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Point-to-point control and trajectory tracking in wheeled mobile robots: some further results and applications

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Abstract: This paper extends previous results in the framework of vehicle control. In particular the paper proposes simple control schemes for driving the vehicle from a given configuration to a prescribed posture in the configuration space, for tracking a time parameterizing path, and for driving a group of vehicles in convoy. We consider the point-to-point and tracking control problems when the controller accounts for the actuator dynamics.

Keywords: Mobile robot, nonholonomic constraints, motion planning, time parameterized path tracking, driving in convoy, actuator dynamics, backstepping.

1. INTRODUCTION

The issues of control and motion planning for nonholonomic wheeled vehicles have been widely pursued. To cite a few examples we list here the works of Murray and Sastry [1993], Laumond *et al.* [1994] and Teel *et al.* [1995], which consider various theoretical issues and the work of Oriolo et al. [2002], which considers also some practical issues of the relevant mobile robot.

This paper considers some applications and further extensions of a previously established motion planning procedure by Ailon *et al.*, [2005]. Though in the present case the model under consideration is nonlinear and quite complicated, the simplicity of the motion planning procedure follows from the concept of flatness Fliess *et al.* [1995], which is useful in solving nonholonomic motion planning problems, and the concept of polynomial controllability Ailon *et al.* [1986]; Ailon and Langholz [1986]; Aeyels [1987] which has been established for linear systems, but in fact will play here a major role. It will be shown that the underlying approach provides useful analytical tools and convenient framework for synthesizing controllers for various control tasks in the considered vehicle model.

We first demonstrate an application of the approach to the control problem of motion of a group of vehicle in a convoy-like fashion, Canudas-de-Wit and NDoudi-Likoho [2000]. As indicated in this latter paper, such a control strategy is important where a multi-body system is required to move in clustered environments such as mine, factories, automatic highway systems, and in military applications associated with automated vehicle missions. In our control approach the convoy leader objective is to track a time parametrized path, namely, a geometric path with an associated timing law that dictates the rate of advancement of the ground vehicle convoy.

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Next, we consider the extension of the proposed control scheme in case the motor dynamics are taken into consideration, Adams [1998], Wang *et al.* [2006]. We assume that in most practical situations it is unrealistic to see the vehicle velocities as input variables that can be selected arbitrarily by the user. In fact the forward and steering velocities are generated by dynamic systems, and thus are not independent variables. Hence we evaluate here some analytical results associated with the underlying control schemes while the motor dynamics regarded as a first order linear dynamic system. Indeed, this does not cover important effects like saturations, transmission and other nonlinear characteristics, but it gives further insight on the effectiveness of the proposed control schemes. Using the backstepping approach proposed by Krstic *et al.* [1995], the new analytical controller becomes more complicated, but its performance effectiveness are still assured. While we consider here a two-wheel differentially driven robot, the present approach can be extended to other robot models, like a car-like mobile robot.

2. PRELIMINARIES

We consider control problems in a mobile robot with two driven wheels (and a free castor wheel) with nonholonomic constraints, whose schematic model is given in Fig. 1.

The state-space model of the system is

$$\begin{aligned} \dot{x} &= u_1 \cos\theta \\ \dot{y} &= u_1 \sin\theta \\ \dot{\theta} &= u_2 \end{aligned} \tag{1}$$

where $\{x, y\}$ denotes the location of the midpoint of the rear axle in the configuration space, and θ is the angle between the x-axis and the central reference line on the cart frame, and u_1 and u_2 are respectively the *driving* and the *steering* velocities of the cart. The initial configuration of the mobile robot at time $t_0 = 0$ is the triple $\{x_0, y_0, \theta_0\}$.

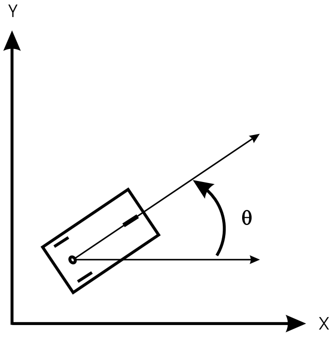

Fig. 1. A schematic model of a wheeled mobile robot.

