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A Finite Step Scheme for General Near-Optimal Control –the Deterministic Case

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Abstract: This paper reports a finite step scheme for the computation of near-optimal control of general nonlinear control systems. It is developed based on the first order estimation of the system and the associated adjoint variational equation. The search is an extension of the standard steepest descent method to the functional case based on the variation of the Hamiltonian function H. Convergence analysis are included to show this scheme does converge to a desired admissible control in finite steps. Consistency of the approximation of the associated adjoint equation is also discussed. A linear quadratic control example and some numerical simulations are also included for illustration purpose.

1. INTRODUCTION

There have been quite a lot of research conducted to address the computation of optimal control of deterministic systems since 1960. Among them, the well-known Maximum Principle and dynamic programming naturally lead to the numerical solutions to the equations obtained by these powerful theoretical results. However, such solution is not satisfactory because their associated convergence is for the resulted equations and not directly pointing to the optimal control problem itself.

An early historical survey has been done by Polak in Polak [1973]. It summarizes the computational methods in Hilbert space and the associated gradient method and Newton-Raphson method. With the Frechet differential, the gradient of the functional to be minimized can be shown to be the partial derivative of the Hamiltonian function. Similar methods is reported again recently in Roberts [2002], where the primary and adjoint equations are combined together to form a 2-D system.

Parametrization method, in corporate with wavelets or spline functions, has been a useful tool for numerical computation of optimal control. See Teo et al. [1992] and references cited therein. The control is first parameterized using a class of spline function, wavelets, or piece-wise linear functions as the simplest case. Then, the optimal control problem is converted into the nonlinear optimization problem for these parameters with constraints. By increasing the number of the function basis used in the parametrization, the resulted controls form a sequence. This sequence can be shown to be convergent weakly to the optimal control.

In Schwartz and Polak [1996], an epigraphic convergence concept is defined as a frame to analyze the approximation of problems in a Hilbert space with a sequence of problems in finite dimensional subspaces of this Hilbert space. To some aspects, this approach can be considered as an extension or abstraction of the parametrization method. The epigraphic method is naturally extended to handle the distributed system in Pironneau and Polak [2002].

Recently, near-optimality for optimal control problem has attracted attention. See Zhou [1998] for details. Among many advantages claimed in these articles, the solution to near-optimal control is usually easier to obtain and of better properties such as smoothness.

In Jiang [2005], a steepest descent algorithm is proposed for the computation of stochastic near-optimal control. Here, the similar approach is proposed for deterministic near-optimal control. Comparing to existing results for numerical computation of optimal control, the proposed approach has several advantages. First, the algorithm is proposed for near-optimal solution instead of the optimal one. This leads to a finite-step algorithm. By using a progressive precision sequence, the proposed algorithm can also be used to compute the optimal control. Second, for most algorithms available for optimal control computation, only week convergence property is guaranteed. Let along the convergence rate. In our proposed algorithm, the error bound is explicitly obtained, which guarantees the convergence of the cost value. Third, as a small but meaningful variation of the steepest descent algorithm, the stepsize in the proposed algorithm can be time-varying, where that of all available versions are constant. This variation equips the algorithm more flexibility. Due to the length limit, some theoretical proofs to results in this paper are omitted. Interested reader may contact the author for details.

2. PROBLEM STATEMENTS AND PRELIMINARIES

Let us first introduce notations that will be used in this paper:

$u(\cdot)$: the m-dimensional control input. Lebesgue measurable from $[t_0, T] \subset \Re$ to a given compact set $\Gamma \subset \Re^m$. The bound of Γ is also denoted as Γ_b.

$U_{ad}[t_0, T]$: the set of admissible controls.

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$M^\top$: the transpose of matrix or vector M.
$\mid a \mid$: the norm of a vector or a matrix a. It is the sum of absolution value of its components.
$\frac{\partial \rho}{\partial z}$: the partial derivative of a vector function ρ with respect to a vector variable z. More specifically, $\left(\frac{\partial \rho}{\partial z}\right)_{i,j} = \frac{\partial \rho_i}{\partial z_j}$.
a_i : the i-th row vector of a matrix a.
$a_{\cdot i}$: the i-th column vector of a matrix a.
(a_{ij}) : A matrix that its (i,j)-th element is a_{ij}.

