Proceedings of the 17th World Congress
The International Federation of Automatic Control
Seoul, Korea, July 6-11, 2008

Synchronization of a Class of Multi-agent Systems with Large Population ⋆

Zhixin LIU * Lei GUO *

* Institute of Systems Science, Chinese Academy of Sciences,
Beijing 100080, P.R. China
(Emails: Lzx@amss.ac.cn, Lguo@amss.ac.cn)

Abstract: Multi-agent systems arise from diverse fields of natural and artificial systems, and a typical case is that each agent has the tendency to behave as other agents do in its neighborhood described by a disk or ball. This is actually reflected by the well-known Vicsek model. Since this model is of fundamental importance in understanding the multi-agent systems, it has attracted much attention from researchers in recent years. In this paper, we will present a comprehensive theoretical analysis of the nonlinear Vicsek model in a random framework, without imposing any connectivity conditions on the system trajectories as did in most of the previous investigations. To be precise, we will show that for any givenmodel parameters, i.e., the interaction radius r and the agents' moving velocity v, the overall system will synchronize as long as the population size is large enough, which justifies the phenomenon observed previously in simulations by Vicsek et al. (1995). The proof is based on the recent work of Tang and Guo (2007) for linearized Vicsek model, and involves the use of spectral graph theory and multi-array martingale estimation theory.

1. INTRODUCTION

Recently, the collective behavior of multi-agent systems has drawn much attention from researchers in diverse fields, including biology, physics, mathematics, and control theory(cf. Parrish (2002), Vicsek et al. (1995), Jadbabaie et al. (2003)). A basic problem is to understand how locally interacting agents lead to collective behavior of the overall multi-agent systems.

A typical phenomenon in multi-agent systems is that each agent has the tendency to behave as other agents do in its neighborhood(cf. Vicsek et al. (1995), Brien (1989)).This is actually reflected by the local interaction rule in the well-known Vicsek model, which possesses some key features of multi-agent systems, such as dynamic behavior and changing neighbors, besides the local interactions. The Vicsek model, studied by Vicsek et al. (1995) from the viewpoint of statistical mechanics, was used to investigate the gathering, transport and phase transition in nonequilibrium systems, and also applied in biology systems involving clustering and migration. Through computer simulations, Vicsek et al. (1995) showed that the system will synchronize when the density is large and the noise is small. A similar phenomenon was observed by Buhl et al. (2006) when they studied the behavior of locust swarm through experiments.

Though the Vicsek model looks simple, the nonlinear coupled relations in the model make the theoretical analysis quite complicated. Jadbabaie et al. (2003) initiated a theoretical study for the synchronization of the Vicsek model, but with linearized heading equations. They showed that the system will synchronize if the associated neighbor graphs are jointly connected in a certain sense, which stimulated considerable research interests in this direction (cf., e.g., Cucker and Smale (2007),Saber (2006) Ren and Beard (2005), Moreau (2005)). However, how to remove or verify the troublesome connectivity condition on the associated dynamical graphs is still a difficult and challenging issue in theory.

A preliminary step towards the above issue seems to have been made by Liu and Guo (2007a), where a sufficient but rough parameter condition to guarantee the synchronization of the Vicsek model is given in a deterministic framework. A sufficient parameter condition for the synchronization is also given by Cucker and Smale (2007), but for a modified Vicsek model with global interactions. A significant step towards the comprehensive analysis of the Vicsek model is made recently by Tang and Guo (2007), where a random framework is introduced in the analysis of the linearized Vicsek model. They proved that the overall system will synchronize with large probability as long as the size of the population is large enough, without imposing connectivity condition on the system trajectories.

This paper will establish a similar result as that of Tang and Guo (2007), but for the nonlinear Vicsek model. In comparison with Tang and Guo (2007), a key issue now is to deal with the nonlinearity arising from the heading equations. We will give a comprehensive theoretical analysis in a random framework with large population, by using some basic facts in Tang and Guo (2007) and multi-array martingale estimation theory. We will prove that for any given model parameters, i.e., given the interaction radius r and the agents' moving velocity v, the overall system will synchronize as long as the size n of the population is large enough.

⋆ This work was supported by the National Natural Science Foundation of China under grants No. 60574068 and No. 60221301.

2. MAIN RESULTS

First, we will introduce the model to be studied in this paper, assuming the readers are familiar with some basic knowledge of algebraic graph theory(cf. Godsil and Royle (2001), Chung (2000)).