In what follows we shall apply a pair of functions $f : [0, \infty) \to \Re$ and $g : [x_0, \infty) \to \Re$, with $f \in C^2(0, \infty)$, and $g \in C^3(x_0, \infty)$. Let $\alpha(x) \doteq dg/dx$. Since in motion $x = x(t)$ to simplify notations we write $\alpha(t) = (dg/dx)(t)$ and similarly $\dot{\alpha}(t) = \left[(d^2g/dx^2)(t)\right]\dot{x}(t)$. In our analysis $g(x)$ denotes a geometric path in the plane. Let $x(t) = f(t)$. We consider here a forward motion in the right-hand side of the xy plane, that is $f(t) > 0$ and $\dot{f}(t) \geq \rho$ for all $t > 0$ for an arbitrary selected constant $\rho > 0$, and $\eta \leq \alpha(x) \leq -\eta$ for all $x \geq x_0$ and arbitrary large constant $\eta > 0$. (This restrictions could be removed by applying a coordinate transformation during motion.)

We apply the concept of *flatness* Fliess *et al.* [1995]. It is possible to define a pair $\{f^*(t), g^*(x)\}$ that, together with their derivatives determine the system state variables and inputs signals. That is $x^*(t) = f^*(t)$ and $y^*(t) = (g^* \circ f^*)(t) = g^*(x^*(t))$ imply $\theta^*(t) = \tan^{-1}\alpha^*(t)$, $\alpha^*(t) = (dg^*/dx)(t)$, and $u^*(t) = [u_1^*(t), u_2^*(t)]^T$ is given by

$$u_1^*(t) = \dot{f}^*(t)/\cos(\theta^*(t))$$
$$u_2^*(t) = \dot{\alpha}^*(t)\cos^2\theta^*(t). \qquad (2)$$

If initially $\{x_0, y_0, \theta_0\} = \{x_0^*, y_0^*, \theta_0^*\}$ the input $u^*(t) = [u_1^*(t), u_2^*(t)]^T$ in (2) drives the vehicle along the path $g^*(x^*)$ with $x^*(t) = f^*(t)$ namely,

$$\dot{x}^* = u_1^*\cos\theta^*$$
$$\dot{y}^* = u_1^*\sin\theta^*$$
$$\dot{\theta}^* = u_2^*. \qquad (3)$$

Let Ξ be the real vehicle with the state-space equation (1) and denote a 'dummy vehicle' that satisfies (3) by Ξ^*. Let $e = [e_1, e_2, e_3]^T = [x - x^*, y - y^*, \theta - \theta^*]^T$ be the error vector-valued function between the configurations of Ξ and Ξ^*. Substracting (3) from (1), the error dynamic can be represented by:

$$\begin{bmatrix}\dot{e}_1\\ \dot{e}_2\\ \dot{e}_3\end{bmatrix} = \begin{bmatrix}u_1\cos\theta - u_1^*\cos\theta^*\\ u_1\sin\theta - u_1^*\sin\theta^*\\ u_2 - u_2^*\end{bmatrix}. \qquad (4)$$

Let the input signals in (1) be given by

$$u_1 = [u_1^*\cos\theta^* - \gamma e_1]/\cos(e_3 + \theta^*)$$
$$u_2 = u_2^* - ae_2 - be_3 \qquad (5)$$

where $\{a, b\}$ is a pair of constants. Substituting (5) in (1) we have by elementary trigonometric identities

$$\dot{e}_1 = -\gamma e_1$$
$$\dot{e}_2 = -\gamma e_1\tan(e_3 + \theta^*) + u_1^*\sin e_3/(\cos(e_3 + \theta^*))$$
$$\dot{e}_3 = -ae_2 - be_3. \qquad (6)$$

