k is given by

$$x_k|Z_k \sim \mathcal{N}(\cdot; m_k, P_k), \tag{35}$$

where

$$m_k = m_{k|k-1} + K_k(z_k - \eta_k) \tag{36}$$

$$P_k = P_{k|k-1} - K_k(C_{1,k}P_{k|k-1} + D_{12,k}R_{\Delta x,k|k-1}), \tag{37}$$

with

$$\eta_k = C_{1,k}m_{k|k-1} + D_{12,k}\bar{w}_{\Delta k|k-1} \tag{38}$$

$$\begin{aligned} K_k = \big(&P_{k|k-1}C_{1,k}^T + R_{\Delta x,k|k-1}^T D_{12,k}^T\big)\big(C_{1,k}P_{k|k-1}C_{1,k}^T + \\ &D_{11,k}R_kD_{11,k}^T + D_{12,k}R_{\Delta k|k-1}D_{12,k}^T + \\ &C_{1,k}R_{\Delta x,k|k-1}^T D_{12,k}^T + D_{12,k}R_{\Delta x,k|k-1}C_{1,k}^T\big)^{-1}. \end{aligned} \tag{39}$$

Proof. Given the mean $m_{k|k-1}$ and covariance $P_{k|k-1}$ of $x_k|Z_{k-1}$, the estimate of $w_{\Delta k}$ and the cross-covariance with x_k are computed as (20)-(22). From these estimates the distribution of z_k and the cross-covariance with x_k are determined. On arrival of the data z_k, the estimate of $x_k|Z_k$ follows from (5) with mean and covariance given by (36)-(37). □

Propositions 1 and 2 give closed form expressions for the recursive computation of the moments of $x_k|Z_{k-1}$ and $x_k|Z_k$ respectively under standard assumptions. The propositions are similar to Kalman prediction and data update steps with the addition of the terms involving the moments of the auxiliary random variable $w_{\Delta k-1}$ in Proposition 1 and moments of $w_{\Delta k|k-1}$ in Propositions 2.

4. SIMULATION RESULTS

In this section we present simulation results for two examples to demonstrate the performance of the proposed filter. Since UKF gives a performance that is better than that using EKF in general Julier and Uhlmann [1997, 2004], Lefebvre et al. [2002], we consider the performance of UKF as a benchmark in example I. For the problem considered in example II, EKF and UKF do not converge to a solution and a comparison is therefore not possible.

Example I Suppose $x_k \in \mathbb{R}^2$ denotes the state of a system at time k where $x_k = [\,x_k(1)\;\; x_k(2)\,]^T$ evolves in a nonlinear fashion with time in the following manner,

$$x_{k+1} = \begin{bmatrix} 0 & 1 \\ -x_k^2(1) & -0.1 \end{bmatrix} x_k + \begin{bmatrix} 0 & 0 \\ 10 & 0 \end{bmatrix} w_k, \tag{40}$$

where $w_k \sim \mathcal{N}(\cdot; 0, R_w)$ with $R_w = 0.03I_2$ and I_a denotes $a \times a$ identity matrix. The measurement model (4) is given by

$$z_k = [\,1\;\;0\,]\,x_k + v_k, \tag{41}$$

where $v_k \sim \mathcal{N}(\cdot; 0, R_v)$ with $R_v = 0.7$. The nonlinear state space model (40)-(41) can be represented in the LFT form (26)-(29) with

$$\begin{aligned} &A_k = \begin{bmatrix} 0 & 1 \\ 0 & -0.1 \end{bmatrix}, \quad B_{1,k} = \begin{bmatrix} 0 & 0 \\ 10 & 0 \end{bmatrix}, \quad B_{2,k} = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}, \\ &C_{1,k} = [\,1\;\;0\,], \quad D_{11,k} = 1, \quad D_{12,k} = 0, \\ &C_{2,k} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad D_{22,k} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}. \end{aligned} \tag{42}$$

