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Remarks on ISS and Integral-ISS Stabilization with Positive Controls

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Abstract: We consider nonlinear systems with both control and disturbance inputs. The main problem addressed is the design of state feedback control laws, achieving ISS and integral-ISS disturbance attenuation, with restricted control under the assumption that an appropriate control Lyapunov function is known. Our results generalize the previous results on ISS and integral-ISS stabilization to the restricted cases when controls are only allowed to take positive or bounded positive values.

1. INTRODUCTION

Since formulated in the seminal paper Sontag [1989a], the notion of input-to-state stability (ISS) has found wide applications, especially in the area of feedback design and analysis to achieve ISS disturbance attenuation for closed-loop systems, see for instance, the textbooks Krstić et al. [1995], Khalil [2002], Isidori [1999] and references therein. One of the variations of ISS, the notion of integral-ISS, was introduced and studied in Sontag [1998], Angeli et al. [2000a,b]. Applications of the integral-ISS property can be found in Angeli [1999], Arcak et al. [2002], Jiang et al. [2004], Liberzon et al. [2002]. As remarked in Sontag [1998] and Angeli et al. [2000a], the integral-ISS property is strictly weaker than the ISS property in the sense that the ISS property implies the integral-ISS property but not vice-versa. As a consequence, it is much more feasible to achieve integral-ISS property than to achieve ISS property, especially in the case of restricted controls.

Most of the past work on ISS and integral-ISS stabilization was carried out for systems with controls that can take arbitrary values. In practice, however, controls are often constrained. It is thus interesting to investigate feedback design and analysis for different types of stability properties with restricted controls. A primary concern of our work is about ISS and integral-ISS stabilization by using controls that only take positive values. Systems with positive controls have appeared in applications including physical systems, ecological systems, and human behavior patterns, see for instance, DeAngelis et al. [1986], Mailleret et al. [2004], Mulsum [1968], Benvenuti et al. [2003] and references there. Very often the controllers of such systems only interact positively with variables that have intrinsically a constant sign, for instance, flight attitudes, level of liquid in tanks, biological variables in an ecological system. There is a rich literature in the study of systems with positive controls, in particular in the linear case, see for instance, de Leenheer and Aeyels [2001], Smimov [1999], Rami and Tadio [2005], and Frias et al. [2005]. Some recent work on nonlinear systems with positive controls includes Bastin and Praly [1999] (on dissipative mass-balance systems) and Kaliora and Astolfi [2002] (on a class of nonlinear cascades in feedforward form). For an excellent collection of papers on related topics about positive systems and monotone systems, see Benvenuti et al. [2003]. The results to be developed in our work will be based on the approach of control Lyapunov functions.

Control Lyapunov functions (CLF) have been used to design control laws for various class of nonlinear controllers. One notable example is the universal formula proposed in the work Sontag [1989b]. It is also shown in Praly et al. [1991] that CLF can be used to design adaptive stabilizers for linearly parameterized nonlinear systems. The results in Sontag [1989b] was extended in Lin and Sontag [1991] to deal with bounded controls, and in Lin and Sontag [1995] to deal with positive (bounded and unbounded) controls. In the work Malisoff and Sontag [2000], the authors considered a stabilization problem with controls taking bounded values in different norms. In the direction of disturbance attenuation, the work in Lin and Sontag [1991] was extended in Liberzon [1999] to achieve ISS and integral-ISS stabilization with bounded controls.

We will consider in this work the integral-ISS and ISS stabilization by using positive controls or bounded positive controls, assuming the existence of an appropriate CLF. While the design of smooth feedback laws developed in Liberzon [1999] used a patching argument (by an abstract function based on a partition of unity), we are able to develop the feedback laws based on the universal formula without the patching argument.

This paper is organized as follows. In Section 2, we review the notions of ISS, integral-ISS, control Lyapunov functions, and other related notions that will be used in the work. In Section 3 we discuss notions on control Lyapunov functions with positive and bounded positive controls, and we show how to construct feedback laws to achieve the integral-ISS property based on given integral-ISS control Lyapunov functions. We also provide some sufficient conditions for a control Lyapunov function for the zero-disturbance system to be an integral-ISS control Lyapunov function. In Section 4 we present some concluding remarks.