The Vicsek model is composed of n autonomous agents (or subsystems or particles) moving in the plane with the same absolute velocity, and with each agent's heading updated according to the vector average of its neighbors. The neighbors of an agent $i(1 \le i \le n)$ at time t are those which lie within a circle of radius $r(r > 0)$ centered at the agent i's current position. Denote the neighbors of the agent i at time t as $\mathcal{N}_i(t)$, *i.e.*

$$\mathcal{N}_i(t) = \{j \mid d_{ij}(t) < r\}, \tag{1}$$

where $d_{ij}(t) = \sqrt{(x_i(t) - x_j(t))^2 + (y_i(t) - y_j(t))^2}$, and $(x_i(t), y_i(t))$ is the position of the agent i at time t. It is easy to see that each agent is a neighbor of itself. Each agent moves in the plane with the same constant absolute velocity $v(v > 0)$, so its position is updated according to the following equation:

$$\begin{cases} x_i(t+1) = x_i(t) + v\cos\theta_i(t+1) \\ y_i(t+1) = y_i(t) + v\sin\theta_i(t+1) \end{cases} \quad \forall i, \tag{2}$$

where $\theta_i(t)$ is the heading of the agent i at time t, which is updated according to the following formula,

$$\theta_i(t+1) = \arctan\frac{\sum_{j\in\mathcal{N}_i(t)}\sin\theta_j(t)}{\sum_{j\in\mathcal{N}_i(t)}\cos\theta_j(t)}. \tag{3}$$

Obviously, the dynamical behavior of the above system is determined completely by the initial states, the moving velocity v and the neighborhood radius r. Furthermore, there is a complicated nonlinear relationship between positions and headings of all agents, which makes a complete theoretical analysis quite involved.

Note that in the Vicsek model, the neighbors of each agent will change over time, and we may use a graph sequence $G_t = \{V, E_t\}$ to describe the evolution of the underlying system dynamics, where $V = \{1, 2, \cdots, n\}$ is the set of agents' indices (vertices), and E_t is the edge set which will change over time. Edges are formed in the following way: if $d_{ij}(t) < r$, then we define an edge between i and j, denoted by $(i,j) \in E_t$. Obviously, the neighbor graphs formed in this way are undirected, and contain loops. The degree, minimum degree and maximum degree of graph G_t are denoted by $d_i(t)(1 \le i \le n), d_{min}(t)$ and $d_{max}(t)$ respectively. And the adjacency matrix, degree matrix and the normalized Laplacian of graph G_t are denoted by $A(t), T(t)$ and $\mathcal{L}(t)$ respectively. The eigenvalues of $\mathcal{L}(t)$ are usually arranged in the following way: $0 = \lambda_0(t) \le \lambda_1(t) \cdots \le \lambda_{n-1}(t)$, and $\overline{\lambda}(t) = \max\{|1-\lambda_1(t)|, |\lambda_{n-1}(t) - 1|\}$ is the spectral gap of the graph G_t.

The main purpose of this paper is to study the synchronization property of the above Vicsek model in the sense that the headings of all agents converge to the same value, i.e., there exists a constant θ, such that $\lim_{t\to\infty}\theta_i(t) = \theta$, for all agents i.

We will analyze the synchronization behavior of the above multi-agent system in the following random framework, which is a natural assumption on the initial states of the system.

Assumption 2.1. The initial positions and headings of all agents are mutually independent, with positions uniformly and independently distributed in the unit square $\mathcal{S}$, and with headings uniformly and independently distributed in $[-\pi + \varepsilon_0, \pi - \varepsilon_0]$, where $\varepsilon_0 \in (0, \pi)$ is a fixed angle.

It is worth noting that the interval of the initial headings in the above assumption is much larger than that used in Liu and Guo (2007b), but may not be further relaxed to $(-\pi, \pi]$ (see Liu and Guo (2006)). Under Assumption 2.1, the initial graph G_0 is a random geometric graph which has some nice properties (see Section 3.1), which enable us to establish the following theorem whose proof will be given in Section 3.

Theorem 2.1. Under Assumption 2.1, for any given velocity $v > 0$ and neighborhood radius $r > 0$, the multi-agent system described by (1)-(3) will synchronize almost surely when the population size n is large enough.

3. PROOF OF THEOREM 2.1

First of all, for analyzing the heading equation (3), we rewrite it into the following equivalent form:

$$\tan\theta_i(t+1) = \sum_{j\in\mathcal{N}_i(t)}\frac{\cos\theta_j(t)}{\sum_{k\in\mathcal{N}_i(t)}\cos\theta_k(t)}\tan\theta_j(t), \tag{4}$$

and put (4) into the following compact matrix form:

$$\tan\theta(t+1) = P(t)\tan\theta(t), \tag{5}$$

where $\tan\theta(t) \triangleq (\tan\theta_1(t), \cdots, \tan\theta_n(t))^\tau$, $P(t) \triangleq (p_{ij}(t))$ is defined as follows:

$$p_{ij}(t) = \begin{cases} \dfrac{\cos\theta_j(t)}{\sum_{k\in\mathcal{N}_i(t)}\cos\theta_k(t)}, & \text{if } (i,j) \in E_t; \\ 0, & \text{otherwise.} \end{cases} \tag{6}$$

We will also need the linearized version (around $\theta_i(t) = 0$), which is denoted by $P^0(t) = (p^0_{ij}(t))$ and defined explicitly by

$$p^0_{ij}(t) = \begin{cases} \dfrac{1}{n_i(t)}, & \text{if } (i,j) \in E_t; \\ 0, & \text{otherwise,} \end{cases} \tag{7}$$

where $n_i(t)$ is the cardinality of $\mathcal{N}_i(t)$.

