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Integral Sliding Mode Output Tracking Controller for Sampled-Data Systems

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Abstract: A new integral type sliding surface (ISM) is design for sampled-data systems for output tracking. ISM surface design is based on Output Feedback. Discrete-time control based on ISM achieves good tracking performance while allowing the pole assignment of m poles, where m is positive integer, which are otherwise zero in a deadbeat design. It will be shown in this work that, the discrete-time version of the sliding mode control based on the integral type sliding surface results in two scenarios: A tracking error of $O(T^2)$ if the discrete-time system is minimum-phase, and a tracking error of $O(T)$ if the original system does not satisfy minimum-phaseness, but, rather a modified version of the system. In this work T is the sampling-time. A simulation example demonstrates the validity of the proposed scheme.

1. INTRODUCTION

Sliding mode control is a very popular robust control method owing to its ease of design and robustness to 'matched' disturbances. However, full state information is required in the controller design which is a drawback since in most practical applications only the output measurement is available. To solve this problem, focus was placed on output feedback based sliding mode control Żak et al. (1993)-Lai et al. (2003). Two approaches arose: a design based on obervers to construct the missing states, Edwards and Spurgeon (1996), Slotine et al. (1987), the other design focused on using only the output measurement, Żak et al. (1993), El-Khazali and DeCarlo (1995). Both approaches present certain strengths and limitations.

Computer implementation of control algorithms presents a great convenience and has, hence, caused the research in the area of discrete-time control to intensify. This also necessitated a rework in the sliding mode control strategy for sampled-data systems. Most of the discrete-time sliding mode approaches are based on the availability of full state information, Su et al. (2000)-Abidi et al. (2007). A few approaches did focus on the output measurement, Lai et al. (2003). In Lai et al. (2003), the control design was based on the assumption that the state matrix of a discrete-time system is invertible. This is true for sampled-data systems. In this work we will focus on state based approaches as well as expand upon the work of Lai et al. (2003) by focusing on arbitrary reference tracking of a linear time invariant system with matched disturbance.

Delays in the state or disturbance estimation in sampled-data systems is an inevitable phenomenon and must be studied carefully. In Abidi et al. (2007) it was shown that in the case of delayed disturbance estimation a worst case accuracy of $O(T)$ can be guaranteed for deadbeat sliding mode control design and a worst case accuracy of $O(T^2)$ for integral sliding mode control. While deadbeat response is a desired phenomenon, deadbeat control is undesirable in practical implementation due to the overlarge control action required. In Abidi et al. (2007) the integral sliding mode design avoided the deadbeat response by eliminating the poles at zero. In this work, we extend the integral sliding mode design to output tracking problems.

A challenging issue in output tracking control is to perform arbitrary reference tracking when only output measurement is available. To accomplish the task of arbitrary reference tracking a controller based on output feedback with a state observer will be designed. The objective is to drive the output tracking error to a certain neighbourhood of the origin. For this purpose a discrete-time integral sliding surface (ISM) is proposed. It will be shown that this approach produces a worst case error of $O(T)$.

2. PROBLEM FORMULATION

2.1 System Properties

Consider the following continuous-time system with a nominal linear-time-invariant model and matched disturbance

$$\begin{aligned}\dot{\mathbf{x}}(t) &= A\mathbf{x}(t) + B(\mathbf{u}(t) + \mathbf{f}(t)) \\ \mathbf{y}(t) &= C\mathbf{x}(t)\end{aligned} \tag{1}$$

where the state $\mathbf{x} \in \Re^n$, the output $\mathbf{y} \in \Re^m$, the control $\mathbf{u} \in \Re^m$, and the disturbance $\mathbf{f} \in \Re^m$ is assumed smooth and bounded. The discretized counterpart of (1) can be given by

$$\begin{aligned}\mathbf{x}_{k+1} &= \Phi\mathbf{x}_k + \Gamma\mathbf{u}_k + \mathbf{d}_k \\ \mathbf{y}_k &= C\mathbf{x}_k, \quad \mathbf{y}_0 = \mathbf{y}(0)\end{aligned} \tag{2}$$

where

$$\Phi = e^{AT}, \quad \Gamma = \int_0^T e^{A\tau} d\tau B$$

$$\mathbf{d}_k = \int_0^T e^{A\tau} B\mathbf{f}((k+1)T - \tau) d\tau,$$

and T is the sampling period. Here the disturbance $\mathbf{d}_k$ represents the influence accumulated from kT to $(k+1)T$; in the sequel, it shall directly link to $\mathbf{x}_{k+1} = \mathbf{x}((k+1)T)$. From the definition of Γ it can be shown that

$$\Gamma = BT + \frac{1}{2!}ABT^2 + \frac{1}{3!}A^2BT^3 + \cdots$$
$$= BT + MT^2 + O(T^3) \Rightarrow BT = \Gamma - MT^2 + O(T^3) \quad (3)$$

where M is a constant matrix. From (3), it can be concluded that the magnitude of Γ is of the order $O(T)$.

Based on the smoothness assumption on the disturbance $\mathbf{f}(t)$, several useful properties were derived in Abidi et al. (2007):

Property 1. The discretized disturbance satisfies:

P 1.

$$\mathbf{d}_k = \int_0^T e^{A\tau}B\mathbf{f}((k+1)T - \tau)d\tau = \Gamma\mathbf{f}_k + \frac{1}{2}\Gamma\mathbf{v}_kT + O(T^3)$$

, where $\mathbf{v}_k = \mathbf{v}(kT)$, $\mathbf{v}(t) = \frac{d}{dt}\mathbf{f}(t)$. Note that the magnitude of the mismatched part in the disturbance $\mathbf{d}_k$ is of the order $O(T^3)$.