The feedback connection has the simple structure $\Delta(x_k) = x_k(1)I_2$. The true trajectory of the state x_k for 50 time steps is shown in Fig. 1. For an initial estimate of x_0 with mean $\bar{x}_0 = [\,0,\; 0\,]^T$ and covariance $R_{x,0} = I_2$, the estimate of the state at each time step given by the proposed filter along with the true trajectory are shown in Fig. 2. In Fig. 3 the covariance of the error in the estimate obtained using the proposed method and UKF are shown. Fig. 3 (a) depicts that the error using the proposed technique is bounded within 0.5 for the duration of the simulation while using UKF the error repeatedly exceeds this bound. A similar trend observed in Fig. 3 (b) suggests that the proposed filter gives better approximation than UKF.

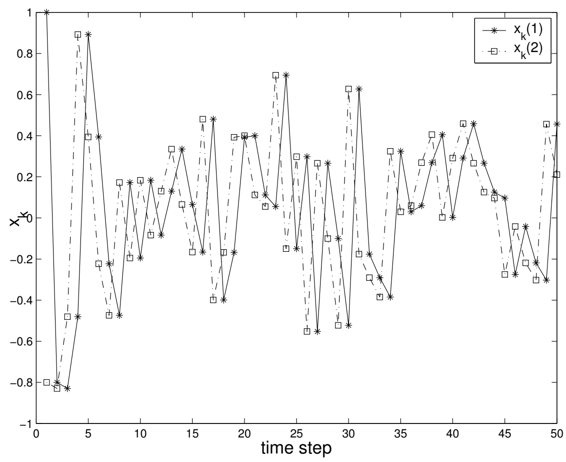

Fig. 1. Trajectory of the state $x_k = [\,x_k(1)\;\; x_k(2)\,]^T$.

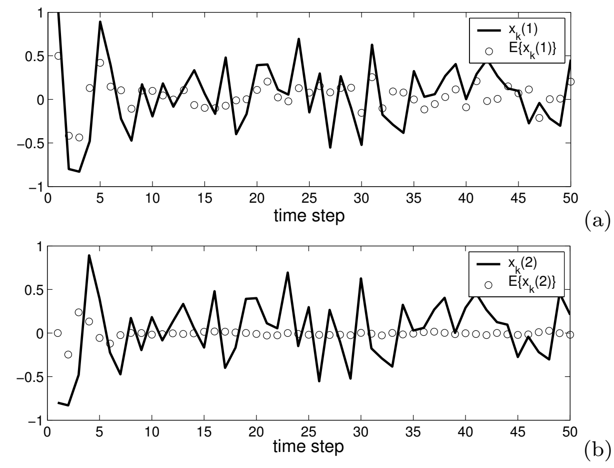

Fig. 2. True trajectory and the estimate of the state $\mathbb{E}\{x_k|Z_k\}$ given by the proposed filter.

4.1 Example II

Consider the highly nonlinear system

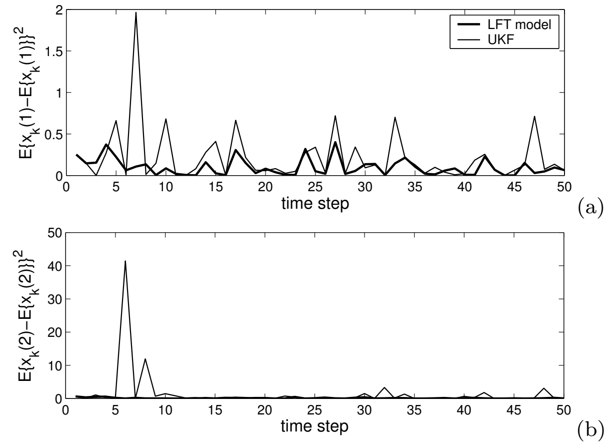

Fig. 3. Covariance of the error in the estimate using the proposed filter and UKF.