The following result Ailon *et al.* [2005] will play a major role in this study. Consider the quadratic function

$$V(e) = \frac{1}{2}e^T\begin{bmatrix}\delta & 0 & 0\\ 0 & \delta_1 & 1\\ 0 & 1 & 1\end{bmatrix}e. \qquad (7)$$

A pair of constants $\delta > 0$ and $\delta_1 > 1$ can be computed, and controller gains $\gamma, a, b > 0$ can be selected such that $V(e)$ in (7) is a *Lyapunov function*, that is

$$\dot{V}(e) \leq -W(e, t) < 0 \qquad (8)$$

in a domain $D \subset \Re^3$ that contains $e = 0$ in which $W(e, t)$ is positive definite for all $t \geq 0$ and $e \in D$.

3. THE LEADER-FOLLOWING CONTROLLER

We consider now the problem of n vehicles driving in a convoy, Canudas-de-Wit and NDoudi-Likoho [2000]. The objective is to control a group of vehicles Ξ_1, Ξ_2, $\cdots$, Ξ_n that will converge to a convoy led by the vehicle Ξ^*. We assume knowledge of (smooth) path $g^*(x)$ and forward velocity $u_1^*(t)$ of Ξ^*. We say that the vehicle Ξ_i satisfies the convoy rule if (i) it moves along the path $g^*(x)$ generated by the convoy leader, and (ii)- the traveled distance along the curve $g^*(x)$ to the neighboring vehicle Ξ_{i-1} in front of Ξ_i is l_i.

Consider a *fictitious convoy of vehicles* denoted by Ξ_1^*, Ξ_2^*, $\cdots$, Ξ_n^* that follow the real leading vehicle, namely, Ξ^*. We assume that all starred vehicles move along the $g^*(x)$ path with the same forward velocity of Ξ^*, namely, u_1^*. This implies that if the initial traveled distance along $g^*(x)$ between vehicles Ξ_i^* and Ξ_{i-1}^* with $\Xi_0^* \doteq \Xi^*$ is $l_i(0) = l_i$, then $l_i(t) = l_i$ for all $t \geq 0$. Since $\tan\theta_i = dg^*(x_i)/dx_i$ the steering velocity u_2^* of the vehicle Ξ_i^* is determined as follows. Firstly $\dot{\theta}_i/\cos^2\theta_i = \dot{x}_i\left[d^2g^*(x_i)/dx_i^2\right]$. But $\dot{x}_i = u_1^*\cos\theta_i$ and therefore $u_{2i}^* = \dot{\theta}_i = u_1^*\left[d^2g^*(x_i)/dx_i^2\right]\cos^3\theta_i$. Using $\cos\alpha = 1/\sqrt{1 + \tan^2\alpha}$ we have

$$u_{2i}^* = u_1^*\frac{d^2g^*(x_i)}{dx_i^2}\frac{1}{\left[1 + (dg^*(x_i)/dx_i)^2\right]^{3/2}}. \qquad (9)$$

Hence, the functions $g^*(x)$ and $u_1^*(t)$ associated with the leading vehicle Ξ^* determine explicitly the input signals of the fictitious vehicles Ξ_i^*, $i = 1, 2, \cdots, n$.

Now, the control objective reduces to the original tracking problem in Section 2. Since the state and input variables of the vehicle Ξ_i^* satisfy

$$\dot{x}_i^* = u_1^*\cos\theta_i^*$$
$$\dot{y}_i^* = u_1^*\sin\theta_i^*$$
$$\dot{\theta}_i^* = u_{2i}^* \qquad (10)$$

it is possible to define the input signals to the real vehicle Ξ_i according to (5) (with the starred variables associated with the vehicle Ξ_i^*) such that $\Xi_i \to \Xi_i^*$. Since this is true for all $i = 1, 2, \cdots, n$ all of the vehicles Ξ_i will converge to the convoy, as required.