P 2. $\mathbf{d}_k = O(T)$.

P 3. $\mathbf{d}_k - \mathbf{d}_{k-1} = O(T^2)$.

P 4. $\mathbf{d}_k - 2\mathbf{d}_{k-1} + \mathbf{d}_{k-2} = O(T^3)$.

Property 2. Assume

$$\mathbf{e}_{k+1} = \Lambda\mathbf{e}_k + \boldsymbol{\delta}_k$$

where matrix Λ is asymptotically stable, the magnitude of $\boldsymbol{\delta}_k$ is of the order $O(T^3)$. Then the magnitude of $\mathbf{e}_k$ is of the order $O(T^2)$.

The primary control objective is to desgign an appropriate controller $\mathbf{u}_k$, such that the output $\mathbf{y}_k$ of (2) can follow an arbitrary trajectory $\mathbf{r}_k$ whose magnitue is of the order $O(1)$.

It is worth to highlight that arbitrary trajectory tracking differs significantly from regulation or set-point control problems. Comparing output tracking for arbitrary trajectory with output regulation or set-point control, the minimum-phase property of the plant (2) is in general a necesary condition for the former but not so for the latter.

Let the control law be $\mathbf{u}_k = -K\mathbf{x}_k + G(q)\mathbf{r}_k$, where $G(q)$ is a design tranfer matrix, q is a forward shifting operator. Substituting the control law into (2) yields

$$\mathbf{y}_k = C\left(qI_m - \Phi + \Gamma K\right)^{-1}\Gamma G(q)\mathbf{r}_k \quad (4)$$

where $I_m \in \Re^m$ is a unity matirx. From (4) we can see that for the precise tracking of an arbitrary reference $\mathbf{r}_k$, $G(q)$ must be the inverse of $C(qI_m - \Phi + \Gamma K)^{-1}\Gamma$. Since K is selected such that $(\Phi - \Gamma K)$ is stable, the only concern is that the inverse $C(qI_m - \Phi + \Gamma K)^{-1}\Gamma$ will contain the zeros of (Φ, Γ, C) and, therefore, will require that the system be minimum-phase. In this work it will be shown that it is possible to avoid this constraint but with the loss of accuracy.

Thus, we can summarize the control objective as follows:

To design a discrete-time integral sliding manifold and a discrete-time SMC law that will stabilize the sampled-data system (2) and achieve as precisely as possible arbitrary output reference tracking. Meanwhile the closed-loop dynamics of the sampled-data system has m closed-loop poles assigned to desired locations.

3. OUTPUT FEEDBACK OUTPUT TRACKING ISM

In this section we will discuss the output feedback based output tracking controller.

3.1 Controller Design

In order to proceed we will first define a reference model

$$\begin{aligned}\mathbf{x}_{r,k+1} &= (\Phi - K_1)\mathbf{x}_{r,k} + K_2\mathbf{r}_{k+1} \\ \mathbf{y}_{r,k} &= C\mathbf{x}_{r,k}\end{aligned} \quad (5)$$

where $\mathbf{x}_{r,k} \in \Re^n$ is the state vector, $\mathbf{y}_{r,k} \in \Re^m$ is the output vector, and $\mathbf{r}_k \in \Re^m$ is a bounded reference trajectory. The matrices K_1 and K_2 are both functions of (Φ, Γ, C) and K_1 is selected such that $(\Phi - K_1)$ is stable. The selection criteria for the matrices K_1 and K_2 will be discussed in detail in §4.4.

Now consider a new sliding surface

$$\begin{aligned}\boldsymbol{\sigma}_k &= D(\mathbf{x}_{r,k} - \mathbf{x}_k) + \boldsymbol{\varepsilon}_k \\ \boldsymbol{\varepsilon}_k &= \boldsymbol{\varepsilon}_{k-1} + ED(\mathbf{x}_{r,k-1} - \mathbf{x}_{k-1})\end{aligned} \quad (6)$$

where $D = C\Phi^{-1}$, $\boldsymbol{\sigma}_k, \boldsymbol{\varepsilon}_k \in \Re^m$ are the switching function and integral vectors, $E \in \Re^{m\times m}$ is an integral gain matrix. Note that $D\mathbf{x}_k = C\Phi^{-1}(\Phi\mathbf{x}_{k-1} + \Gamma\mathbf{u}_{k-1} + \mathbf{d}_{k-1}) = \mathbf{y}_{k-1} + D(\Gamma\mathbf{u}_{k-1} + \mathbf{d}_{k-1})$ is independent of the states.

The equivalent control law can be derived from $\boldsymbol{\sigma}_{k+1} = 0$. From (6) $\boldsymbol{\varepsilon}_k = \boldsymbol{\sigma}_k - D(\mathbf{x}_{r,k} - \mathbf{x}_k)$, we have

$$\begin{aligned}\boldsymbol{\sigma}_{k+1} &= D(\mathbf{x}_{r,k+1} - \mathbf{x}_{k+1}) + \boldsymbol{\varepsilon}_{k+1} \\ &= D(\mathbf{x}_{r,k+1} - \mathbf{x}_{k+1}) + \boldsymbol{\varepsilon}_k + ED(\mathbf{x}_{r,k} - \mathbf{x}_k) \\ &= D(\mathbf{x}_{r,k+1} - \mathbf{x}_{k+1}) + \boldsymbol{\sigma}_k - D(\mathbf{x}_{r,k} - \mathbf{x}_k) + ED(\mathbf{x}_{r,k} - \mathbf{x}_k) \\ &= D\mathbf{x}_{r,k+1} - D\mathbf{x}_{k+1} + \boldsymbol{\sigma}_k - \Lambda D(\mathbf{x}_{r,k} - \mathbf{x}_k)\end{aligned} \quad (7)$$