The following sets are useful in the proof of theorem 2.1,

$$\mathcal{R}_j = \{i : (1-\eta)r \le d_{ij}(0) \le (1+\eta)r\}, \tag{8}$$

$$\mathcal{N}'_j(t) = \{i : d_{ij}(t) < 2r\}, \quad t \ge 0, \tag{9}$$

where $d_{ij}(t)$ is the distance between agents i and j at time t, and $0 < \eta < 1$ is a constant. Denote R_j, and $n'_j(t)$ as the cardinality of $\mathcal{R}_j$ and $N'_j(t)$ respectively, and $R_{max} \triangleq \max_j R_j$.

3.1 The Asymptotic Properties of G_0

In this section, we will provide the estimations of some characteristics of both the initial graph G_0 and the headings at time $t = 1$. The following lemma is given by Tang and Guo (2007).

Lemma 3.1. For the random geometric graph G_0.

1) The maximum degree and minimum degree satisfy

$$\min\left(\frac{\pi}{64}, \frac{\pi r^2}{4}\right) n \le d_{min}(0) \le d_{max}(0) \le n, \quad a.s.;$$

2) The number of agents in the ring defined via (8) is bounded by

$$R_{max} \le 4n\pi\eta r^2(1 + o(1)), \quad a.s.; \tag{10}$$

3) The spectral gap of the initial graph satisfies

$$\bar{\lambda}(0) \le 1 - \frac{\pi r^2}{512(r + \sqrt{6})^4}(1 + o(1)), \quad a.s.$$

Remark 3.1. For the random geometric graph G_0, we have for large n

i)

$$\min\left(\pi r^2, \frac{\pi}{64}\right) n \le \min_i n_i'(0) \le \max_i n_i'(0) \le n, \ a.s., \tag{11}$$

where $n_i'(0)$ is the cardinality of $\mathcal{N}_i'(0)$ defined by (9).

ii) By Lemma 3.1 and the above remark, there are positive constants α and β such that

$$d_{max}(0) \le \beta d_{min}(0)(1 + o(1)),$$
$$R_{max} \le \alpha d_{min}(0)(1 + o(1))$$

hold almost surely, which can actually be taken as

$$\beta = \max\left(\frac{64}{\pi}, \frac{4}{\pi r^2}\right),$$
$$\alpha = \max\left(16, 256r^2\right)\eta.$$

Thus we can take the constant η small enough, such that $\alpha = \frac{1}{4}$.

By using the multi-array martingale estimation theorem (see Corollary 3.1 in Liu and Guo (2007b)), we can get the following estimations of the initial headings:

Lemma 3.2. For large n, we have

$$1) \max_{1\le i\le n}\Big|\sum_{j\in\mathcal{N}_i(0)} \sin\theta_j(0)\Big| \le C_1 b_n, \quad a.s.$$

$$2) \max_{1\le i\le n}\Big|\sum_{j\in\mathcal{N}_i(0)} (\cos\theta_j(0) - C_2)\Big| \le C_3 b_n, \quad a.s.$$

where $b_n = \sqrt{n\log n}(1 + o(1))$, and

$$C_1 = 3\left\{\frac{1}{2} + \frac{\sin 2\varepsilon_0}{4(\pi - \varepsilon_0)}\right\}^{1/2}; \quad C_2 = \frac{\sin\varepsilon_0}{\pi - \varepsilon_0};$$

$$C_3 = 3\left\{\frac{1}{2} - \frac{\sin 2\varepsilon_0}{4(\pi - \varepsilon_0)} - \left(\frac{\sin\varepsilon_0}{\pi - \varepsilon_0}\right)^2\right\}^{1/2}.$$

The proofs are similar to those of Lemma 3.3 in Liu and Guo (2007b), we omit it due to space limitations. Using the above lemma, we can get the following results.

Corollary 3.1. For $\theta_i(1)$ defined by (3), we have for large n

$$1) \max_{1\le i\le n} |\tan\theta_i(1)| \le C_4\frac{b_n}{n}, \quad a.s.;$$

$$2) \max_{1\le i\le n} |\cos\theta_i(t) - 1| \le C_5\frac{b_n}{n}, \quad a.s., \quad \forall t \ge 1,$$

where C_4 and C_5 are some constants depending on ε_0 and r only.