Remark. While the fictitious vehicles Ξ_i^* have the same forward velocity $u_1^*(t)$ (as of the leader Ξ^*), the forward velocities of the real vehicles Ξ_i are, in general different (but converge to u_1^*).

Example. In this example we demonstrate the control approach for driving a group of three carts in convoy. The results are given in Fig. 2.

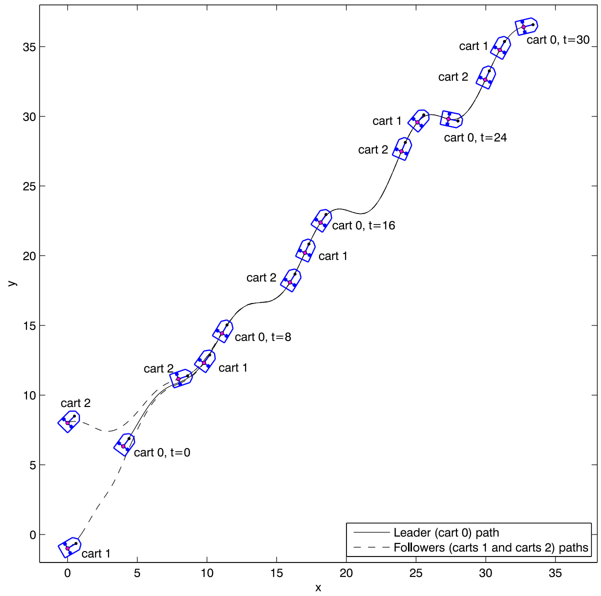

Fig. 2. Three carts driving in convoy.

4. DYNAMIC ACTUATORS

Let ω_r and ω_l be the angular velocities of the right and the left wheels, respectively. The wheel radius is r and the distance between the centers of the two wheels is h. In case of motion without slipping we have

$$\begin{aligned} u_1 &= (\omega_r + \omega_l)\, r/2 \\ u_2 &= (\omega_l - \omega_r)\, r/h. \end{aligned} \tag{11}$$

The considered models of the actuators are

$$\dot{\omega}_i = -\mu\omega_i + \beta v_i; i = r, l \tag{12}$$

where μ and β are positive constants and v_i the input signals to the actuators. Hence, the resulting dynamic of the input signals can be described as follows

$$\begin{aligned} \dot{u}_1 &= (\dot{\omega}_r + \dot{\omega}_l) r/2 = (-\mu\omega_r + \beta v_r - \mu\omega_l + \beta v_l) r/2 \\ &= -\mu u_1 + \beta\,(v_r + v_l)\, r/2 \\ \dot{u}_2 &= (\dot{\omega}_l - \dot{\omega}_r) r/h = (-\mu\omega_l + \beta v_l + \mu\omega_r - \beta v_r) r/h \\ &= -\mu u_2 + \beta\,(v_l - v_r)\, r/h. \end{aligned} \tag{13}$$

Defining $v_1 \doteq \beta\,(v_r + v_l)\, r/2$, $v_2 \doteq \beta\,(v_l - v_r)\, r/h$ we have

$$\begin{aligned} \dot{u}_1 &= -\mu u_1 + v_1 \\ \dot{u}_2 &= -\mu u_2 + v_2. \end{aligned} \tag{14}$$

The augmented state-space model of the kinematic vehicle with the actuator dynamics is thus given by:

$$\begin{aligned} \dot{x} &= u_1 \cos\theta \\ \dot{y} &= u_1 \sin\theta \\ \dot{\theta} &= u_2 \\ \dot{u}_1 &= -\mu u_1 + v_1 \\ \dot{u}_2 &= -\mu u_2 + v_2. \end{aligned} \tag{15}$$

The state vector of (15) is $\varphi = [x, y, \theta, u_1, u_2]^T$ and the input is $v = [v_1, v_2]^T$.