where $\Lambda = I - E$. Substituting the system dynamics (2) into (7) yields

$$\begin{aligned}\boldsymbol{\sigma}_{k+1} &= D\mathbf{x}_{r,k+1} - D(\Phi\mathbf{x}_k + \Gamma\mathbf{u}_k + \mathbf{d}_k) + \boldsymbol{\sigma}_k - \Lambda D(\mathbf{x}_{r,k} - \mathbf{x}_k) \\ &= \mathbf{a}_k - D\Gamma\mathbf{u}_k - D\mathbf{d}_k\end{aligned} \quad (8)$$

where $\mathbf{a}_k = -(D\Phi - \Lambda D)\mathbf{x}_k + (D\mathbf{x}_{r,k+1} - \Lambda D\mathbf{x}_{r,k})$.

Letting $\boldsymbol{\sigma}_{k+1} = 0$, solving for the equivalent control $\mathbf{u}_k^{eq}$, we have

$$\begin{aligned}\mathbf{u}_k^{eq} &= (D\Gamma)^{-1}(\mathbf{a}_k - D\mathbf{d}_k) \quad (9)\\ &= -(D\Gamma)^{-1}(D\Phi - \Lambda D)\mathbf{x}_k + (D\Gamma)^{-1}(D\mathbf{x}_{r,k+1} - \Lambda D\mathbf{x}_{r,k}) \\ &\quad -(D\Gamma)^{-1}D\mathbf{d}_k.\end{aligned}$$

Controller (10) is not implementable as it requires *a priori* knowledge of the disturbance. Thus, the estimate of the disturbance should be used

$$\mathbf{u}_k = (D\Gamma)^{-1}(\mathbf{a}_k - D\hat{\mathbf{d}}_k) \quad (10)$$

where $\hat{\mathbf{d}}_k$ is the disturbance estimation.

3.2 Disturbance Observer Design

Note that according to *Property 1*, the disturbance can be written as

$$\mathbf{d}_k = \Gamma \mathbf{f}_k + \frac{1}{2}\Gamma \mathbf{v}_k T + O(T^3) = \Gamma \boldsymbol{\eta}_k + O(T^3) \quad (11)$$

where $\boldsymbol{\eta}_k = \mathbf{f}_k + \frac{1}{2}\mathbf{v}_k T$. If $\boldsymbol{\eta}_k$ can be estimated, then the estimation error of $\mathbf{d}_k$ would be $O(T^3)$ which is acceptable in practical applications.

Define the observer

$$\begin{aligned} \mathbf{x}_{d,k} &= \Phi \mathbf{x}_{d,k-1} + \Gamma \mathbf{u}_{k-1} + \Gamma \hat{\boldsymbol{\eta}}_{k-1} \\ \mathbf{y}_{d,k-1} &= C\mathbf{x}_{d,k-1} \end{aligned} \quad (12)$$

where $\mathbf{x}_{d,k-1} \in \Re^n$ is the observer state vector, $\mathbf{y}_{d,k-1} \in \Re^m$ is the observer output vector, $\hat{\boldsymbol{\eta}}_{k-1} \in \Re^m$ is the disturbance estimate and will act as the 'control input' to the observer, therefore we can write $\hat{\mathbf{d}}_{k-1} = \Gamma \hat{\boldsymbol{\eta}}_{k-1}$. Since the disturbance estimate will be used in the final control signal, it must not be overly large. Therefore, it is wise to avoid a deadbeat design. For this reason we design the disturbance observer based on an integral sliding surface

$$\begin{aligned} \boldsymbol{\sigma}_{d,k} &= \mathbf{e}_{d,k} - \mathbf{e}_{d,0} + \boldsymbol{\varepsilon}_{d,k} \\ \boldsymbol{\varepsilon}_{d,k} &= \boldsymbol{\varepsilon}_{d,k-1} + E_d \mathbf{e}_{d,k-1} \end{aligned} \quad (13)$$

where $\mathbf{e}_{d,k} = \mathbf{y}_k - \mathbf{y}_{d,k}$ is the output estimation error, $\boldsymbol{\sigma}_{d,k}, \boldsymbol{\varepsilon}_{d,k} \in \Re^m$ are the sliding function and integral vectors, and E_d is an integral gain matrix.

Following the similar design of (?), let $\boldsymbol{\sigma}_{d,k} = 0$ we can derive the virtual equivalent control $\mathbf{u}_{k-1} + \hat{\boldsymbol{\eta}}_{k-1}$, thus

$$\hat{\boldsymbol{\eta}}_{k-1} = (C\Gamma)^{-1}\left[\mathbf{y}_k - \Lambda_d \mathbf{e}_{d,k-1} - C\Phi \mathbf{x}_{d,k-1} + \boldsymbol{\sigma}_{d,k-1}\right] - \mathbf{u}_{k-1} \quad (14)$$

where $\Lambda_d = I_m - E_d$.

In practice, the quantity $\mathbf{y}_{k+1}$ is not available at the time instance k when computing $\hat{\eta}_k$. Therefore we can only compute $\hat{\eta}_{k-1}$, and in the control law (10) we use the delayed estimate $\hat{\mathbf{d}} = \Gamma \hat{\boldsymbol{\eta}}_{k-1}$.

The stability and convergence properties of the observer (12) and the disturbance estimation (14) are analyzed in the theorem 1.

Theorem 1. The observer outputs $\mathbf{y}_{d,k}$ converge asymptotically to the true outputs $\mathbf{y}_k$, and the disturbance estimate $\hat{\mathbf{d}}_{k-1}$ converges to the actual disturbance $\mathbf{d}_{k-1}$ with the precision order $O(T^2)$.