Proof. 1) By Lemma 3.1, we have for large n

$$\frac{b_n}{d_i(0)} = O\left(\frac{b_n}{n}\right) = O\left(\sqrt{\frac{\log n}{n}}\right)$$
$$= o(1), \quad a.s., \qquad i = 1, \cdots, n. \tag{12}$$

Therefore, by (3),(12) and Lemma 3.2, we have

$$\max_{1\le i\le n} |\tan\theta_i(1)| = \max_{1\le i\le n}\frac{|\sum_{j\in\mathcal{N}_i(0)}\sin\theta_j(0)|}{|\sum_{j\in\mathcal{N}_i(0)}\cos\theta_j(0)|}$$
$$\le \frac{C_1 b_n}{C_2 d_{min}(0) - C_3 b_n} = O\left(\frac{b_n}{n}\right), \quad a.s.$$

This completes the first inequality of the corollary.

2) First we will consider the asymptotic properties of $\max_{1\le i\le n}(1 - \cos\theta_i(1))$. By (3), for any agent i, we have

$$\cos\theta_i(1) = \frac{\sum_{j\in\mathcal{N}_i(0)}\cos\theta_j(0)}{\Big\{\Big(\sum_{j\in\mathcal{N}_i(0)}\sin\theta_j(0)\Big)^2 + \Big(\sum_{j\in\mathcal{N}_i(0)}\cos\theta_j(0)\Big)^2\Big\}^{\frac{1}{2}}},$$

So by (12) and Lemma 3.2, and the following elementary inequality:

$$\sqrt{a^2 + b^2} \le a + b, \quad a, b \ge 0,$$

we have

$$1 \ge \cos\theta_i(1)$$
$$\ge \frac{C_2 d_i(0) - C_3 b_n}{\{(C_2 d_i(0) + C_3 b_n)^2 + (C_1 b_n)^2\}^{1/2}}$$
$$\ge \frac{C_2 d_i(0) - C_3 b_n}{(C_2 d_i(0) + C_3 b_n) + C_1 b_n}$$
$$= \frac{C_2 - C_3 g_{in}}{C_2 + (C_3 + C_1)g_{in}}, \quad a.s., \tag{13}$$

where

$$g_{in} = \frac{b_n}{d_i(0)} = O\left(\frac{b_n}{n}\right) = o(1), \quad a.s. \tag{14}$$

Furthermore, by (13) and (14), we can get

$$\max_i |\cos\theta_i(1) - 1| \le \max_i \frac{(2C_3 + C_1)g_{in}}{C_2 + (C_3 + C_1)g_{in}}$$
$$= O\left(\frac{b_n}{n}\right) = o(1), a.s. \tag{15}$$

Moreover, by this and 1) of Corollary 3.1, it is easy to see that for large n,

$$\theta_i(1) \in (-\pi/2, \pi/2), \quad \forall i. \tag{16}$$

By (3), we know that $\min_{1\le i\le n} \cos\theta_i(t)$ is non-decreasing for $t \ge 1$, so we have

$$\max_{1\le i\le n}(1-\cos\theta_i(t)) \le \max_{1\le i\le n}(1-\cos\theta_i(1)), \quad \forall t \ge 1,$$

which in conjunction with (15) yields the desired result 2). This completes the proof. ■

3.2 Dealing with the Nonlinearity.

Tang and Guo (2007) proved that the linearized Vicsek model will synchronize as long as the population size is large enough. While the difference between the Vicsek model and its linearized version is the nonlinearity resulting from the heading update equations (3). In this subsection, we will mainly deal with this nonlinearity. Set

$$\Delta P_1(t) \triangleq P^0(t) - P(t) = (\delta p_{ij}(t)), \tag{17}$$

We will estimate the entries $\delta p_{ij}(t)$ first. By the definition of $P(t)$ and $P^0(t)$ (see (6) and (7)), we have

$$\delta p_{ij}(t) = \begin{cases} \dfrac{\cos\theta_i(t)}{\sum_{j\in\mathcal{N}_i(t)}\cos\theta_j(t)} - \dfrac{1}{n_i(t)}, & \text{if } (i,j)\in E_t; \\ 0, & \text{otherwise.} \end{cases}$$