5. STATE-TO-STATE CONTROL

Let $\bar{\varphi}(0) \doteq \bar{\varphi}_0 = [x_0, y_0, \theta_0]^T$ and $\bar{\varphi}(t_f) \doteq \bar{\varphi}_f = [x_f, y_f, \theta_f]^T$ be initial and final configuration vectors. Recall that we consider forward velocity and therefore $u_1(0) > 0$.

Proposition 5.1. Let $-\pi/2 < \theta_0, \theta_f < \pi/2$, and $0 \le x_0 < x_f$. Suppose that a pair of functions $f : [0, t_f] \to \Re$ and $g : [x_0, x_f] \to \Re$ can be selected such that the following holds ($\alpha = dg/dx$):

$$\begin{aligned} f(0) &= x_0; f(t_f) = x_f \\ g(x_0) &= y_0; g(x_f) = y_f \\ \tan^{-1}(\alpha(0)) &= \theta_0; \tan^{-1}(\alpha(t_f)) = \theta_f \\ \dot{f}(0) &= u_{10}\cos(\theta_0)\,; \dot{\alpha}(0) = u_{20}/\cos^2\theta_0 \end{aligned} \tag{16}$$

with $\dot{f}(t) \ge \rho$ for all $t \in [0, t_f]$ for some positive ρ. Set

$$\begin{aligned} x(t) &= f(t) \\ y(t) &= (g \circ f)(t) = g(x(t)) \end{aligned} \tag{17}$$

Then the actuators input vectors $v = [v_1, v_2]^T$ with

$$\begin{aligned} v_1 &= \left(\ddot{f} + \mu\dot{f}\right)/\cos\theta + \dot{f}\dot{\alpha}\alpha\cos\theta \\ v_2 &= \left(\ddot{\alpha} + \mu\dot{\alpha} - 2\dot{\alpha}^2\alpha\cos^2\theta\right)\cos^2\theta \end{aligned} \tag{18}$$

transfers the vehicle from the configuration $\bar{\varphi}_0$ to $\bar{\varphi}_f$.

Proof. The proof follows from the *flatness property* where the (fictitious) flat output $y_f(t) = [x(t), y(t)]^T$ satisfies (17). Observing (17) we have from the first two equations of (15) (recall that we should have $\dot{f} > 0$) $\theta(t) = \tan^{-1}(\alpha(t))$ and $u_1(t) = \dot{f}(t)/\cos\theta(t)$; from the third equation $u_2(t) = \dot{\theta}(t) = \dot{\alpha}(t)\cos^2\theta(t)$ and finally substituting the obtained results for u_i and $\dot{u}_i$ in the last two equations of (15) we have that $y_f(t)$ and its derivatives "generate" the inputs v_i in (18). Going in the opposite direction the input signals v_i in (18) generates the trajectory that satisfies the boundary conditions (16). ◇◇

Assume a pair of initial and final states $\{\bar{\varphi}_0, \bar{\varphi}_f\}$ that satisfies the conditions in Proposition 5.1 is given. We illustrate here a possible motion planning procedure for driving the vehicle from $\bar{\varphi}_0$ to $\bar{\varphi}_f$. To this end we consider the following pair of functions

$$f(t) = \sum_{i=0}^{m} d_i t^i;\;\; g(x) = \sum_{i=0}^{n} a_i \exp(-i\lambda x). \tag{19}$$

where $\lambda > 0$ is an arbitrary constant (which dominates the rate of convergence.) The constants d_i and a_i and the integers m and n are yet to be determined. Since $\alpha(t) = (dg/dx)(t)$, (19) yields

$$\alpha(t) = -\sum_{i=0}^{n} a_i i\lambda \exp\left(-i\lambda f(t)\right)$$
$$\dot{\alpha}(t) = \dot{f}(t)\sum_{i=0}^{n} a_i i^2\lambda^2 \exp\left(-i\lambda f(t)\right). \qquad (20)$$

Using (19) and (20) we consider next the boundary conditions. Observing the equations in (16) and (19) we have:

$$d_0 = x_0;\ \sum_{i=0}^{m} d_i t_f^i = x_f$$
$$d_1 = u_{10}\cos\theta_0;\ \sum_{i=1}^{m-1} d_i i t^{i-1} \geq \rho. \qquad (21)$$

Remark. Since $x_0 < x_f$ and $0 < u_{10}$ it is always possible to determine a polynomial $f(t)$ in (19) that satisfies (21) (for some $\rho > 0$).