Proof:

Substituting (14) into (12), and using the relation $\mathbf{e}_{d,k-1} = C(\mathbf{y}_{k-1} - \mathbf{y}_{d,k-1})$, yield

$$\begin{aligned} \mathbf{x}_{d,k} = &\left[\Phi - \Gamma(C\Gamma)^{-1}(C\Phi - \Lambda_d C)\right]\mathbf{x}_{d,k-1} + \Gamma(C\Gamma)^{-1}[\mathbf{y}_k - \Lambda_d \mathbf{y}_{k-1}] \\ &+ \Gamma(C\Gamma)^{-1}\boldsymbol{\sigma}_{d,k-1}. \end{aligned} \quad (15)$$

Since the virtual control $\mathbf{u}_{k-1} + \hat{\boldsymbol{\eta}}_{k-1}$ is chosen such that $\boldsymbol{\sigma}_{d,k} = 0$ for any $k > 0$, (15) renders to

$$\mathbf{x}_{d,k} = \left[\Phi - \Gamma(C\Gamma)^{-1}(C\Phi - \Lambda_d C)\right]\mathbf{x}_{d,k-1} + \Gamma(C\Gamma)^{-1}[\mathbf{y}_k - \Lambda_d \mathbf{y}_{k-1}]. \quad (16)$$

The second term on the right hand side of (16) can be expressed as

$$\begin{aligned} \Gamma(C\Gamma)^{-1}[\mathbf{y}_k - \Lambda_d \mathbf{y}_{k-1}] = &\Gamma(C\Gamma)^{-1}(C\Phi - \Lambda_d C)\mathbf{x}_{k-1} \\ &+ \Gamma \mathbf{u}_{k-1} + \Gamma(C\Gamma)^{-1}C\mathbf{d}_{k-1} \end{aligned}$$

by using the relations $\mathbf{y}_k = C\Phi \mathbf{x}_{k-1} + C\Gamma \mathbf{u}_{k-1} + C\mathbf{d}_{k-1}$ and $\mathbf{y}_{k-1} = C\mathbf{x}_{k-1}$. Therefore (16) can be rewritten as

$$\begin{aligned} \mathbf{x}_{d,k} = &\Phi \mathbf{x}_{d,k-1} + \Gamma(C\Gamma)^{-1}(C\Phi - \Lambda_d C)\Delta \mathbf{x}_{d,k-1} \\ &+ \Gamma \mathbf{u}_k + \Gamma(C\Gamma)^{-1}C\mathbf{d}_{k-1} \end{aligned} \quad (17)$$

where $\Delta \mathbf{x}_{d,k-1} = \mathbf{x}_{k-1} - \mathbf{x}_{d,k-1}$.

Further subtracting (17) from the system (2) we obtain

$$\begin{aligned} \Delta \mathbf{x}_{d,k} = &\left[\Phi - \Gamma(C\Gamma)^{-1}(C\Phi - \Lambda_d C)\right]\Delta \mathbf{x}_{d,k-1} \\ &+ [I - \Gamma(C\Gamma)^{-1}C]\mathbf{d}_{k-1} \end{aligned} \quad (18)$$

where $[I - \Gamma(C\Gamma)^{-1}C]\mathbf{d}_{k-1}$ is $O(T^3)$ because
$[I - \Gamma(C\Gamma)^{-1}C][\Gamma \boldsymbol{\eta}_{k-1} + O(T^3)] = [I - \Gamma(C\Gamma)^{-1}C]O(T^3) = O(T^3)$.

Applying the *Property 2*, $\Delta \mathbf{x}_{d,k-1} = O(T^2)$.

From (18) we can see that the stability of the disturbance observer depends only on the system matrix $\left[\Phi - \Gamma(C\Gamma)^{-1}(C\Phi - \Lambda_d C)\right]$ and is guaranteed by the selection of the matrix Λ_d and the fact that system (Φ, Γ, C) is minimum phase. It should also be noted that the residue term $[I - \Gamma(C\Gamma)^{-1}C]\mathbf{d}_{k-1}$ in the state space is orthogonal to the output space, as $C[I - \Gamma(C\Gamma)^{-1}C]\mathbf{d}_{k-1} = 0$. Therefore premultliplication of (18) with C yields the output tracking error dynamics

$$\mathbf{e}_{d,k} = \Lambda_d \mathbf{e}_{d,k-1} \quad (19)$$

which is asymptotically stable through choosing a stable matrix Λ_d.

Finally we discuss the convergence property of the estimate $\hat{\mathbf{d}}_{k-1}$. Subtracting (12) from (2) with one-step delay, we obtain

$$\Delta \mathbf{x}_{d,k} = \Phi \Delta \mathbf{x}_{d,k-1} + \Gamma(\boldsymbol{\eta}_{k-1} - \hat{\boldsymbol{\eta}}_{k-1}) + O(T^3). \quad (20)$$

Premultiplying (20) with C, and substituing (19) that describes $C\Delta \mathbf{x}_{d,k}$, yield

$$\hat{\boldsymbol{\eta}}_{k-1} = \boldsymbol{\eta}_{k-1} + (C\Gamma)^{-1}(C\Phi - \Lambda_d C)\Delta \mathbf{x}_{d,k-1} + (C\Gamma)^{-1}O(T^3). \quad (21)$$

The first term on the right hand side of (21) is $O(T)$ because $\Delta \mathbf{x}_{d,k-1} = O(T^2)$ but $(C\Gamma)^{-1} = O(T^{-1})$. As a result, from (21) we can conclude that $\hat{\boldsymbol{\eta}}_{k-1}$ approaches $\boldsymbol{\eta}_{k-1}$ with the precision $O(T)$. In terms of the relationship

$$\mathbf{d} - \hat{\mathbf{d}} = \Gamma(\boldsymbol{\eta} - \hat{\boldsymbol{\eta}}) + O(T^3)$$

and $\Gamma = O(T)$, we conclude $\hat{\mathbf{d}}_{k-1}$ converges to $\mathbf{d}_{k-1}$ with the precision of $O(T^2)$.