Consider the following nonlinear control system:

$$\begin{cases} \dot{x}(t) &= f(t, x(t), u(t)), \\ x(t_0) &= x_0, \end{cases} \tag{1}$$

where f is measurable in (t, x, u). The objective of optimal control is to find the admissible control to minimize the cost function given by:

$$J(u(\cdot)) = \int_{t_0}^{T} L(t, x(t), u(t))dt + h(x(T)), \tag{2}$$

where L is a measurable function in (t, x, u) and h is a function of $x(T)$. If the time interval and the initial state value of considering can change, J is also a function of t_0, x_0 and T. We also let the optimal cost, usually called value function, be denoted as V or $V(t_0, x_0, T)$.

We say that an admissible control u^ϵ is near optimal if the value of the corresponding cost is near the value of V. More specifically, for given $\epsilon > 0$, u^ϵ is ϵ-optimal if $\mid J(u(\cdot)) - V \mid \leq \epsilon$. In some cases it is not easy to justify the real meaning of near-optimal for one single fixed ϵ. We consider a positive sequence converges to zero, say, $\mathcal{E} := \{\epsilon_n\} \to 0+$. A sequence of admissible controls $\{u_n(\cdot)\}$ is called $\mathcal{E}$-optimal if $\mid J(u_n) - V \mid \leq \epsilon_n$. In this study, we need the following assumption:

Assumption 1. **(A1)**. f and L are measurable in (t, x, u) and continuously differentiable in x and u. **(A2)**. h is continuously differentiable. **(A3)**. There is a constant C such that the following Lipschitz type conditions are satisfied:

$$\begin{aligned} &\mid \rho(t, x, u) \mid \leq C(1 + \mid x \mid), \\ &\mid \rho(t, x, u) - \rho(t, x', u') \mid + \mid \frac{\partial \rho}{\partial x}(t, x, u) - \frac{\partial \rho}{\partial x}(t, x', u') \mid \\ &+ \mid \frac{\partial \rho}{\partial u}(t, x, u) - \frac{\partial \rho}{\partial u}(t, x', u') \mid \leq C(\mid x - x' \mid + \mid u - u' \mid), \\ &\mid h(x) \mid \leq C(1 + \mid x \mid), \\ &\mid h(x) - h(x') \mid + \mid \frac{\partial h}{\partial x}(x) - \frac{\partial h}{\partial x}(x') \mid \leq C \mid x - x' \mid, \end{aligned}$$

where $\rho = f, L$.

In the proof of the proposed algorithm, we also need the following result borrowed from Ekeland [1974]:

Lemma 1. (Ekeland's Principle) Let (S, d) be a complete metric space and $\rho(\cdot) : S \to \Re^1$ be lower-semicontinuous and bounded below. For $\epsilon \geq 0$, if $u^\epsilon \in S$ satisfies $\rho(u^\epsilon) \leq \inf_{u \in S} \rho(u) + \epsilon$. Then, for any $\lambda > 0$, there exists a $u^\lambda \in S$ such that

$$\rho(u^\lambda) \leq \rho(u^\epsilon), \;\; d(u^\lambda, u^\epsilon) \leq \lambda,$$

$$\rho(u^\lambda) \leq \rho(u) + \frac{\epsilon}{\lambda} d(u^\lambda, u^\epsilon),$$

where $d(u^\lambda, u^\epsilon)$ is the distance in the metric space.

3. ADJOINT EQUATION, MAXIMUM PRINCIPLE, AND CONDITIONS TO NEAR-OPTIMALITY

For a given admissible control input $\bar{u}(t)$, let the trajectory of the equation (1) be denoted as $\bar{x}(t)$. Then, the first order variational equation for the optimal control problem defined by (1) and (2) are defined as the following differential equation with initial condition:

$$\begin{aligned} \dot{y}_1(t) = &\frac{\partial f}{\partial x}(t, \bar{x}(t), \bar{u}(t)) y_1(t) \\ &+ f(t, \bar{x}(t), u(t)) - f(t, \bar{x}(t), \bar{u}(t)) \end{aligned} \tag{3}$$

$$y_1(t_0) = 0, \tag{4}$$

This equation is closely related to the first order expansion of the cost function (2). More specifically, let the cost be defined as:

$$\begin{aligned} J_1(u(\cdot)) = E[&\int_{t_0}^{T} \frac{\partial L}{\partial x}(t, \bar{x}(t), \bar{u}(t)) y_1(t) dt \\ &+ \frac{\partial h}{\partial x}(\bar{x}(T)) y_1(T)], \end{aligned} \tag{5}$$