Equation (16) yields the following set of equations

$$\sum_{i=0}^{n} a_i \exp(-i\lambda x_0) = y_0$$
$$\sum_{i=0}^{n} a_i \exp(-i\lambda x_f) = y_f$$
$$\sum_{i=0}^{n} a_i i\lambda \exp(-i\lambda x_0) = -\tan\theta_0$$
$$\sum_{i=0}^{n} a_i i\lambda \exp(-i\lambda x_f) = -\tan\theta_f$$
$$\sum_{i=0}^{n} a_i i^2\lambda^2 \exp\left(-i\lambda x_0\right) = u_{20}/\left(u_{10}\cos^3\theta_0\right). \qquad (22)$$

Choose $n \geq 4$ and define $\tau_0 \doteq e^{-\lambda x_0}$ and $\tau_f \doteq e^{-\lambda x_f}$. Then (22) becomes (for simplicity we take $\lambda = 1$)

$$\begin{bmatrix} 1 & \tau_0 & \tau_0^2 & \tau_0^3 & \tau_0^4 & \tau_0^5 & \cdots \\ 1 & \tau_f & \tau_f^2 & \tau_f^3 & \tau_f^4 & \tau_f^5 & \cdots \\ 0 & \tau_0 & 2\tau_0^2 & 3\tau_0^3 & 4\tau_0^4 & 5\tau_0^5 & \cdots \\ 0 & \tau_f & 2\tau_f^2 & 3\tau_f^3 & 4\tau_f^4 & 5\tau_f^5 & \cdots \\ 0 & \tau_0 & 4\tau_0^2 & 9\tau_0^3 & 16\tau_0^4 & 25\tau_0^5 & \cdots \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_4 \\ \vdots \end{bmatrix}$$
$$= \begin{bmatrix} y_0 \\ y_f \\ -\tan\theta_0 \\ -\tan\theta_f \\ u_{20}/\left(u_{10}\cos^3\theta_0\right) \end{bmatrix}. \qquad (23)$$

Proposition 5.2. The matrix $H(\tau_0, \tau_f) \in \Re^{5\times(n+1)}$ on the left hand-side of (23) is of full raw rank. In particular for $n = 4$ the equation has a unique solution for $\mathbf{a} = [a_0, a_1, \cdots, a_4]^T$.

Proof. Applying a sequence of elementary row operations, and recalling the mean value theorem it can be shown that the first five columns of H are linearly independent for any $0 < \tau_f < \tau_0$, which asserts the claim. $\diamondsuit\diamondsuit$

6. TRAJECTORY FOLLOWING

This section presents a simple open loop control procedure for driving towards a desired geometric path $g^*(x)$, and a closed-loop control scheme for trajectory following.

6.1 Open loop control for tracking time parameterized paths

Let $g^*(x)$ be an assigned path in the right-hand side of the (x, y) plane. Let $f(t) \doteq f^*(t)$ with $\dot{f}^*(t) \geq \rho > 0$ for $t \geq 0$, be a polynomial function (see (19)) that satisfies the following conditions (see (21)):

$$d_0 = x_0;\ d_1 = u_{10}\cos\theta_0. \qquad (24)$$

Set $x(t) = x^*(t) = f^*(t)$. Let Ξ^* be a fictitious vehicle that moves along the path $g^*(x)$. That is, Ξ^* satisfies (3). Being a fictitious system we may determine arbitrarily the initial coordinate x_0^* of Ξ^*. Thus we set $x_0^* = x_0$. This implies that $y_0^* = g^*(x_0^*)$. By previous results and the flatness property the pair $\{f^*(t), g^*(x^*(t))\}$ determines $\theta^*(t)$ and $u_i^*(t)$ for $t \geq 0$. The objective is to drive the vehicle Ξ so that it will converge to the fictitious one Ξ^* which moves along the path $g^*(x)$.