Remark 1: At the time k, we can guarantee the convergence of $\hat{\boldsymbol{\eta}}_{k-1}$ to $\boldsymbol{\eta}_{k-1}$ with the precision $O(T)$. In other words, we can guarantee the convergence of the disturbance estimate at the time k, $\hat{\mathbf{d}}_k$, to the actual disturbance at time $k-1$, $\mathbf{d}_{k-1}$, with the precision $O(T^2)$. This result is consistent with the state-based estimation presented in §3 in which $\hat{\mathbf{d}}_k$ is made equal to $\mathbf{d}_{k-1}$. Comparing differences between the state-based and output-based disturbance estimation, the former has only one-step delay with perfect precision, whereas the latter is asymptotic with $O(T^2)$ precision.

3.3 Stability Analysis

To analyze the stability of the closed-loop system, substitute $\mathbf{u}_k$ in (10) into the plant (2) leading to the closed-loop equation in the sliding mode

$$\mathbf{x}_{k+1} = \left[\Phi - \Gamma(D\Gamma)^{-1}(D\Phi - \Lambda D)\right]\mathbf{x}_k + \mathbf{d}_k - \Gamma(D\Gamma)^{-1}D\hat{\mathbf{d}}_k \\ +\Gamma(D\Gamma)^{-1}[D\mathbf{x}_{r,k+1} - \Lambda D\mathbf{x}_{r,k} + \boldsymbol{\sigma}_k]. \quad (22)$$

The stability of the above sliding equation is summarized in Theorem 2.

Theorem 2. Using the control law (10) the sliding mode is

$$\boldsymbol{\sigma}_{k+1} = D(\hat{\mathbf{d}}_k - \mathbf{d}_k).$$

Further, the state tracking error $\Delta\mathbf{x}_k = \mathbf{x}_{r,k} - \mathbf{x}_k$ is bounded if system (Φ, Γ, D) is minimum-phase and the eigenvalues of the matrix Λ are within the unit circle.

Proof:

In order to verify the first part of theorem 2, rerwite the dynamics of the slidng mode (8)

$$\boldsymbol{\sigma}_{k+1} = \mathbf{a}_k - D\Gamma\mathbf{u}_k - D\mathbf{d}_k = \mathbf{a}_k - D\Gamma\mathbf{u}_k^{eq} - D\mathbf{d}_k + D\Gamma(\mathbf{u}_k^{eq} - \mathbf{u}_k) \\ = D\Gamma(\mathbf{u}_k^{eq} - \mathbf{u}_k),$$

where we use the property of equivalent control $\boldsymbol{\sigma}_{k+1} = \mathbf{a}_k - D\Gamma\mathbf{u}_k^{eq} - D\mathbf{d}_k = 0$. Comparing two control laws (10) and (10), we obtain

$$\boldsymbol{\sigma}_{k+1} = D(\hat{\mathbf{d}}_k - \mathbf{d}_k).$$

Note that if there is no disturbance or we have perfect estimation of the disturbance, then $\boldsymbol{\sigma}_{k+1} = 0$ as desired. From the results of theorem 1 and *Property 1*

$$\hat{\mathbf{d}}_k - \mathbf{d}_k = \Gamma\hat{\boldsymbol{\eta}}_{k-1} - \Gamma\boldsymbol{\eta}_k + O(T^3) \\ = \Gamma(\hat{\boldsymbol{\eta}}_{k-1} - \boldsymbol{\eta}_{k-1}) - \Gamma(\boldsymbol{\eta}_k - \boldsymbol{\eta}_{k-1}) + O(T^3) \to O(T^2)$$

as $k \to \infty$. Thus $\boldsymbol{\sigma}_{k+1} \to O(T^2)$ which is acceptable in practice.

To prove the boundedness of the state tracking error $\Delta\mathbf{x}_k$, first derive the state error dynamics. Subtracting both sides of (22) from the refernece model (5), and substituting $\boldsymbol{\sigma}_k = D(\hat{\mathbf{d}}_{k-1} - \mathbf{d}_{k-1})$, yields

$$\Delta\mathbf{x}_{k+1} = \left[\Phi - \Gamma(D\Gamma)^{-1}(D\Phi - \Lambda D)\right]\Delta\mathbf{x}_k \\ +[I - \Gamma(D\Gamma)^{-1}D](K_2\mathbf{r}_{k+1} - K_1\mathbf{x}_{r,k}) - \boldsymbol{\zeta}_k \quad (23)$$

where

$$\boldsymbol{\zeta}_k = \mathbf{d}_k - \Gamma(D\Gamma)^{-1}D(\hat{\mathbf{d}}_k - \hat{\mathbf{d}}_{k-1} + \mathbf{d}_{k-1}). \quad (24)$$

To verify the stability of the system we define the following lemma:

Lemma 3. The eignvalues of $[\Phi - \Gamma(C\Gamma)^{-1}(C\Phi - \Lambda C)]$ are the eigenvalues of Λ and the non-zero eigenvalues of $[\Phi - \Gamma(C\Gamma)^{-1}C\Phi]$.