2. PRELIMINARIES

Throughout this work, we use $|\xi|$ to denote the Euclidean norm for $\xi \in \mathbb{R}^n$, and use $\|u\|$ to denote the L_∞ norm of u on $[0, \infty)$. A function $\varphi : \mathbb{R}^n \to \mathbb{R}$ is called almost smooth if it is C^∞ away from 0, and continuous everywhere.

A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is *of class* $\mathcal{K}$ if it is continuous, positive definite, and strictly increasing; and is *of class* $\mathcal{K}_\infty$ if it is also unbounded. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be of *class* $\mathcal{KL}$ if for each fixed $t \geq 0$, $\beta(\cdot, t)$ is of class $\mathcal{K}$, and for each fixed $s \geq 0$, $\beta(s, t)$ decreases to 0 as $t \to \infty$.

2.1 ISS and Integral-ISS

In this section we briefly review the notions of ISS and integral-ISS. For more detailed discussions, we refer the reader to Sontag [1989a], Sontag and Wang [1995a], Sontag [1998] and Angeli et al. [2000a].

Consider a nonlinear system as follows:

$$\dot{x} = F(x, w), \tag{1}$$

where $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^k$ is a C^∞ map, and $F(0, 0) = 0$. For each t, $x(t)$ takes values in $\mathbb{R}^n$. The disturbance w is a measurable, locally essentially bounded function that takes values in $\mathbb{R}^d$. We use $x(\cdot, x_0, w)$ to denote the solution of the system with the initial condition $x(0) = x_0$ and the disturbance w.

System (1) is *input-to-state stable* (ISS) if there are $\beta \in \mathcal{KL}$ and $\kappa \in \mathcal{K}$ such that

$$|x(t, x_0, w)| \leq \beta(|x_0|, t) + \kappa(\|w\|) \quad \forall\, t \geq 0, \tag{2}$$

for all x_0 and all w.

System (1) is *integral-input-to-state stable* (integral-ISS) if there exist some functions $\sigma, \kappa \in \mathcal{K}_\infty$ and $\beta \in \mathcal{KL}$ such that

$$\sigma(|x(t, x_0, w)|) \leq \beta(|x_0|, t) + \int_0^t \kappa(|w(s)|)\, ds \quad \forall\, t \geq 0, \tag{3}$$

for all x_0 and all w.

A C^∞ function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is called an *integral-ISS-Lyapunov function* of (1) if it is proper, positive definite, and for some positive definite function α and some function $\gamma \in \mathcal{K}$, it holds that

$$DV(x) f(x, w) \leq -\alpha(|x|) + \gamma(|w|) \quad \forall\, x, w. \tag{4}$$

If the function α in (4) can be chosen to be a $\mathcal{K}_\infty$ function, then V is called an *ISS-Lyapunov function* of (1).

Clearly, an ISS-Lyapunov function is always an integral-ISS Lyapunov function. See Sontag and Wang [1995a] and Angeli et al. [2000a] for the following results:

Theorem 1. Consider a system as in (1).

(1) The system is ISS if and only if it admits an ISS-Lyapunov function.
(2) The system is integral-ISS if and only if it admits an integral-ISS-Lyapunov function. □

Statement (2) of Theorem 1 can be modified to:

Lemma 2.1. System (1) is integral-ISS if and only if it admits a smooth, proper, and positive definite Lyapunov function V such that

$$DV(x) f(x, w) < \gamma(|w|) \qquad \forall\, x \neq 0,\ \forall\, w. \tag{5}$$

Proof. Assume the system is integral-ISS. Then it admits an integral-ISS Lyapunov function that satisfies (4) which trivially implies (5).