The adjoint equation associated with the first order solution is given by:

$$\begin{cases} \dot{p}(t) = & -\frac{\partial f}{\partial x}(t, \bar{x}(t), \bar{u}(t))^\top p(t) \\ & -\frac{\partial L}{\partial x}(t, \bar{x}(t), \bar{u}(t)), \\ p(T) = & \frac{\partial h}{\partial x}(\bar{x}(T)). \end{cases} \tag{6}$$

By using the integral by part method, the following relationships can be established:

$$\begin{aligned} J_1(u(\cdot)) = \int_{t_0}^{T} & p(t)^\top [f(t, \bar{x}(t), u(t)) \\ & - f(t, \bar{x}(t), \bar{u}(t))] dt. \end{aligned} \tag{7}$$

Define the Hamiltonian function H as:

$$H(t, x, v, p) = L(t, x, v) + p^\top f(t, x, v). \tag{8}$$

Then, the so-called Maximum Principle and the major results in Zhou [1998] can be summarized by the following theorem:

Theorem 1. Assume that conditions in Assumption 1 in previous section are all satisfied.
(1). The necessary condition for an admissible control $\bar{u}(t)$ minimizing the cost function J is that $\bar{u}(\cdot)$ minimizes the functional $\int_{t_0}^{T} H(t, \bar{x}(t), u(t), \bar{p}(t)) dt$, where $\bar{x}$ is the system trajectory corresponding to $\bar{u}$.
(2). There exists a constant $C_1 > 0$, such that an admissible control $\bar{u}(t)$ and its corresponding solution $\bar{x}(t)$ is ϵ-optimal. Then, the following inequality is satisfied:

$$\int_{t_0}^{T} H(t,\bar{x}(t),\bar{u}(t),\bar{p}(t))dt \leq$$

$$\inf_{u\in U_{ad}[t_0,T]} \int_{t_0}^{T} H(t,\bar{x}(t),u(t),\bar{p}(t))dt + C_1\epsilon^{1/2} \quad (9)$$

(3). There exists a constant C_1, which is related to the Lipschitz constant C, the time interval $[t_0,T]$ and the value range Γ of admissible controls but independent of ϵ. Assume the Hamiltonian function $H(t,x(t),u(t),p)$ defined in (8) and the function $h(x)$ in the terminal cost of the cost function given by (2) are convex in with respect to t and x, respectively. If for some $\epsilon > 0$, there is an admissible control $\bar{u}$ such that the inequality (9) holds. Then, $J(\bar{u}(\cdot)) \leq \inf_{u\in U_{ad}[t_0,T]} J(u(\cdot)) + C_2\epsilon^{\frac{1}{4}}$.

In the proof of the result corresponding to (3) of Theorem 1 in Zhou [1998], one can see that $\frac{\partial H}{\partial u}$ to be small is a result of the inequality (9). In this paper, whether $\int_{t_0}^{T} \mid \frac{\partial H}{\partial u} \mid dt$ is small is the major criterion that our scheme based on. For this concern, we need the following theorem:

Theorem 2. Assume that conditions in Assumption 1 in previous section are all satisfied.
(1). There exists a constant $C_2 > 0$ independent of ϵ, such that, if $u^\epsilon(\cdot)$ is an ϵ-optimal control, the following estimation holds:

$$\int_{t_0}^{T} \mid \frac{\partial H}{\partial u}(t,x^\epsilon(t),u^\epsilon(t),p^\epsilon) \mid dt < C_2\epsilon^{\frac{1}{4}}. \quad (10)$$

(2). Assume the Hamiltonian function H defined in (8) and the function $h(x(t))$ in the terminal cost of the cost function given by (2) are convex. Then, there is a constant $C_3 > 0$ independent of ϵ such that for any $\epsilon > 0$, as long as an admissible control $\bar{u}(\cdot)$ satisfies

$$\int_{t_0}^{T} \mid \frac{\partial H}{\partial u}(t,\bar{x},\bar{u},\bar{p}) \mid dt < C_3\epsilon, \quad (11)$$

$\bar{u}$ is guaranteed to be an $\epsilon^{1/2}$-optimal control.