Proof See Appendix. The stability of (24) is dependent on the system matrix $\left[\Phi - \Gamma(D\Gamma)^{-1}(D\Phi - \Lambda D)\right]$. From Lemma 1 the closed-loop poles of (24) are the eigenvalues of Λ and the open-loop zeros of the system (Φ, Γ, D). Thus, m poles of the closed-loop system can be selected by the proper choice of the matrix Λ while the remaining poles are stable only if the system (Φ, Γ, D) is minimum-phase. Note that both $\mathbf{r}_{k+1}$ and $\mathbf{x}_{r,k}$ are reference signals and are bounded. Therefore we need only to show the boundedness of $\boldsymbol{\zeta}_k$ which is

$$\boldsymbol{\zeta}_k = \Gamma\boldsymbol{\eta}_k + O(T^3) - \Gamma(D\Gamma)^{-1}D[\Gamma\hat{\boldsymbol{\eta}}_{k-1} - \Gamma\hat{\boldsymbol{\eta}}_{k-2} + \Gamma\boldsymbol{\eta}_{k-1} + O(T^3)] \\ = \Gamma(\boldsymbol{\eta}_k - \hat{\boldsymbol{\eta}}_k) + \Gamma(\boldsymbol{\eta}_{k-1} - \hat{\boldsymbol{\eta}}_{k-1}) - \Gamma(\boldsymbol{\eta}_{k-2} - \hat{\boldsymbol{\eta}}_{k-2}) \\ -\Gamma(\boldsymbol{\eta}_{k-1} - \boldsymbol{\eta}_{k-2}) + O(T^3). \quad (25)$$

From theorem 1 and *Property 1*, all terms in the bracket on the right hand side of (25) approach $O(T^2)$. Note also $\Gamma = O(T)$, thus $\boldsymbol{\zeta}_k = O(T)O(T^2) + O(T^3) = O(T^3)$.

3.4 Reference Model Selection and Tracking Error Bound

We have established the stability condition for the closed-loop system, but, have not yet established the tracking error bound. From (24) it can be seen that the tracking error bound is dependent on the disturbance estimate $\hat{\mathbf{d}}_k$ as well as the selection of K_1 and K_2. Up to this point, not much was discussed in terms of the selection of the reference model (5). As it can be seen from (24) the selection of the reference model can effect the overall tracking error bound. Since we consider an arbitrary reference $\mathbf{r}_k$, the reference model must be selected such that its output is the reference signal $\mathbf{r}_k$. To achieve this requirement, we explore two possible selections of the reference model.

Reference model based on (Φ, Γ, C) being minimum-phase

For this reference model select the matrices $K_1 = \Gamma(C\Gamma)^{-1}C\Phi$ and $K_2 = \Gamma(C\Gamma)^{-1}$ and the reference model (5) can be written as

$$\mathbf{x}_{r,k+1} = \left[\Phi - \Gamma(C\Gamma)^{-1}C\Phi\right]\mathbf{x}_{r,k} + \Gamma(C\Gamma)^{-1}\mathbf{r}_{k+1} \\ \mathbf{y}_{r,k} = C\mathbf{x}_{r,k} = \mathbf{r}_k. \quad (26)$$

It can be easily seen from (26) that it is stable only if the matrix $\left[\Phi - \Gamma(C\Gamma)^{-1}C\Phi\right]$ is stable, i.e., the system (Φ, Γ, C) is minimum-phase. Substituting the selected matrices K_1 and K_2 into (24) and using the fact that $\left[I - \Gamma(D\Gamma)^{-1}D\right]\Gamma = 0$, we obtain

$$\Delta\mathbf{x}_{k+1} = \left[\Phi - \Gamma(D\Gamma)^{-1}(D\Phi - \Lambda_d D)\right]\Delta\mathbf{x}_k - \boldsymbol{\zeta}_k \quad (27)$$

where $\boldsymbol{\zeta}_k = O(T^3)$ according to Theorem 4.

According to *Property 2*, the ultimate error bound on $\|\Delta\mathbf{x}_k\|$ will be one order higher than the bound on $\boldsymbol{\zeta}_k$ due to convolution. Since the bound on $\boldsymbol{\zeta}_k$ is $O(T^3)$, the ultimate bound on $\|\Delta\mathbf{x}_k\|$ is $O(T^2)$. Thus, the ultimate bound on the output tracking error is

$$\|\mathbf{e}_k\| \le \|C\|\|\Delta\mathbf{x}_k\| = O(T^2). \quad (28)$$

From the result (28) we conclude that the output feedback controller (10) and the reference model (26) are capable of maintaining a similar performance to the state feedback ISM with the added constraint of having to satisfy both (Φ, Γ, C) and (Φ, Γ, D) minimum-phase.

Reference model based on (Φ, Γ, D) *being minimum-phase*

In the case that it is only possible to satisfy (Φ, Γ, D) minimum-phase, a different reference model needs to be selected. For this new reference model, select the matrices $K_1 = \Gamma(D\Gamma)^{-1}D\Phi$ and $K_2 = \Gamma(D\Gamma)^{-1}$. Then the reference model (5) can be written as

$$\begin{aligned} \mathbf{x}_{r,k+1} &= \left[\Phi - \Gamma(D\Gamma)^{-1}D\Phi\right]\mathbf{x}_{r,k} + \Gamma(D\Gamma)^{-1}\mathbf{r}_{k+1} \\ \mathbf{y}_{r,k} &= D\mathbf{x}_{r,k} = \mathbf{r}_k. \end{aligned} \tag{29}$$

The matrix $\left[\Phi - \Gamma(D\Gamma)^{-1}C\Phi\right]$ is stable only if (Φ, Γ, D) is minimum-phase. Substituting the selected matrices K_1 and K_2 into (24), and using the property $[I - \Gamma(D\Gamma)^{-1}D]\Gamma = 0$, we have

$$\Delta\mathbf{x}_{k+1} = \left[\Phi - \Gamma(D\Gamma)^{-1}(D\Phi - \Lambda D)\right]\Delta\mathbf{x}_k - \boldsymbol{\zeta}_k. \tag{30}$$

We can see from (30) that the tracking error bound is only dependent on the disturbance estimtation $\boldsymbol{\zeta}_k$.