On the other hand, suppose (1) admits a Lyapunov function satisfying (5). Then the disturbance-free system $\dot{x} = f(x, 0)$ is globally asymptotically stable. By Theorem 1 of Angeli et al. [2000a], the closed-loop system is integral-ISS. ∎

2.2 CLF and Universal Formulas

Consider a system as follows:

$$\dot{x} = f(x) + g(x)u, \tag{6}$$

where $f, g : \mathbb{R}^n \to \mathbb{R}^n$ are C^∞ maps, and $f(0) = 0$. For each t, $x(t)$ takes values in $\mathbb{R}^n$. For simplicity, we consider the case of single input. We assume that controls are restricted to take values in some subset $\mathcal{U}$ of $\mathbb{R}$:

$$u(t) \in \mathcal{U} \subseteq \mathbb{R}.$$

Different subsets U impose different constraints for feedback design problems. For the preliminary discussions, below $\mathcal{U}$ is only assumed to be a subset of $\mathbb{R}$. Specific structures on the set $\mathcal{U}$ will be posted later.

We say that a system as in (6) is stabilizable if there is a feedback function $u = k(x)$ which is almost smooth, satisfying the property that $k(x) \in \mathcal{U}$, that stabilizes the closed-loop system in the sense that the system

$$\dot{x} = f(x) + g(x)k(x).$$

is globally asymptotically stable.

Let $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ be proper, positive definite, and almost smooth. We say that V is a *control Lyapunov function* (CLF) for (6) with the control value set $\mathcal{U}$ if V satisfies the following:

$$\inf_{u \in \mathcal{U}} \{DV(x) f(x) + DV(x) g(x) u\} < 0 \quad \forall\, x \neq 0 \tag{7}$$

Let $a(x) = DV(x) f(x)$, $b(x) = DV(x) g(x)$. Then (7) can be rewritten as

$$\inf_{u \in \mathcal{U}} \{a(x) + b(x) u\} < 0 \quad \forall\, x \neq 0. \tag{8}$$

We say that a CLF V satisfies the *small control property* (SCP) if for every $\varepsilon > 0$, there exists some $\delta > 0$ such that for every $0 < |x| < \delta$ there exists some $|\mu| < \varepsilon$ such that

$$a(x) + b(x)\mu < 0.$$

Instead of saying V satisfies the small control property, we will sometimes say that the pair $(a(x), b(x))$ satisfies the small control property.

The following result can be found in Sontag [1989b] for the case when $\mathcal{U} = \mathcal{U}_0 := \mathbb{R}$; in Lin and Sontag [1991] for the case when $\mathcal{U} = \mathcal{U}_1 := (-1, 1)$; and in Lin and Sontag [1995] for the cases when $\mathcal{U} = \mathcal{U}_2 := (0, 1)$ and when $\mathcal{U} = \mathcal{U}_3 := (0, \infty)$.

Theorem 2. Consider system (6) with the control value set $\mathcal{U}_i$ for $i = 0, 1, 2, 3$. The system is stabilizable by an almost smooth feedback $u = k_i(x)$ if and only if the system admits a CLF satisfying (8) with $\mathcal{U} = \mathcal{U}_i$ and the SCP property. ∎

To develop our results for disturbance attenuation, we provide below the explicit feedback laws associated with Theorem 2 for the cases when $\mathcal{U} = \mathcal{U}_2$ or $\mathcal{U} = \mathcal{U}_3$. For

$\mathcal{U}_2$, the stabilizing feedback is given by $u = k_2(x) = \kappa_2(a(x), b(x))$, where

$$\kappa_2(a, b) = \tilde{\kappa}_2(r, \theta - (3/2)\pi),$$

and where (r, θ) $(0 < \theta < 2\pi)$ is the polar coordinate representation of (a, b), and

$$\tilde{\kappa}_2(r, \theta) = \frac{\chi(r, \theta) - \chi(r, -\pi)}{\chi(r, \pi/4) - \chi(r, -\pi)},$$

with

$$\chi(r, \theta) = \left(\frac{2}{\pi} \arctan\left(\frac{\theta}{r}\right) + 1\right) \theta. \tag{9}$$

For the case of $\mathcal{U}_3$, the stabilizing feedback law is given by $u = k_3(x) = \kappa_3(a(x), b(x)) = \tilde{\kappa}_3(r, \theta)$, where

$$\tilde{\kappa}_3(r, \theta) = \frac{3\pi}{2(\pi - 2\theta)} \left(\chi(r, \theta) - \chi(r, -\pi)\right), \tag{10}$$

where χ is as defined in (9).