The proof to this theorem is omitted to save space.

4. A FUNCTIONAL STEEPEST DESCENT SCHEME

Now we are ready to present a finite step computation scheme for the near-optimal control of the problem given by (1) and (2). More specifically, given a small positive real number ϵ, an ϵ-optimal solution can be obtained in finite step using the algorithm proposed in the subsequent paragraph. However, the convergence proof is arranged in next section for the ease of reading.

Functional Steepest Descent Algorithm
Initial step: Start with an arbitrarily selected admissible control $\bar{u}(\cdot)$.
Updating rule:
(1) Calculate $\bar{x}(t)$ according to the equation (1) .
(2) Solve the first order adjoint equation (6) for $\bar{p}(t)$.
(3) Update $\bar{u}(t)$ with

$$\bar{u}(t) - \lambda(t)\frac{\partial H}{\partial u}(t,\bar{x}(t),\bar{u}(t),\bar{p}(t)), \quad (12)$$

where $\lambda(t)$ is a positive scaler function bounded by a some constant $C_{\bar{u}(\cdot)}$.

An example: Linear Quadratic Gaussian Control Consider the following problem:

$$\begin{cases} \dot{x}(t) = A(t)x(t) + B(t)u(t) \\ x(t_0) = x_0, \end{cases} \quad (13)$$

$$J(u(\cdot)) = \frac{1}{2}\int_{t_0}^{T} [< L(t)x(t), x(t) > + \frac{1}{2} < Gx(T), x(T) >$$
$$+2 < M(t)x(t), u(t) > + < N(t)u(t), u(t) >]dt, \quad (14)$$

where L, M, N, G are matrices of appropriate dimensions, and $< \cdot, \cdot >$ denote the inner product in Euclidean space.

Now the adjoint equations become:

$$\begin{cases} \dot{p}(t) = -\left\{A^\top(t)p(t) + \frac{1}{2}L(t)x(t) + u^\top(t)M(t)\right\} \\ p(T) = Gx(T), \end{cases} \quad (15)$$

The Hamiltonian function H is the following:

$$H(t,x,u,p) = \frac{1}{2} < L(t)x(t), x(t) > + < M(t)x(t), u(t) >$$
$$+\frac{1}{2} < N(t)u(t), u(t) > +p^\top(t)[A(t)x(t) + B(t)u(t)]. \quad (16)$$

It can be calculated that:

$$\frac{\partial H(t,x,u,p)}{\partial u} = M(t)x(t) + N(t)u(t) + B^\top(t)p(t). \quad (17)$$

Apparently, the proposed scheme is a normal iterative method to solve the equation:

$$M(t)x(t) + N(t)u(t) + B^\top(t)p(t) = 0$$

for u.

Considering the special case where $M = 0$, let the symmetric matrix $P(t)$ is defined by:

$$-\dot{P}(t) = P(t)A(t) + A^\top(t)P(t)$$
$$-P(t)B(t)N^{-1}(t)B^\top(t)P(t) + L(t).$$

Then, at the equilibrium solution where

$$u(t) = -N^{-1}(t)[M(t)x(t) + B^\top(t)p(t)],$$

it can be easily checked by simple computation that $p(t) = P(t)x(t)$ satisfies the adjoint equation (15). This is a reminiscence to the classical results given in [Anderson and Moore, 1989, Pages 20-26].

5. CONVERGENCE ANALYSIS

In this section, technical results of the convergence analysis will be given. Consider two consecutive controls $u_k(\cdot)$, and $u_{k+1}(\cdot)$ in the iterative computing process using scheme in the Functional Steepest Descent Algorithm. We will analyze the convergence of that algorithm through evaluating the difference $J(u_{k+1}(\cdot)) - J(u_k(\cdot))$. First, we will see how well the first-order approximation can be used to estimate the difference. To this end, the following lemma holds:

Lemma 2. For any two generic consecutive controls calculated based on the updating rule in the algorithm, assume the step size is chosen such that $\int_{t_0}^{T} \mid \lambda_k(t) \mid^2 dt \leq 1$. The following estimation holds:

$$|x_{k+1}(t) - x_k(t) - y_1(t)|^2 \leq C_5(\int_{t_0}^{t} \mid \lambda_k(\tau) \mid^2 d\tau)^2, \quad \forall t \in [t_0, T], \tag{18}$$

where $y_1(t)$ is the first-order approximation defined by the equations (3).