$$x_{k+1} = \big(Q_0 + Q_1 x_k^3(1) + Q_2 x_k^3(2) + Q_3 x_k(1) x_k^2(2) + Q_4 x_k(1) + Q_5 x_k(2)\big) x_k + \begin{bmatrix} -2 & 0 \\ 1 & 0 \end{bmatrix} w_k, \tag{43}$$

$$z_k = 100\,[\,-1\;\;1\,]\,x_k + v_k, \tag{44}$$

where

$$Q_0 = \begin{bmatrix} -0.7 & -1.0 \\ 0.1 & -0.5 \end{bmatrix}, \qquad Q_1 = \begin{bmatrix} 0.3 & 0.2 \\ 0.1 & 0.2 \end{bmatrix},$$
$$Q_2 = \begin{bmatrix} 0.2 & 0.1 \\ 0.2 & 0.3 \end{bmatrix}, \qquad Q_3 = \begin{bmatrix} 0.4 & 0.1 \\ 0.15 & 0.1 \end{bmatrix},$$
$$Q_4 = \begin{bmatrix} 0.25 & 0.25 \\ 0.1 & 0.25 \end{bmatrix}, \qquad Q_5 = \begin{bmatrix} 0.25 & 0 \\ 0.1 & 0.25 \end{bmatrix}, \tag{45}$$

and $w_k \sim \mathcal{N}(\cdot; 0, R_w)$ with $R_w = 0.3 I_2$ and $v_k \sim \mathcal{N}(\cdot; 0, R_v)$ with $R_v = 30$. The system (43)-(44) admits an exact equivalent LFT model (26)-(29) with

$$A_k = \begin{bmatrix} -0.7 & -1.0 \\ 0.1 & -0.5 \end{bmatrix}, \quad B_{1,k} = \begin{bmatrix} -2 & 0 \\ 1 & 0 \end{bmatrix},$$
$$B_{2,k} = \begin{bmatrix} 0.25 & 0 & 0.3 & 0 & 0 & 0.1 & 0.2 & 0.3 & 0 & 0.4 & 0.5 \\ 0.1 & 0 & 0.1 & 0.25 & 0 & 0.3 & 0.2 & 0.3 & 0 & 0.15 & 0.35 \end{bmatrix},$$
$$C_{1,k} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad D_{11,k} = 1, \quad D_{12,k} = 0,$$
$$C_{2,k}^T = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$
$$D_{22,k} = \begin{bmatrix} 0_{1,5} & 0_{1,5} & 0 \\ L & 0_5 & 0_{5,1} \\ M & N & 0_{5,1} \end{bmatrix}, \tag{46}$$

where $0_{a,b}$ denotes $a \times b$ zero matrix, 0_a denotes $a \times a$ zero matrix,

$$L = \begin{bmatrix} I_2 & 0_{2,1} & 0_2 \\ 0_{1,2} & 0 & 0_{1,2} \\ 0_2 & 0_{2,1} & I_2 \end{bmatrix}, M = \begin{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} & 0_{2,1} & \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \\ [\,1\;\;0\,] & 0 & [\,0\;\;0\,] \\ 0_2 & 0_{2,1} & 0_2 \end{bmatrix},$$
$$N = \begin{bmatrix} 0_2 & 0_{2,1} & 0_2 \\ 0_{1,2} & 0 & 0_{2,1} \\ 0_2 & 0_{2,1} & \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \end{bmatrix}, \tag{47}$$

and the feedback connection has the form

$$\Delta(x_k) = \mathrm{diag}\big([\,x_k(1),\, x_k(1),\, x_k(1),\, x_k(2),\, x_k(2),\, x_k(2),\, x_k(2),\, x_k(1),\, x_k(2),\, x_k(2),\, x_k(2)\,]\big). \tag{48}$$

The trajectory of the state x_k as it evolves with time for 50 time steps is shown in Fig. 4. In Fig. 5 the true trajectory and the estimate of the state at each time step using the proposed filter are shown given an initial estimate of x_0 with the first two moments $\bar{x}_0 = [\,0,\, 0\,]^T$ and covariance $R_{x,0} = 0.5 I_2$. The covariance of the error in the estimates is shown in Fig. 6. The results show that the proposed filter gives reasonably good estimates with the covariance of the error within 0.3 for the duration of the simulation.