3. INTEGRAL-ISS STABILIZATION WITH POSITIVE AND BOUNDED POSITIVE CONTROLS

In this section we consider the disturbance attenuation for systems of the following type:

$$\dot{x} = f(x) + g(x)u + p(x)w, \tag{11}$$

where u is the control input which takes values in some $\mathcal{U}$, and w denotes the disturbances.

The work on the problem of integral-ISS stabilization for the case of $\mathcal{U} = \mathcal{U}_1$ was developed in Liberzon [1999]. Below we consider the problem for the cases when $\mathcal{U} = \mathcal{U}_2$ and $\mathcal{U} = \mathcal{U}_3$.

Assume that there is a feedback $u = k(x) \in \mathcal{U}$ that renders the closed-loop system to be integral-ISS, there is a smooth, positive definite, and proper Lyapunov function V such that

$$a(x) + b(x)k(x) + q(x)w \le -\alpha(|x|) + \rho(|w|)$$

where

$$\begin{aligned} &a(x) = DV(x)f(x), b(x) = DV(x)g(x), \\ &q(x) = DV(x)p(x), \end{aligned}$$

$\alpha(\cdot)$ is a positive definite function, and $\rho(\cdot)$ is of class $\mathcal{K}_\infty$. This implies that

$$\inf_{u\in\mathcal{U}} \{a(x) + b(x)u + q(x)w\} \le -\alpha(|x|) + \rho(|w|). \tag{12}$$

For a system as in (11) with the control value set $\mathcal{U}$, we say that a smooth, positive definite, and proper function V is an integral-ISS CLF if V satisfies (12). We say that V satisfies the SCP property if the pair $(a(x), b(x))$ satisfies the SCP property. It follows from Theorem 1 that if a system admits an integral-ISS stabilizing feedback, then it admits an integral-ISS CLF. Assume that the system (11) admits an integral-ISS CLF satisfying (12), and hence,

$$\inf_{u\in\mathcal{U}} \{a(x) + b(x)u + q(x)w - \chi(|w|\} \le -\alpha(|x|),$$

for all $\chi \in \mathcal{K}_\infty$ satisfying $\chi(r) \ge \rho(r)$. Let

$$\omega(x) = \max_{w} \{a(x) + q(x)w - \chi(|w|)\}.$$

Observe that one can always choose χ large enough so that

$$\omega(x) = \max_{|w|\le\pi(|x|)} \{a(x) + q(x)w - \chi(|w|)\} \tag{13}$$

for some function $\pi \in \mathcal{K}_\infty$. For instance, one can simply let χ be a function such that

$$\chi(r) \ge \max\{\rho(r), q_0(r)r\}$$

with $\pi(r) = r$, where $q_0 \in \mathcal{K}_\infty$ is such a function that $|q(x)| \le q_0(|x|)$. Hence

$$\inf_{u\in\mathcal{U}} \{\omega(x) + b(x)u\} \le -\alpha(|x|).$$

Modifying $\omega(x)$ if necessary so that one may assume that ω is a smooth function and

$$\inf_{u\in\mathcal{U}} \{\omega(x) + b(x)u\} \le -\frac{\alpha(|x|)}{2}. \tag{14}$$

One may also choose $\pi(|x|)$ as in (13) small enough around $x = 0$ such that

$$|q(x)|\, \pi(|x|) \le \psi(|x|)(|a(x)| + |b(x)|) \tag{15}$$

in a neighborhood of 0 for some smooth function $\psi \in \mathcal{K}$ (note that the function $|a(x)| + |b(x)|$ is positive definite).

Lemma 3.1. Assume the pair $(a(x), b(x))$ satisfies the SCP property. Suppose π is chosen so that (15) holds. Then the pair $(\omega(x), b(x))$ satisfies the SCP property.