The proof to this lemma is omitted to save space, interested reader may contact the author for details. Now we are ready to present the convergence property of the Functional Steepest Descent Algorithm.

Theorem 3. For given an $\epsilon > 0$, assume that the admissible control value set Γ is big enough so that no ϵ-optimal control can reach the δ_0 neighborhood of the boundary of Γ, where δ_0 is a small positive constant. Then, there exist positive constants α, β and δ such that, if $\lambda_k(t)$ is selected in the interval $[\alpha, \beta]$ and satisfies $\int_{t_0}^{T} \mid \lambda_k(t) \mid^2 dt \leq 1$, the cost value is guaranteed to decrease at least by the amount of δ, by using the proposed scheme starting from any admissible control. Therefore, if the cost function $J(u(\cdot))$ is bounded below, the proposed scheme achieves ϵ-optimal solution in finite steps.

Proof: Consider the cost function difference between two consecutive steps:

$$\begin{aligned}
&J(u_{k+1}(\cdot)) - J(u_k(\cdot)) \\
&= \int_{t_0}^{T} [L(t, \tilde{x}_k(t), u_k(t)) - L(t, x_k(t), u_k(t))]dt \\
&+ \int_{t_0}^{T} [L(t, x_k(t), u_{k+1}(t)) - L(t, x_k(t), u_k(t))]dt \\
&+ \int_{t_0}^{T} \{[L(t, \tilde{x}_k(t), u_{k+1}(t) - L(t, \tilde{x}_k(t), u_k(t))] \\
&-[L(t, x_k(t), u_{k+1}(t)) - L(t, x_k(t), u_k(t))]\} \, dt \\
&+I_1 + I_2 + (h(\tilde{x}_k(T)) - h(x_k(T)),
\end{aligned} \tag{19}$$

where: $\tilde{x}_k(t) = x_k(t) + y_1(t)$,

$$I_1 = \int_{t_0}^{T} [L(t, x_{k+1}(t), u_{k+1}(t) - L(t, \tilde{x}_k(t), u_{k+1}(t))]dt,$$

$$I_2 = (h(x_{k+1}(T)) - h(\tilde{x}_k(T))).$$

By the Lipschitz assumption and Lemma 2, one can see that

$$\mid I_i \mid \leq Const. \cdot (\max_{t \in [t_0, T]} \lambda_k(t))^2, i = 1, 2. \tag{20}$$

Now we can evaluate each items in (19). Let the first three integral items be denoted as E_1, E_2, E_3, respectively. Then:

$$\begin{aligned}
E_1 &= \int_{t_0}^{T} \frac{\partial L}{\partial x}(t, x_k, u_k) y_1(t) dt \\
&+ \frac{1}{2} \int_{t_0}^{T} y_1(t)^\top \frac{\partial^2 L}{\partial x^\top \partial x}(t, x_k(t), u_k(t)) y_1(t) dt \\
&+ I_3,
\end{aligned} \tag{21}$$

$$\begin{aligned}
E_3 &= \int_{t_0}^{T} [\frac{\partial L}{\partial x}(t, x_k, u_{k+1}) - \frac{\partial L}{\partial x}(t, x_k, u_k)] y_1(t) dt \\
&+ \frac{1}{2} \int_{t_0}^{T} y_1(t)^\top [\frac{\partial^2 L}{\partial x^\top \partial x}(t, x_k(t), u_{k+1}(t)) \\
&- \frac{\partial^2 L}{\partial x^\top \partial x}(t, x_k(t), u_k(t))] y_1(t) dt + I_4,
\end{aligned} \tag{22}$$

where $\mid I_i \mid \leq Const. \cdot (\mid y_1(t) \mid^2 + \mid y_2(t) \|^2), i = 3, 4$.

By the same token, we can see that

$$\begin{aligned}
h(\tilde{x}_k(T)) - h(x_k(T) &= [\frac{\partial h}{\partial x}(x_k(T)) y_1(T)] \\
&+ \frac{1}{2} \left(y_1(T)^\top \frac{\partial^2 h}{\partial x^\top \partial x}(x(T)) y_1(T) \right) + I_5,
\end{aligned} \tag{23}$$

where $I_5 \leq Const. \cdot \mid y_1(t) \mid^2$.