Note that $b_n/n = \sqrt{\log n/n}(1+o(1)) = o(1)$. So for any i and j such that $(i,j) \in E_t$, by 2) of Corollary 3.1, we have

$$\begin{aligned} |\delta p_{ij}(t)| &= \left|\frac{\sum_{j\in\mathcal{N}_i(t)}(\cos\theta_i(t)-\cos\theta_j(t))}{n_i(t)\sum_{j\in\mathcal{N}_i(t)}\cos\theta_j(t)}\right| \\ &\le \frac{2C_5 n_i(t)\frac{b_n}{n}}{n_i^2(t)(1-C_5\frac{b_n}{n})} = \frac{2C_5}{n_i(t)}\frac{b_n}{n}(1+o(1)) \\ &= \frac{2C_5}{n_i(t)}\sqrt{\frac{\log n}{n}}(1+o(1)), \quad a.s., \end{aligned} \tag{18}$$

where $n_i(t)$ is the cardinality of the set $\mathcal{N}_i(t)$. Next, we will estimate $\|\Delta P_1(t)\|$. Denote

$$\lambda_m(t) \triangleq \lambda_{max}\big(\Delta P_1(t)\cdot\Delta P_1(t)^\tau\big).$$

By Gěrschgorin Disk Theorem (cf. Horn and Johnson (1985)), for any eigenvalues λ of the matrix $\Delta P_1(t) \cdot \Delta P_1(t)^\tau$, we have

$$\Big|\lambda - \sum_{l=1}^{n}\delta p_{il}(t)\delta p_{il}(t)\Big| \le \sum_{i\ne j}\sum_{l=1}^{n}|\delta p_{il}(t)\delta p_{jl}(t)|,$$

so we have

$$\lambda_m(t) \le \max_{1\le i\le n}\sum_{j=1}^{n}\sum_{l=1}^{n}|\delta p_{il}(t)\delta p_{jl}(t)|. \tag{19}$$

In order to estimate $\lambda_m(t)$, we need to introduce the following sets:

$$\mathcal{N}_{ij}(t) = \{l : d_{il}(t) < r,\ d_{jl}(t) < r\}$$

and

$$\mathcal{Z}_i(t) = \{j : \mathcal{N}_{ij}(t) \ne \varnothing\}.$$

Moreover denote $n_{ij}(t)$ and $z_i(t)$ as the cardinality of $\mathcal{N}_{ij}(t)$ and $\mathcal{Z}_i(t)$ respectively. Obviously, we have

$$\max_{1\le i,j\le n} n_{ij}(t) \le \min\{n_i(t), n_j(t)\} \le d_{max}(t) \tag{20}$$

and

$$\max_{1\le i\le n} z_i(t) \le \max_{1\le i\le n} n_i'(t), \tag{21}$$

where $n_i'(t)$ is the cardinality of $\mathcal{N}_i'(t)$ defined by (9). By (19) and the above notations, we have:

$$\begin{aligned} \|\Delta P_1(t)\| &= \sqrt{\lambda_m(t)} \\ &\le \max_{1\le i\le n}\sum_{j\in Z_i(t)}\sum_{l\in\mathcal{N}_{ij}(t)}|\delta p_{il}(t)\delta p_{jl}(t)| \\ &\le \sqrt{\max_{1\le i,j\le n} n_{ij}(t)\max_{1\le i\le n} z_i(t)\max_{ij:(i,j)\in E_t}|\delta p_{ij}(t)|^2}. \end{aligned}$$

By substituting (18),(20) and (21) into the above inequality, we have

$$\|\Delta P_1(t)\| \le \frac{2C_5}{d_{min}(t)}\sqrt{\frac{(\log n)d_{max}(t)\max_{1\le i\le n} n_i'(t)}{n}}(1+o(1)), \ a.s. \tag{22}$$

Using the asymptotic properties of the characteristics given in Section 3.1, we can get the following proposition.

Proposition 3.1. Under Assumption 2.1, there exists a positive constant η, such that the following propositions hold almost surely for large n:

1) For any agents i and j, their distance satisfies

$$|d_{ij}(t) - d_{ij}(0)| \le \eta r(1+o(1)), \quad \forall t \ge 1. \tag{23}$$

2) The spectral gap of the initial graph G_0 satisfies:

$$\begin{aligned} &\overline{\lambda}(0) + \sqrt{\beta}\,(\varepsilon_1(t)+\varepsilon_2) \\ &\le 1 - \frac{\pi r^2}{3\cdot 512(r+\sqrt{6})^4}(1+o(1)), \quad \forall t \ge 1, \end{aligned} \tag{24}$$

where

$$\varepsilon_1(t) \triangleq \sup_{1\le s\le t}\|\,\Delta P_1(s)\,\|, \tag{25}$$

$$\varepsilon_2 \triangleq \frac{\min(\sqrt{\pi/64}, \sqrt{\pi r^2/4})\pi r^2}{3\cdot 512(r+\sqrt{6})^4}, \tag{26}$$

where $\Delta P_1(s)$ is defined by (17).