Proof. First note that the SCP property is equivalent to the property that for any $\varepsilon > 0$, there exists some $\delta > 0$ such that

$$a(x) - \varepsilon\, |b(x)| < 0 \qquad \forall\, 0 < |x| < \delta. \tag{16}$$

Let $\varepsilon > 0$. Applying (16) to $\varepsilon/4$, one sees that there exists some $\delta > 0$ such that for every $0 < |x| < \delta$,

$$a(x) - \frac{\varepsilon}{4}\, |b(x)| < 0.$$

Without loss of generality, we assume that $\varepsilon < 1$. Shrinking δ if necessary, one may assume that $\psi(\delta) < \varepsilon/4$ (where ψ is as in (15)), and hence, for all $0 < |x| < \delta$,

$$|q(x)|\, \pi(|x|) \le \frac{\varepsilon}{2}(|a(x)| + |b(x)|). \tag{17}$$

Let $0 < |x| < \delta$. If $a(x) \le 0$, then

$$\begin{aligned} &a(x) + \frac{\varepsilon}{2}(|a(x)| + |b(x)|) - \varepsilon\, |b(x)| \\ &\le \frac{a(x)}{2} - \frac{\varepsilon}{2}\, |b(x)| < 0. \end{aligned}$$

If $a(x) > 0$, then

$$\begin{aligned} &a(x) + \frac{\varepsilon}{2}(|a(x)| + |b(x)|) - \varepsilon\, |b(x)| \\ &\le 2a(x) - \frac{\varepsilon\, |b(x)|}{2} < 0. \end{aligned}$$

In both cases, one has

$$a(x) + \frac{\varepsilon}{2}(|a(x)| + |b(x)|) - \varepsilon\, |b(x)| < 0 \tag{18}$$

for all $0 < |x| < \delta$. Note that

$$\begin{aligned} \omega(x) &= \max_{|w|\le\pi(|x|)} \{a(x) + q(x)w - \chi(|w|)\} \\ &\le a(x) + |q(x)|\, \pi(|x|). \end{aligned}$$

Hence, it follows from (17) and (18) that

$$\omega(x) - |b(x)|\,\varepsilon < 0 \qquad \forall\, 0 < |x| < \delta.$$

This shows that the pair $(\omega(x), b(x))$ satisfies the SCP property. ∎

Theorem 3. Consider a system as in (11) with either $\mathcal{U} = \mathcal{U}_2$ or $\mathcal{U} = \mathcal{U}_3$. If V is an integral-ISS CLF for the system (11) satisfying the small control property, then there is a feedback law which is almost smooth for the system so that the closed-loop system is integral-ISS.

Proof. Assuem V is an integral-ISS CLF for the system that satisfies (12). Choose the function π so that the resulted $\omega(x)$ is smooth, satisfies (14), and the pair $(\omega(x), b(x))$ satisifes the SCP property. Applying Theorem 2 for the cases when $\mathcal{U} = \mathcal{U}_2$ and $\mathcal{U} = \mathcal{U}_3$ respectively, one sees that there exists an almost smooth feedback law $u = k_i(x)$ such that $k_i(x) \in \mathcal{U}_i$ for each $i = 2, 3$ and that

$$\omega(x) + b(x)k_i(x) \le -\frac{\alpha(x)}{2}.$$

Hence, along the trajectories of the closed-loop system, one has

$$\frac{d}{dt}V(x(t)) = a(x(t)) + b(x(t))k_i(x(t)) + q(x(t))w(t)$$
$$\le \omega(x(t)) + b(x(t))k_i(x(t)) + \chi(|w|) < \chi(|w|)$$

for all $x \neq 0$. By Lemma 2.1, the closed-loop system is integral-ISS. ∎

3.1 Remarks on CLF's

In this section we consider the question when a CLF for the 0-disturbance system

$$\dot{x} = f(x) + g(x)u \tag{19}$$

of (11) is an integral-ISS Lyapunov function for (11).

Let V be a CLF for (19). Then

$$\inf_{\mathcal{U}}\{a(x) + b(x)u\} \le -\alpha(|x|).$$

When applied to system (11), one has

$$\inf_{\mathcal{U}}\{a(x) + b(x)u + q(x)w\} \le -\alpha(|x|) + q(x)w.$$

The next result applies to a system as in (11) for any given control value set $\mathcal{U}$:

Proposition 3.2. Consider system (11). Suppose V is a CLF for the corresponding 0-disturbance system (19). Assume that there is some function $\nu \in \mathcal{K}$ such that

(1) $\displaystyle\int_0^\infty \frac{1}{1+\nu(s)}\,ds = \infty$; and

(2) $|q(x)| \le \nu(V(x))$.