To this end we set

$$y = g(x) \doteq g^*(x^*) - \sum_{i=0}^{n} a_i \exp\left(-i\lambda x^*\right) \qquad (25)$$

Fix $\lambda > 0$ (the factor that dominates the rate of convergence). Next, compute the constants a_i such that all the initial conditions are satisfied. That is (observe (22) and recall that by assumption the forward velocity is positive)

$$\sum_{i=0}^{n} a_i \exp(-i\lambda x_0) = g^*(x_0^*) - y_0$$
$$\sum_{i=0}^{n} a_i i\lambda \exp(-i\lambda x_0) = -\tan\theta_0^* + \tan\theta_0$$
$$\sum_{i=0}^{n} a_i i^2\lambda^2 \exp\left(-i\lambda x_0\right) = u_{20}^*/U_0^* - u_{20}/U_0 \qquad (26)$$

where $U_0^* \doteq u_{10}^*\cos^3\theta_0^*$ and $U_0 \doteq u_{10}\cos^3\theta_0$. Writing the last equation in a compact form (we take again $\lambda = 1$):

$$\begin{bmatrix} 1 & \tau_0 & \tau_0^2 & \cdots \\ 0 & \tau_0 & 2\tau_0^2 & \cdots \\ 0 & \tau_0 & 4\tau_0^2 & \cdots \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \end{bmatrix}$$
$$= \begin{bmatrix} g^*(x_0^*) - y_0 \\ -\tan\theta_0^* + \tan\theta_0 \\ u_{20}^*/\left(u_{10}^*\cos^3\theta_0^*\right) - u_{20}/\left(u_{10}\cos^3\theta_0\right) \end{bmatrix}. \qquad (27)$$

The matrix on the left-hand side is of full raw rank and for $n = 2$ it has a unique solution for $\mathbf{a} = [a_0, a_1, a_2]^T$.

Using the pair of functions $f(t) = f^*(t)$ and $g(x)$ in (25) and the flatness property one can determine the input $v = [v_1, v_2]^T$ in (18) such that the resulting flat output is $x(t) = x^*(t)$ and $y(t) = g(x(t)$ with (recall that $x_0 \geq 0$ and $f^*(t)$ increases monotonically) $\lim_{t\to\infty} y(t) = g^*(x^*(t)$. Hence the vehicle Ξ converges to Ξ^* exponentially.

Remark. Since $f^*(t)$ is determined *a-priori* we have here an open loop control system.

6.2 A closed-loop control scheme for tracking a time parameterized path

We recall that $e = [e_1, e_2, e_3]^T = [x - x^*, y - y^*, \theta - \theta^*]^T$ we have using (1), (3) and (15) (note that $\theta = e_3 + \theta^*$)

$$\begin{aligned} \dot{e}_1 &= u_1 \cos(e_3 + \theta^*) - u_1^* \cos\theta^* \\ \dot{e}_2 &= u_1 \sin(e_3 + \theta^*) - u_1^* \sin\theta^* \\ \dot{e}_3 &= u_2 - u_2^* \\ \dot{u}_1 &= -\mu u_1 + v_1 \\ \dot{u}_2 &= -\mu u_2 + v_2. \end{aligned} \tag{28}$$