Proof: Denote

$$\delta(t) = \max_{i,j}\{\tan\theta_i(t) - \tan\theta_j(t)\}, \tag{27}$$

which reflects the error of synchronization. Using Corollary 3.1, we have for large n

$$\delta(1) \le \sqrt{2}\max_i|\tan\theta_i(1)| \le \sqrt{2}C_4\frac{b_n}{n}, \ a.s.; \tag{28}$$

$$\|\tan\theta(1)\| \le \sqrt{n}\max_i|\tan\theta_i(1)| \le \sqrt{n}C_4\frac{b_n}{n}, \ a.s. \tag{29}$$

Moreover, by (3) and (16), we can see that $\delta(t), t \geq 1$ is a non-increasing sequence. These facts will be used in the proof to follow.

By the position update law (2), we can deduce that

$$|d_{ij}(t+1) - d_{ij}(t)| \leq v\delta(t+1), \ \forall \ t \geq 0, \tag{30}$$

where $\delta(t)$ is defined by (27).

We will use induction to prove the theorem, and consider the case for $t = 1$ first. By setting $t = 0$ in (30), and using (28), we have

$$\begin{aligned} &|d_{ij}(1) - d_{ij}(0)| \leq v\delta(1) \\ &\leq v\sqrt{2}C_4\frac{b_n}{n} = \sqrt{2}vC_4\sqrt{\frac{\log n}{n}}(1+o(1)) \\ &\leq \eta r, \quad a.s., \end{aligned} \tag{31}$$

Obviously, the last inequality holds for large n. So (23) holds for $t = 1$.

Now, we prove that (24) holds for $t = 1$. By (31), we see that the number of each agent's neighbors changed at time $t = 1$ in comparison with its neighbors at the initial time is bounded by R_{max} defined via (8). So by Lemma 3.1, we have

$$\begin{aligned} &d_{min}(1) \geq d_{min}(0) - R_{max} \\ &\geq \min\left(\frac{\pi r^2}{4}, \frac{\pi}{64}\right) n - 4n\pi\eta r^2 \end{aligned} \tag{32}$$

By taking the constant η small enough and substituting (32) into (22), we can get

$$\begin{aligned} &\|\Delta P_1(1)\| = \sqrt{\lambda_m(1)} \\ &\leq \frac{2C_5}{d_{min}(1)}\sqrt{\frac{(\log n)d_{max}(1)\max\limits_{1\leq i\leq n} n_i'(1)}{n}}(1+o(1)) \\ &= O\left(\sqrt{\frac{\log n}{n}}\right), \quad a.s.; \end{aligned} \tag{33}$$

Furthermore, we can take n large enough such that

$$\begin{aligned} &\|\Delta P_1(1)\| \\ &\leq \frac{\min(\sqrt{\pi/64}, \sqrt{\pi r^2/4})\pi r^2}{3 \cdot 512(r+\sqrt{6})^4} \end{aligned} \tag{34}$$

holds almost surely. By (34), Lemma 3.1, Remark 3.1 and definitions (25) and (26) for $\varepsilon_1(1)$ and ε_2, we have

$$\begin{aligned} &\overline{\lambda}(0) + \sqrt{\beta}\,(\varepsilon_1(1) + \varepsilon_2) \\ &\leq \overline{\lambda}(0) + \max\left(\sqrt{\frac{64}{\pi}}, \sqrt{\frac{4}{\pi r^2}}\right)(\varepsilon_1(1) + \varepsilon_2) \\ &\leq 1 - \frac{\pi r^2}{3 \cdot 512(r+\sqrt{6})^4}(1+o(1)), \quad a.s. \end{aligned}$$

So (24) holds for $t = 1$.

Next, we assume that for all $s \leq t$, (23) and (24) hold, i.e., for any i and j, we have

$$|d_{ij}(s) - d_{ij}(0)| \leq \eta r, \quad a.s., \tag{35}$$

and

$$\begin{aligned} &\overline{\lambda}(0) + \sqrt{\beta}(\varepsilon_1(s) + \varepsilon_2) \\ &\leq 1 - \frac{\pi r^2}{3 \cdot 512(r+\sqrt{6})^4}(1+o(1)), \quad a.s. \end{aligned} \tag{36}$$