Applying the estimation (21), (22) and (23) to the cost difference, with some intermediate results in Lemma 2 , we can reach that

$$\begin{aligned}
&J(u_{k+1}(\cdot)) - J(u_k(\cdot)) = J_1(u_{k+1}(\cdot)) + J_2(u_{k+1}(\cdot)) \\
&+ \frac{1}{2} \int_{t_0}^{T} y_1^\top \frac{\partial^2 L}{\partial x^\top \partial x}(t, x_k, u_k) y_1 dt \\
&+ \int_{t_0}^{T} [L(t, x_k, u_{k+1}) - L(t, x_k, u_k)]dt \\
&+ \int_{t_0}^{T} \left[\frac{\partial L}{\partial x}(t, x_k, u_{k+1}) - \frac{\partial L}{\partial x}(t, x_k, u_k) \right] y_1 dt \\
&+ \frac{1}{2} \int_{t_0}^{T} y_1^\top \left[\frac{\partial^2 L}{\partial x^\top \partial x}(t, x_k, u_{k+1}) - \frac{\partial^2 L}{\partial x^\top \partial x}(t, x_k, u_k) \right] y_1 dt \\
&+ \frac{1}{2} \left[y_1(T) \frac{\partial^2 h}{\partial x^\top \partial x}(x_k(T)) y_1(T) \right] + I_5,
\end{aligned} \tag{24}$$

where $\mid I_5 \mid \leq Const. \cdot \mid \lambda_k(t) \mid^2$. We can rewrite this equation as:

$$\begin{aligned}
&J(u_{k+1}(\cdot)) - J(u_k(\cdot)) \\
&= \int_{t_0}^{T} [\mathcal{H}^{x_k, u_k}(t, x_k, u_{k+1}) - \mathcal{H}^{x_k, u_k}(t, x_k, u_k)]dt + I_5.
\end{aligned} \tag{25}$$

As such, there exists a constant $C^\star > 0$ such that

$$| J(u_{k+1}) - J(u_k) - \int_{t_0}^{T} \frac{\partial H}{\partial u}(t, x_k, u_k, p_k)(u_{k+1} - u_k)dt |$$
$$\leq C^\star \mid \lambda_k \mid^2 . \quad (26)$$

On the other hand, it can be proved that

$$(\int_{t_0}^{T} \mid \frac{\partial H}{\partial u}(t, x_k, u_k, p_k) \mid dt)^2 \leq$$
$$\frac{T - t_0}{2} \int_{t_0}^{T} \frac{\partial H}{\partial u}(t, x_k, u_k, p_k)[\frac{\partial H}{\partial u}(t, x_k, u_k, p_k)]^\top dt.$$

for given $\epsilon > 0$, if $\int_{t_0}^{T} \mid \frac{\partial H}{\partial u}(t, x_k, u_k, p_k) \mid dt < C_3\epsilon$, we know that u_k is ϵ-optimal control. Assume that

$$\int_{t_0}^{T} \frac{\partial H}{\partial u}(t, x_k, u_k, p_k)[\frac{\partial H}{\partial u}(t, x_k, u_k, p_k)]^\top dt \geq \frac{2C_3^2\epsilon^2}{T - t_0},$$

otherwise u_k is already ϵ-optimal. Then, choose α and β such that

$$0 \leq \alpha \leq \beta < 1, \beta < \frac{C_3^2\epsilon^2}{C^\star(T - t_0)}. \quad (27)$$

It can be checked that for any $\lambda_k(t)$ selected in the interval $[\alpha, \beta]$, there holds:

$$J(u_{k+1}) < J(u_k) - \frac{C_3^2\epsilon^2}{T - t_0}\alpha. \quad (28)$$

Therefore, the claim is proved.

6. CONSISTENT APPROXIMATION SOLUTIONS TO ADJOINT DIFFERENTIAL EQUATIONS

The convergence analysis conducted in the last section is based on the assumption that precise solution to the adjoint equation can be available. In this section, the approximation of the adjoint differential equation is discussed. Rather than to develop an approximation algorithm, the objective is to consider the precision required for a consistent approximation algorithm so as to achieve desired near optimal solution to the original stochastic control problem.