Then the function W given by $W = \rho \circ V$ is an integral-ISS function, where $\rho(s) = \int_0^s \frac{1}{1+\nu(s)}\,ds$, for (11). Further more, W satisfies the SCP condition if and only if V satisfies the SCP condition.

Proof. For the function W given by $W = \rho \circ V$, it holds that

$$DW(x)f(x) + DW(x)g(x)u + DW(x)p(x)w$$
$$= \frac{a(x)}{1+\nu(V(x))} + \frac{b(x)}{1+\nu(V(x))}u + \frac{q(x)}{1+\nu(V(x))}w$$
$$\le \tilde{a}(x) + \tilde{b}(x)u + |w|\,,$$

where

$$\tilde{a}(x) = \frac{a(x)}{1+\nu(V(x))}, \tilde{b}(x) = \frac{b(x)}{1+\nu(V(x))}.$$

Note then that

$$\inf_{u\in\mathcal{U}}\{\tilde{a}(x) + \tilde{b}(x)u + |w|\} \le -\frac{\alpha(|x|)}{1+\rho(V(x))} + |w|\,. \tag{20}$$

From this one can see that W is an integral-ISS CLF for (11). It should be clear that $(a(x), b(x))$ satisfies the SCP condition if and only if $(\tilde{a}(x), \tilde{b}(x))$ satisfies the SCP condition. ∎

It can easily be observed that for the Lyapunov function W, one can let $\omega(x) = \tilde{a}(x)$. Hence, if one applies the feedback laws $k_i(x)$ $(0 \le i \le 3)$ by using W as the CLF to (11), then the closed-loop system is integral-ISS.

Corollary 3.3. Consider a system as in (11) with $\mathcal{U} = \mathcal{U}_i$ $(0 \le i \le 3)$. Assume that the 0-disturbance system (19) admits a CLF for which b satisfies conditions (1) and (2) in Proposition 3.2. Then the system (11) is integral-ISS stabilized by $u = k_i(x)$ with W as the CLF. □

Below we consider a system as (11) with $\mathcal{U} = \mathcal{U}_3$ $(= (0, \infty))$. Let the set $\mathcal{B}_3 = \{x : b(x) \ge 0\}$. Assume V is a CLF for the 0-disturbance system (19), and so

$$\inf_{u>0}\{a(x) + b(x)u\} \le -\alpha(|x|). \tag{21}$$

This implies that

$$x \neq 0, x \in \mathcal{B}_3 \;\Rightarrow\; a(x) < 0.$$

Assume that

$$\lim_{\substack{x\to\infty \\ x\in\mathcal{B}_3}} \frac{-a(x)}{|q(x)|} = +\infty \tag{22}$$

where we make the convention that $\frac{-a(x)}{q(x)} = \infty$ at the points where $q(x) = 0$. (The meaning of the limit along $\mathcal{B}_3$ is that values of the function is large as long as $|\xi|$ is large and $x \in \mathcal{B}_3$. If $\mathcal{B}_3$ happens to be bounded, this condition is vacuous.)

By the proof of Theorem 2 in the work Sontag and Wang [1995b], one sees that there exists some smooth function $\psi \in \mathcal{K}_\infty$ such that

$$a(x) + \psi(|x|)\,|q(x)| < -\alpha(|x|) \quad \forall\, x \in \mathcal{B}_3,$$

that is,

$$b(x) \ge 0 \;\Rightarrow a(x) + \psi(|x|)\,|q(x)| \le -\alpha(|x|). \tag{23}$$

Let $\chi(r) = \max_{|x|=r}\{\psi(|x|)q(|x|)\}$. Then

$$a(x) + q(x)w \le a(x) + \psi(|x|)\,|q(x)| + \chi(r),$$

and hence,

$$\inf_{u>0}\{a(x) + q(x)w + b(x)u - \chi(|w|)\}$$
$$\le \inf_{u>0}\{a(x) + \psi(|x|)\,|q(x)| + b(x)u - \chi(|w|)\}.$$

By (23), one sees that

$$\inf_{u>0}\{a(x) + q(x)w + b(x)u - \chi(|w|)\} \le -\alpha(|x|).$$

This shows that V is an integral-ISS CLF for the system (11).