However, the disturbance observer requires (Φ, Γ, C) to be minimum-phase, hence is not implementable in this case. Without the disturbance estimator, noticing *Property 1*, (25) becomes

$$\begin{aligned} \boldsymbol{\zeta}_k &= \mathbf{d}_k - \Gamma(D\Gamma)^{-1}\mathbf{d}_{k-1} \\ &= \mathbf{d}_k - \mathbf{d}_{k-1} + [I - \Gamma(D\Gamma)^{-1}D](\Gamma\boldsymbol{\eta}_{k-1} + O(T^3)) \\ &= O(T^2) + O(T^3) = O(T^2). \end{aligned} \tag{31}$$

As the result, the closed-loop system is

$$\Delta\mathbf{x}_{k+1} = \left[\Phi - \Gamma(D\Gamma)^{-1}(D\Phi - \Lambda D)\right]\Delta\mathbf{x}_k + O(T^2) \tag{32}$$

By the *Proprty 2*, the ultimate bound on $\|\Delta\mathbf{x}_k\| = O(T)$, and therefore, the ultimate bound on the tracking error is

$$\|\mathbf{e}_k\| \leq \|D\|\|\Delta\mathbf{x}_k\| = O(T). \tag{33}$$

While this approach gives a less precise output tracking performance, it only requires (Φ, Γ, D) to be minimum-phase and can be used in the cases (Φ, Γ, C) is not minimum-phase.

4. SIMULATION EXAMPLE

Consider the following SISO system

$$A = \begin{bmatrix} 10 & 1 \\ -10 & -10 \end{bmatrix}, \quad B = \begin{bmatrix} 4 \\ 4.2 \end{bmatrix}, \quad C = [\,1 \;\; 0\,].$$

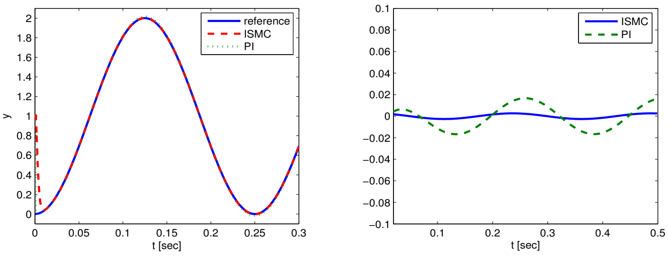
The sampled-data system obtained at a sampling time of $T = 1ms$ is

$$\Phi = \begin{bmatrix} 1.01 & -0.001 \\ -0.01 & 0.99 \end{bmatrix}, \quad \Gamma = \begin{bmatrix} 0.0040 \\ 0.0042 \end{bmatrix}, \quad C = [\,1 \;\; 0\,].$$

The zero of (Φ, Γ, C) is $z = 0.989$ while the zero of (Φ, Γ, D) is $z = 0.989$ as well and, therefore, the system satisfies both (Φ, Γ, C) and (Φ, Γ, D) minimum phase. Controller (10) and the reference with (Φ, Γ, C) minimum phase will be used as it achieves the best performance. The design parameter E is selected as 0.25 to ensure that the remaining pole is $z = 0.75$. The disturbance acting on the system will be non-smooth and given by

$$f = \begin{cases} 10 & if \;\; x_2 < 0 \\ 0 & if \;\; x_2 = 0 \\ -10 & if \;\; x_2 > 0 \end{cases} \tag{34}$$

The system is simulated under the influence of the disturbance f in (34) for tracking a reference $r_k = 1 + sin(8\pi kT - \pi/2)$. The result is compared to a PI controller. From Fig.1a and Fig.1b the performance of the output feedback approach can be seen. It is clear that the tracking error is as much better than the PI controller. Also note from Fig.2 the the control input at close to $t = 0$ is much larger for the PI control than the ISMC. Now, consider the following



(a) (b)

Fig. 1. Performance of ISMC and PI controllers

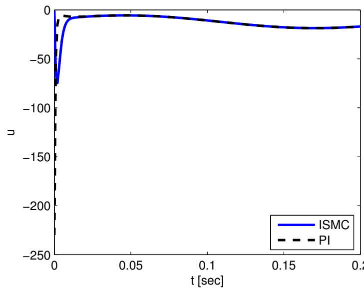


Fig. 2. Control input for ISMC and PI controllers

system

$$A = \begin{bmatrix} -60 & -10 \\ 10 & -10 \end{bmatrix}, \quad B = \begin{bmatrix} 4 \\ 4.2 \end{bmatrix}, \quad C = [\,1 \;\; 0\,].$$

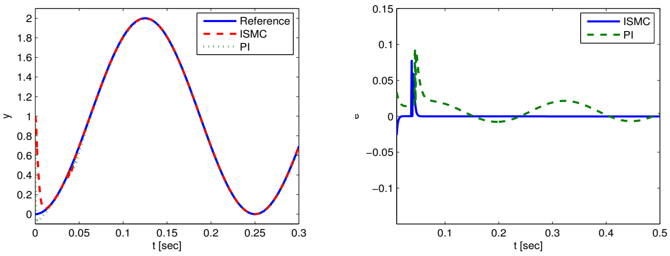
After sampling the system at $T = 1ms$ the system parameters become

$$\Phi = \begin{bmatrix} 0.9417 & -0.0097 \\ 0.0097 & 0.9900 \end{bmatrix}, \quad \Gamma = \begin{bmatrix} 0.0039 \\ 0.0042 \end{bmatrix}, \quad C = [\,1 \;\; 0\,].$$