Consider the case where two pairs of an approximated solution ($\hat{p}_k(t)$ is obtained to the adjoint equation (6) at the step k using the proposed iterative scheme. Let the solution to the equation (3) with $u(t)$ replaced by $\hat{u}_{k+1}$ be denoted as $\hat{y}_1(t)$, where $\hat{u}_{k+1}$ is the next step control computed using the approximation solutions $\hat{p}_k$. For simplicity, we also denote all other variables associated with these approximation solution using a hat symbol $\hat{}$. Then, considering the convergence analysis in last section, one can see that the difference of cost functions between two consecutive steps can always be decomposed in the form of (19) regardless of whether $\tilde{x}_k$ is an good approximation of x_{k+1} or not. However, (20) needs to be verified in this case. Similarly, one can see that (21), (22), and (23) hold with y_1 replaced by $\hat{y}_1$. Notice the estimation to I_3, I_4, I_5 are still correct. In order to obtain the estimation on I_1, I_2 and obtain (25), we need the following lemma:

Lemma 3. Given an admissible control $\bar{u}(\cdot)$, let the corresponding state trajectory be denoted as $\bar{x}(\cdot)$, and the solution to the adjoint equation (6) be denoted as $\bar{p}(\cdot)$. Assume $\hat{p}(\cdot)$ is an approximated solution to those adjoint equations such that

$$\int_{t_0}^{T} \mid \hat{p}(t) - \bar{p}(t) \mid^2 dt < \eta, \quad (29)$$

where η is a given small positive real number. Then, the following results hold:
(1). $\int_{t_0}^{t} \left| \frac{\partial H}{\partial u}(\tau, \hat{x}_k, \hat{u}_k, \hat{p}_k(\hat{u}_k)) \right|^2 d\tau$ is bounded by a constant. This constant is depend on the Lipschitz constant C and the approximation error η.
(2). There is a constant $\hat{C}$ such that the auxiliary cost function J_1 defined in (5) satisfies the following estimation:

$$\mid J_1(u(\cdot)) - \hat{J}_i(u(\cdot)) \mid < \hat{C}\eta, \quad (30)$$

where $\hat{J}_1$ is the corresponding cost defined in (7) with $p(t)$ replaced by $\hat{p}(t)$.

The proof is omitted to save space.

Applying the claim (1) in Lemma 3, the following result corresponding to Lemma 2 can be obtained:

Corollary 4. Under the assumption in Lemma 3, for the first order approximation of the trajectory $x(\cdot)$ at $(\hat{x}_k, \hat{u}_k)$ based on the approximation solution of the corresponding adjoint equation, there exists a constant $\hat{C}_5$ such that:

$$E\,|\hat{x}_{k+1}(t) - \hat{x}_k(t) - \hat{y}_1(t)|^2 \leq \hat{C}_5 E(\int_{t_0}^{t} \mid \lambda_k(\tau) \mid^2 d\tau)^2,$$
$$\forall t \in [t_0, T]. \quad (31)$$

where $\hat{C}_5$ is only depend on the Lipschitz constant C and the approximation error bound η.

Theorem 5. Assume all assumptions and conditions in Theorem 3 are satisfied. Choose $\eta_k \leq \sup_{t\in[t_0,T]} \lambda_k^2(t)$. Then, there exist positive constants α, β and δ such that, if $\lambda_k(t)$ is selected in the interval $[\alpha, \beta]$ and satisfies $E\int_{t_0}^{T} \mid \lambda_k(t) \mid^2 dt \leq 1$, the cost value is guaranteed to decrease at least by the amount of δ, by using the proposed scheme starting from any admissible control with the approximated solutions of the adjoint equation satisfying (29) where the η is replaced by η_k at step k. Therefore, if the cost function $J(u(\cdot))$ is bounded below, the proposed computation scheme achieves ϵ-optimal solution in finite steps.

7. ILLUSTRATIVE SIMULATIONS

Consider the optimal control problem discussed in Schwartz and Polak [1996],and Hager [1976]:

$$\min_{u\in U} \; f(u(\cdot)) := x_2(1),$$
$$\text{s. t. } \dot{x} = \begin{bmatrix} \frac{1}{2}x_1 + u \\ \frac{10}{16}x_1^2 + \frac{1}{2}x_1 u + \frac{1}{2}u^2 \end{bmatrix},$$
$$x(0) = [1, 0]^\top, t \in [0, 1]$$

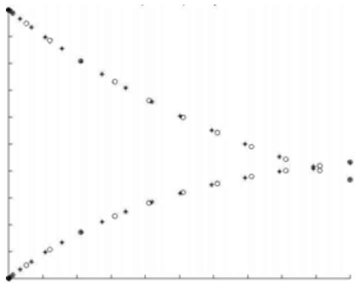

Fig. 1. Trajectories of the system control by the optimal control law and the numerical solution control. The "*"-curve is the optimal one. The "o"-curve is the near-optimal one.