Proposition 3.4. Consider the system (11) with $\mathcal{U} = (0, \infty)$. Assume that V is a CLF for the 0-disturbance system (19). If property (22) holds, then V is an integral-ISS CLF for (11).

If in addition, the function α appeared in (21) is of class $\mathcal{K}_\infty$, then V is an ISS CLF for (11). □

It is worth to note that to have $\alpha \in \mathcal{K}_\infty$ is equivalent to have the property that

$$\lim_{\substack{x\to\infty \\ x\in\mathcal{B}_3}} (-a(x)) = +\infty. \tag{24}$$

Observe that the same argument can be applied to the case of $\mathcal{U} = \mathcal{U}_0\ (= \mathbb{R})$ with $\mathcal{B}_3$ replaced by $\mathcal{B}_0 := \{x : \ b(x) = 0\}$ to establish the following:

Proposition 3.5. Consider the system (11) with $\mathcal{U} = \mathbb{R}$. Assume that V is a CLF for the 0-disturbance system (19). If the following holds:

$$\lim_{\substack{x\to\infty \\ x\in\mathcal{B}_0}} \frac{-a(x)}{|q(x)|} = +\infty, \tag{25}$$

then V is an integral-ISS CLF function for (11).

If in addition, the function α appeared in (21) is of class $\mathcal{K}_\infty$, then V is an ISS CLF for (11). □

Similarly, to have $\alpha \in \mathcal{K}_\infty$ is equivalent to the property

$$\lim_{\substack{x\to\infty \\ x\in\mathcal{B}_0}} (-a(x)) = +\infty. \tag{26}$$

Example 3.6. Consider the system

$$\dot{x} = x^2 - u + xw, \quad u \in (0, \infty). \tag{27}$$

A CLF for the zero-disturbance system $\dot{x} = x^2 - u$ is $V = x^2/2$ with

$$a(x) = x^3, \quad b(x) = -x.$$

Note that since whenever $b(x) > 0$, $a(x) = x^3 < 0$, hence,

$$\inf_{u>0} \{a(x) + b(x)u\} \le |x|^3.$$

For system (27), one has $q(x) = x^2$, and condition (22) holds, and hence, V is also an integral-ISS CLF for (27). In fact, since (24) holds, V is an ISS CLF for (27). To find an ISS stabilizing feedback law, let

$$\begin{aligned}\omega(x) &= \max_{|w|\le|x|/2} \{a(x) + q(x)w\} \\ &= a(x) + \frac{|q(x)x|}{2} = x^3 + \frac{|x^3|}{2},\end{aligned}$$

and

$$\dot{V}(x) \le \omega(x) - xu + |w|^3 .$$

Observe that the pair $(\omega(x), b(x))$ satisfies the SCP property.

To find the polar coordinate representation (r, θ) for $(\omega(x), b(x))$, one has

$$r = \sqrt{\omega^2(x) + b^2(x)} = \begin{cases} |x|\sqrt{\frac{9}{4}x^4 + 1} & \text{if } x \ge 0 \\ |x|\sqrt{\frac{1}{4}x^4 + 1} & \text{if } x < 0 \end{cases}$$

and

$$\theta = \arctan\frac{b(x)}{\omega(x)} = \begin{cases} 2\pi - \arctan\left(\frac{2}{3x^2}\right) & \text{if } x > 0, \\ \pi - \arctan\left(\frac{2}{x^2}\right) & \text{if } x < 0, \end{cases}$$

(where $\arctan s \in (0, \pi/2)$ for $s > 0$). Applying the feedback law $u = k_3(x)$ given by $k_3(x) = \kappa_3(\omega(x), b(x))$, one obtains an almost smooth feedback law that renders the closed-loop system the ISS property. □