We will prove that (23) and (24) hold at time $t+1$. By (30), we have

$$\begin{aligned} &|d_{ij}(t+1) - d_{ij}(0)| \\ &\leq \sum_{s=1}^{t+1}|d_{ij}(s) - d_{ij}(s-1)| \leq v\sum_{j=1}^{t+1}\delta(j), \end{aligned} \tag{37}$$

where $\delta(t)$ is defined by (27), which is a non-increasing sequence for $t \geq 1$. Note that the "linear" time-varying equation for $\tan\theta(t)$ defined by (5) has essentially the same form as that for $\theta(t)$ in Lemma 2 of Tang and Guo (2007), we then have,

$$\begin{aligned} &\delta(s+1) \\ &\leq \sqrt{2\beta}\Big(\overline{\lambda}(0) + \sqrt{\beta}\sup_{1\leq k\leq s} \| \Delta P(k) \| \Big)^s \| \tan\theta(1) \|, \\ &\qquad 0 \leq s \leq t, \end{aligned} \tag{38}$$

where $\Delta P(k) \triangleq P(k) - P^0(0), k \geq 1$ satisfies

$$\begin{aligned} &\sup_{1\leq k\leq s} \| \Delta P(k) \| \\ &\leq \sup_{1\leq k\leq s} \| P(k) - P^0(k) \| + \sup_{1\leq k\leq s} \| P^0(k) - P^0(0) \| . \end{aligned}$$

Now, we estimate $\sup_{1\leq k\leq s}\Delta P(k)$. By definition (25), the first term of the right-hand in the above inequality is exactly $\varepsilon_1(s)$. We now proceed to show that the second term is bounded by ε_2. By the proof of Corollary 1 in Tang and Guo (2007), we can see that it is still applicable. Hence, by Lemma 3.1 and Remark 3.1, the second term satisfies

$$\begin{aligned} &\|P^0(s) - P^0(0)\| \leq \frac{1+\beta}{1-\alpha}\cdot\frac{R_{max}}{d_{min}(0)}(1+o(1)) \\ &\leq \frac{16r^2\left[1+\max(\frac{64}{\pi}, \frac{4}{\pi r^2})\right]}{3\min\left(\frac{1}{64}, \frac{r^2}{4}\right)}\eta, \quad a.s., 1 \leq s \leq t. \end{aligned} \tag{39}$$

So we can take the constant η small enough, such that

$$\begin{aligned} &\|P^0(s) - P^0(0)\| \\ &\leq \frac{\min(\sqrt{\pi/64}, \sqrt{\pi r^2/4})\pi r^2}{3 \cdot 512(r+\sqrt{6})^4}, \quad a.s., 1 \leq s \leq t. \end{aligned} \tag{40}$$

By this, Lemma 3.1, Remark 3.1 and the induction assumption (36), we have

$$\begin{aligned} &\overline{\lambda}(0) + \sqrt{\beta}\sup_{1\leq s\leq t} \| \Delta P(s) \| \\ &\leq \overline{\lambda}(0) + \sqrt{\beta}\left(\sup_{1\leq s\leq t}\varepsilon_1(s) + \varepsilon_2\right) \\ &\leq 1 - \frac{\pi r^2}{3 \cdot 512(r+\sqrt{6})^4}(1+o(1)), \quad a.s. \end{aligned} \tag{41}$$

Now, we are in a position to estimate $v\sum_{j=1}^{t+1}\delta(j)$. Set

$$s_0 \triangleq \min\{s : \sqrt{2}\beta(\lambda')^{s-1} \parallel \tan\theta(1) \parallel \le \delta(1)\},$$

where $\lambda' = 1 - \frac{\pi r^2}{3\cdot 512(r+\sqrt{6})^4}(1+o(1))$, then we have

$$s_0 = \left\lceil \frac{\log \frac{\delta(1)}{\sqrt{2}\beta\|\tan\theta(1)\|}}{\log(\lambda')} + 1 \right\rceil \le \frac{\log \frac{\delta(1)}{\sqrt{2}\beta\|\tan\theta(1)\|}}{\log(\lambda')} + 2.$$

Hence, by (28), (29), (38)and (41), we have for large n

$$v\sum_{s=1}^{t+1}\delta(s) = v\left(\sum_{s=1}^{s_0-1}\delta(s) + \sum_{s=s_0}^{t+1}\delta(s)\right)$$

$$\le v(s_0-1)\delta(1) + v\sqrt{2}\beta(\lambda')^{s_0-1} \parallel \tan\theta(1) \parallel \sum_{s=s_0}^{t+1}(\lambda')^{s-s_0}$$

$$\le v\delta(1)\left(\frac{\log \frac{\delta(1)}{\sqrt{2}\beta\|\tan\theta(1)\|}}{\log\lambda'} + 1 + \frac{1}{1-\lambda'}\right)$$

$$\le \frac{v\delta(1)}{1-\lambda'}\left(2 + \log\frac{\sqrt{2}\beta \parallel \tan\theta(1) \parallel}{\delta(1)}\right)$$

$$= O\left(\sqrt{\frac{\log n}{n}}\log n\right) \le \eta r, \quad a.s., \tag{42}$$

where the following facts are used in the above inequality: $\log\frac{\delta(1)}{\sqrt{\beta} \parallel \tan\theta(1) \parallel} < 0$ and $\log x \le x-1,\ \forall\ 0 < x < 1$. So (23) holds for $s = t+1$ by (37) and (42).