For this system the zero of (Φ, Γ, C) is $z = 1.001$ where as the zero of (Φ, Γ, D) is $z = 0.998$ and, therefore, the output feedback approach with the reference model with (Φ, Γ, D) minimum phase is the only option. Using the same disturbance and reference signals, the system is simulated. As it can be seen from Fig.3a and Fig3b, the performance is quite good and better than that of a PI controller. The same can be said of the control inputs as seen from Fig.4

5. CONCLUSION

This work presents a form of the discrete-time integral sliding control design for sampled-data systems with output tracking. Proper disturbance and state observers were presented. The closed-loop stability of the system was not dependent on either observer and is designed seperately. It was shown that the maximum bound on the tracking



(a) (b)

Fig. 3. Performance ISMC and PI controllers

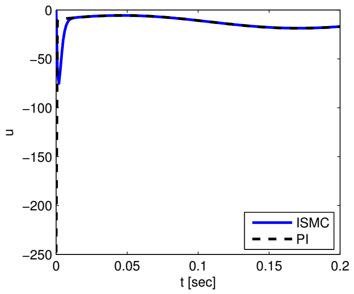


Fig. 4. Control input for ISMC and PI controllers

error is $O(T^2)$ at steady state. It was also shown that even though the state observer produced $O(T)$ estimation error, the ISM state observer approach could still produce $O(T^2)$ tracking error. Simulation comparison with a PI controller suggests the effectiveness of the proposed method.

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Appendix A. PROOF OF LEMMA 1

If the matrices Φ, Γ and D are partitioned as shown

$$\Phi = \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix},\ D = \begin{bmatrix} D_1 & D_2 \end{bmatrix}, \text{and } \Gamma = \begin{bmatrix} \Gamma_1 \\ \Gamma_2 \end{bmatrix}$$

where $(\Phi_{11}, D_1, \Gamma_1) \in \Re^{m\times m}$, $(\Phi_{12}, D_2) \in \Re^{m\times n-m}$, $(\Phi_{21}, \Gamma_2) \in \Re^{n-m\times m}$ and $\Phi_{22} \in \Re^{n-m\times n-m}$. The eigenvalues of $\bar{\Phi}$ are found from

$$\det\left[\lambda I_n - \Phi + \Gamma(D\Gamma)^{-1}(D\Phi - \Lambda D)\right] = 0 \qquad \text{(A.1)}$$

or

$$\det\begin{bmatrix} \lambda I - \Phi_{11} + \Gamma_1'\left(D_1' - \Lambda D_1\right) & -\Phi_{12} + \Gamma_1'\left(D_2' - \Lambda D_2\right) \\ -\Phi_{21} + \Gamma_2'\left(D_1' - \Lambda D_1\right) & \lambda I - \Phi_{22} + \Gamma_2'\left(D_2' - \Lambda D_2\right) \end{bmatrix} = 0 \qquad \text{(A.2)}$$

where $\Gamma_1' = \Gamma_1(D\Gamma)^{-1}$, $\Gamma_2' = \Gamma_2(D\Gamma)^{-1}$, $D_1' = D\begin{bmatrix}\Phi_{11} & \Phi_{21}\end{bmatrix}^T$ and $D_2' = D\begin{bmatrix}\Phi_{12} & \Phi_{22}\end{bmatrix}^T$. If the top row is premultiplied with D_1 and the bottom row is premultiplied with D_2 and the the results summed and used as the new top row, using the fact that $D_1\Gamma_1 + D_2\Gamma_2 = D\Gamma$ the following is obtained

$$\det\begin{bmatrix} (\lambda I_m - \Lambda)D_1 & (\lambda I_m - \Lambda)D_2 \\ -\Phi_{21} + \Gamma_2'\left(D_1' - \Lambda D_1\right) & \lambda I_{n-m} - \Phi_{22} + \Gamma_2'\left(D_2' - \Lambda D_2\right) \end{bmatrix} = 0$$

factoring the term $(\lambda I_m - \Lambda)$ and premultipying the top row with $\Gamma_2(D\Gamma)^{-1}\Lambda$ and adding to the bottom row yields

$$\det(\lambda I_m - \Lambda)\det\begin{bmatrix} D_1 & D_2 \\ -\Phi_{21} + \Gamma_2' D_1' & \lambda I_{n-m} - \Phi_{22} + \Gamma_2' D_2' \end{bmatrix} = 0. \qquad \text{(A.3)}$$

Thus, we can conclude that m eigenvalues of the matrix $\left[\Phi - \Gamma(D\Gamma)^{-1}(D\Phi - \Lambda D)\right]$ are the eigenvlaues of Λ.

Note that the matrix

$$\begin{bmatrix} D_1 & D_2 \\ -\Phi_{21} + \Gamma_2' D_1' & \lambda I_{n-m} - \Phi_{22} + \Gamma_2' D_2' \end{bmatrix}$$

corresponds to a design of $\boldsymbol{\sigma}_{k+1} = D\mathbf{x}_k = 0$ whose closed-loop is governed by the matrix $[\Phi - \Gamma(D\Gamma)D\Phi]$. It is well known that $[\Phi - \Gamma(D\Gamma)D\Phi]$, thus, we conclude that the eigenvalues of $\left[\Phi - \Gamma(D\Gamma)^{-1}(D\Phi - \Lambda D)\right]$ are the eigenvalues of Λ and the eigenvlaues of $[\Phi - \Gamma(D\Gamma)D\Phi]$.