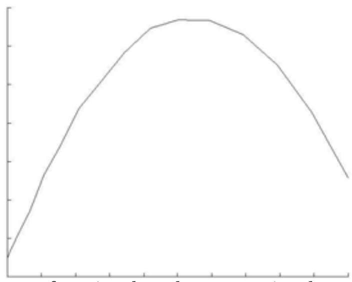

Fig. 2. The difference of optimal and near-optimal controls

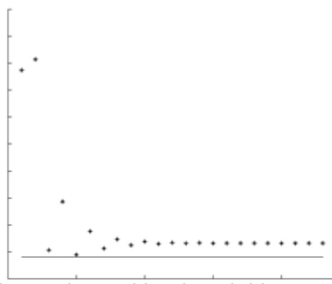

Fig. 3. The optimal cost, denoted by the solid line, and near-optimal costs at each iteration, denoted by "*" dots.

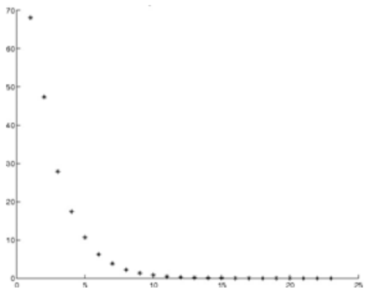

Fig. 4. The norm of the partial derivatives of the Hamiltonian at each iteration.

The optimal solution can be given in analytic form: $u^\star(t) = -(tanh(1-t)+0.5)cosh(1-t)/cosh(1)$, $t \in [0,1]$, with the optimal cost equal to $e^2 sinh(2)/(1+e^2))^2$. As such, convergence of the near-optimal solutions can be easily justified. Applying the algorithm proposed, the step size $\lambda(t)$ is chosen as 0.4 where k is the iteration number. The controlled system is simulated using Runge-Kutta (2,3) method. The control input $u(t)$ is discretized using the grid obtained by Runge-Kutta ODE solver and interpolated piece-wise linearly. We compute the near-optimal control for $\epsilon = 10^{-3}$. The near-optimal control is obtained in 25 steps with the initial control to be zero. The following figures 1 - 8 show the simulation results of the algorithm. It can be seen that the near-optimal control may not approximate the optimal control very well. But the cost function is very close.

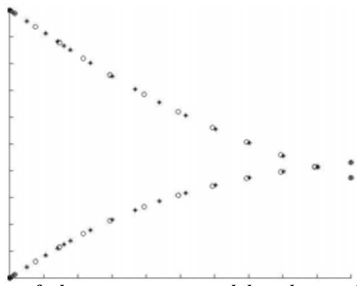

Fig. 5. Trajectories of the system control by the optimal control law and the numerical solution control. Now the control is constrained. The "*"-curve is the optimal one. The "o"-curve is the near-optimal one.

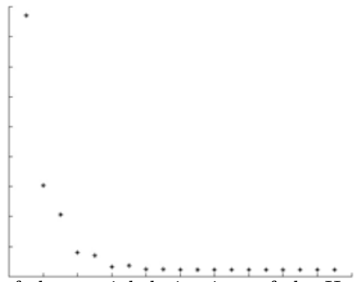

Fig. 6. The norm of the partial derivatives of the Hamiltonian at each iteration for the system with control constraints.

Consider the same problem with a constraint on the control input bounded by $[-1.3, -0.3]$. Now, constant stepsize is no longer applicable. Instead, we choose a time varying stepsize by truncating the line search if it exceeds the bound. Figures 5 and 6 show the simulation results obtained by the first 20 steps. In these simulation experiments, the computation is finished at finite steps. The system trajectories are very close to that under optimal control strategy. The cost function is within the desired neighborhood of the optimal cost. The convergence of control sequence is not demonstrated. However, that issue itself is not of our concern in this paper.

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