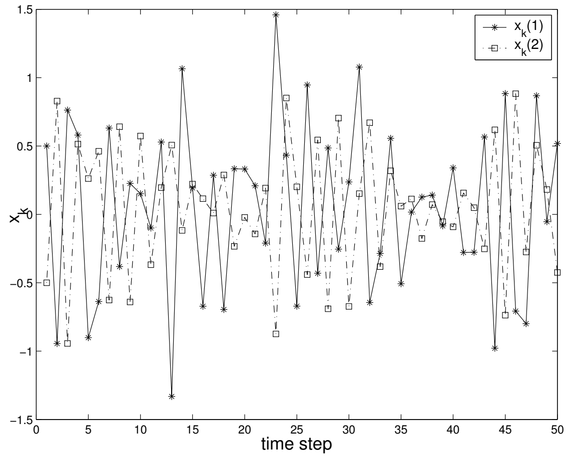

Fig. 4. Trajectory of the state $x_k = [\,x_k(1)\;\; x_k(2)\,]^T$.

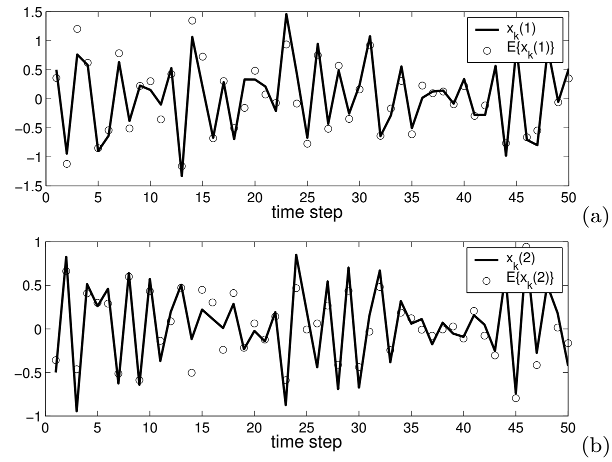

Fig. 5. Trajectory of the state and the estimate $\mathbb{E}\{x_k | Z_k\}$ given by the proposed filter.

5. CONCLUSION

A nonlinear filtering approach for the LFT model is introduced. The LFT model gives an equivalent representation for any smooth nonlinear mapping which makes it flexible enough to accommodate a general class of nonlinear problems. The representation is characterized by a linear structure and a simple feedback connection. By applying the unscented transformation to the feedback path only we derive a closed form solution for recursive estimation of the moments of the state conditional on a data sequence. Simulation results demonstrate that the proposed filter is a promising candidate for nonlinear filtering involving highly nonlinear models.

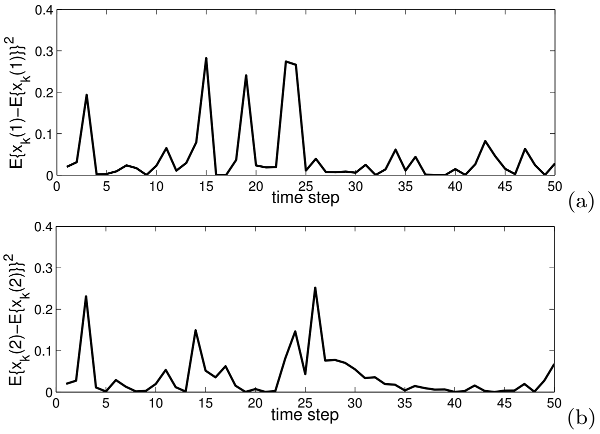

Fig. 6. Covariance of the error in the estimate using the proposed filter.

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