3.2 Integral-ISS Stabilization for the Unrestricted Case

In this section, we make a brief remark on the integral-ISS stabilizability for the case when $U = \mathbb{R}^m$ for some $m > 0$ under a matching condition. For a system as in the following:

$$\dot{x} = f(x) + G(x)u, \tag{28}$$

where f and G are smooth maps, u takes values in $\mathbb{R}^m$, consider the problem when there are disturbances w in the control channel when a feedback law is applied. This leads to the closed-loop system of the following type:

$$\begin{aligned}\dot{x} &= f(x) + G(x)(k(x) + w) \\ &= f(x) + G(x)k(x) + G(x)w.\end{aligned} \tag{29}$$

In Sontag [1989a], it was shown that if one can stablize a system as in (28) in the case of zero-disturbance by a feedback law $u = k_0(x)$ (in particular, by the universal formula if a CLF is given), then one can modify the feedback law to $u = k_1(x)$ to yield the ISS property for the closed-loop system.

Below we show that if the stabilizing feedback law $k_0(x)$ is given by the universal formula, then the closed-loop is automatically integral-ISS.

Let V be a CLF for the 0-disturbance system. The universal formula $u = k_0(x) = \kappa_0(a(x), B(x))$ is given by

$$\kappa_0(a, B) = \begin{cases} -\dfrac{a + \sqrt{a^2 + |B(x)|^4}}{|B|^2} B(x)^T, & \text{if } B \ne 0, \\ 0, & \text{if } B = 0, \end{cases} \tag{30}$$

where $a(x) = DV(x)f(x), B(x) = DV(x)G(x)$.

Under the feedback law $u = \kappa_0(a(x), B(x))$, it holds along the trajectories of (29) that

$$\begin{aligned}\frac{d}{dt}V(x(t)) &= a(x) + B(x)\kappa(a(x), B(x)) + B(x)w \\ &= -\sqrt{a^2(x) + |B(x)|^4} + B(x)w \\ &\le -\sqrt{a^2(x) + |B(x)|^4} + \frac{|B(x)|^2}{2} + \frac{w^2}{2} \\ &\le -\frac{\sqrt{a^2(x) + |B(x)|^4}}{2} + \frac{w^2}{2}\end{aligned}$$

where we have applied the inequality $2|c_1c_2| \le |c_1|^2 + {c_2}^2$ to the term $B(x)w$. Since V is a CLF, the function $\sqrt{a^2(x) + |B|^4(x)}$ is positive definite. This implies that V is an integral-ISS-Lyapunov function for the closed-loop system with w as input. Hence, we have shown the following:

Proposition 3.7. Let V be a CLF for the disturbance-free system of (28) (that is, $\dot{x} = f(x) + G(x)u$). Under the feedback law given by the universal formula (30), the closed-loop system of (29) is integral-ISS with w as input.

It can be seen that the main idea in the simple proof of Proposition 3.7 is the stability margin resulted from the universal formula can be used to dominate the term $B(x)w$. This idea, however, fails to work in the case of positive controls. Consider, for instance the system $\dot{x} = -x + x^3 u$ with the control value set $\mathcal{U}_3$. The system is stabilized by the feedback law $u = 0$. But there is no positive feedback law $u = k(x)$ for which the closed-loop system

$$\dot{x} = -x + x^3 k(x) + x^3 w$$

is integral-ISS.

Remark 3.8. It can be seen from the proof that if $a^2(x) + |B(x)|^4$ is a proper function, then the closed-loop system of (29) is ISS under the feedback law $u = \kappa(a(x), B(x))$. This is because when $a^2(x)+|B(x)|^4$ is proper, the function V becomes an ISS-Lyapunov function for the closed-loop system. □

4. CONCLUSIONS

In this work we consider the problem of disturbance attenuation for the cases when controls are restricted to take positive values or bounded positive values. Our results provide explicit feedback laws to achieve the integral-ISS property for the closed-loop system based on control Lyapunov functions. We also provided some sufficient conditions for a CLF for the 0-disturbance system to be an integral-ISS CLF for the system with disturbances. Our future work will be directed at the stabilization problem for classes of feedforward and feedback form systems with restricted controls.

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