Finally, we prove that (24) holds at time $t+1$. since (23) holds at $t+1$, we see that the number of each agent's neighbors changed at time $t+1$ in comparison with its neighbors at the initial time does not exceed R_{max} defined via (8). Similar to the analysis of (34), we can deduce that

$$\|\Delta P_1(t+1)\| \le \frac{\min(\sqrt{\pi/64}, \sqrt{\pi r^2/4})\pi r^2}{3\cdot 512(r+\sqrt{6})^4}, \quad a.s.$$

Combining this with our induction assumption (36), we can see that (24) holds at time $t+1$. Therefore, by induction, (23) and (24) hold almost surely for all $t \ge 1$. This completes the proof of the proposition. ■

Proof of Theorem 2.1.

By Proposition 3.1, it is easy to prove that the neighbor graphs are connected for any t. So by Theorem 2 in Liu and Guo (2006), we can see that the Vicsek model will synchronize almost surely for large n. Due to space limitations, we omit the details. ■

4. CONCLUDING REMARKS

In this paper, we have shown that for any given model parameters r and v, the multi-agent system described by the Vicsek model will synchronize as long as the population size n is large enough, which provides an analysis for the emergent phenomenon observed by Vicsek *et al.*, without resorting to the connectivity condition imposed on the system dynamics in almost all of the existing studies. Of course, many challenging problems still remain unsolved, for example, the rigorous theoretical analysis for the noise case, nontrivial necessary conditions for synchronization, and the investigation of more complicated models and emergent phenomena, etc.

REFERENCES

D. P. O' Brien (1989). Analysis of the internal arrangement of individuals within crustacean aggregations. *J. Exp. Mar. Bio. Eco.*,128(1): 1-30.

J. Buhl (2006), D. J. T. Sumpter, I. D. Couzin, J. J. Hale, E. Despland, E. R. Miller, and S. J. Simpson. From disorder to order in marching locusts. *Science*, 312: 1402-1406.

F. R. K. Chung (2000). *Spectral Graph Theory.* Providence, RI:AMS.

F. Cucker, and S. Smale (2007). Emergent behavior in flocks. *IEEE Trans. Autom. Control*, 52(5):852-862.

C. Godsil, and G. Royle (2001). *Algebraic Graph Theory.* New York: Springer-Verlag.

J. Han, M. Li, and L. Guo (2006). Soft control on collective behavior of a group of autonomous agents by a shill agent. *J. Systems Science and Complexity*, 19(1):54-62.

R. A. Horn, and C. R. Johnson (1985). *Matrix Analysis*, Cambridge University Press.

A. Jadbabaie, J. Lin, and A. S. Morse (2003). Coordination of groups of mobile autonomous agents using nearest neighbor rules. *IEEE Trans. Autom. Control*, 48(9):988-1001.

Z. X. Liu, and L. Guo (2006). Connectivity and synchronization of multi-agent systems, Proc. 25th Chinese Control Conference, 373-378, Angust 7-11, Hrbin, China.

Z. X. Liu, and L. Guo (2007a). Connectivity and synchronization of Vicsek's model. *Science in China: Series E*, 37(8):979-988.

Z. X. Liu, and L. Guo (2007b). synchronization of Vicsek model with large population. *Proceedings of the 26th Chineses Control Conference*, pages 6-673-6-677, July 26-31, Zhangjiajie, China.

L. Moreau (2005). Stability of multiagent systems with time-dependent communication links. *IEEE Trans. Autom. Control*, 50(2):169-181.

A. Okubo (1986). Dynamical aspects of animal grouping: swarms, schools, flocks and herds. *Adv. Biophys.*, 22: 1-94.

J. K. Parrish, S. V. Viscido, and D. Grunbaum (2002). Self-organized fish schools: an examination of emergent properties. *Biol. Bull.*, 202: 296-305.

M. Penrose (2003). *Random Geometric Graphs.* Oxford University Press.

W. Ren, and R. W. Beard (2005). Consensus seeking in multiagent systems under dynamically changing interaction topologies. *IEEE Trans. Autom. Control*, 50(5):655-661.

R. Olfati-Saber. "Flocking for Multi-Agent Dynamic Systems: Algorithms and Theory," *IEEE Trans. Autom. Control*, 51(3): 401-420

G. G. Tang, and L. Guo (2007). Convergence of a class of multi-agent systems in probabilistic framework. *Journal of Systems Science and Complexity*, 20(2):173-197.

T. Vicsek, A. Czirok, E. B. Jacob, I. Cohen, O. Shochet (1995). Novel type of phase transition in a system of self-driven particles. *Phys. Rev. Lett.*, 75:1226-1229.