We seek for a control scheme such that the vehicle Ξ will converge asymptotically to the fictitious vehicle Ξ^*. Towards this goal we apply the *backstepping* procedure Fierro and Lewis [1997] and recalling the controller proposed in Ailon *et al.* [2005] we extend the approach to the current augmented model. From Section 2, if the actuator dynamics are ignored we have that Ξ with $u = [u_1, u_2]^T$ in (5) converges to Ξ^* exponentially. Hence, rewriting (28)

$$\begin{aligned} \dot{e}_1 &= \left[\frac{(u_1^* \cos\theta^* - \gamma e_1)}{\cos(e_3 + \theta^*)} \right. \\ &\quad \left. + \left(u_1 - \frac{(u_1^* \cos\theta^* - \gamma e_1)}{\cos(e_3 + \theta^*)} \right) \right] \cos(e_3 + \theta^*) \\ &\quad - u_1^* \cos\theta^* \\ &= -\gamma e_1 + \left(u_1 - \frac{(u_1^* \cos\theta^* - \gamma e_1)}{\cos(e_3 + \theta^*)} \right) \cos(e_3 + \theta^*) \\ \dot{e}_2 &= \left[\frac{(u_1^* \cos\theta^* - \gamma e_1)}{\cos(e_3 + \theta^*)} \right. \\ &\quad \left. + \left(u_1 - \frac{(u_1^* \cos\theta^* - \gamma e_1)}{\cos(e_3 + \theta^*)} \right) \right] \sin(e_3 + \theta^*) \\ &\quad - u_1^* \sin\theta^* \\ &= -\gamma e_1 \tan(e_3 + \theta^*) + u_1^* \frac{\sin e_3}{\cos(e_3 + \theta^*)} \\ &\quad + \left(u_1 - \frac{(u_1^* \cos\theta^* - \gamma e_1)}{\cos(e_3 + \theta^*)} \right) \sin(e_3 + \theta^*) \\ \dot{e}_3 &= [u_2^* - a e_2 - b e_3 \\ &\quad + (u_2 - (u_2^* - a e_2 - b e_3))] - u_2^* \\ &= -a e_2 - b e_3 + (u_2 - (u_2^* - a e_2 - b e_3)) \\ \dot{u}_1 &= -\mu u_1 + v_1 \\ \dot{u}_2 &= -\mu u_2 + v_2. \end{aligned} \tag{29}$$

Define

$$\begin{aligned} z_1 &\doteq u_1 - \frac{(u_1^* \cos\theta^* - \gamma e_1)}{\cos(e_3 + \theta^*)} = u_1 - f_1(e_1, e_3, t) \\ z_2 &\doteq u_2 - (u_2^* - a e_2 - b e_3) = u_2 - f_2(e_2, e_3, t). \end{aligned} \tag{30}$$

Using (29) the derivatives of z_i are given by

$$\begin{aligned} \dot{z}_1 &= \dot{u}_1 - \left[\frac{\partial f_1}{\partial t} + \frac{\partial f_1}{\partial e_1} \dot{e}_1 + \frac{\partial f_1}{\partial e_3} \dot{e}_3 \right] = \dot{u}_1 + \psi_1 \\ \dot{z}_2 &= \dot{u}_2 - \left[\frac{\partial f_2}{\partial t} + \frac{\partial f_2}{\partial e_2} \dot{e}_2 + \frac{\partial f_2}{\partial e_3} \dot{e}_3 \right] = \dot{u}_2 + \psi_2 \\ \psi_1 &\doteq - \left[\frac{\partial f_1}{\partial t} + \frac{\partial f_1}{\partial e_1} \dot{e}_1 + \frac{\partial f_1}{\partial e_3} \dot{e}_3 \right] \\ \psi_2 &\doteq - \left[\frac{\partial f_2}{\partial t} + \frac{\partial f_2}{\partial e_2} \dot{e}_2 + \frac{\partial f_2}{\partial e_3} \dot{e}_3 \right] \end{aligned} \tag{31}$$

Note that the terms $\dot{e}_i$ are defined by the right-hand sides of (29). Hence, at this stage we obtain the dynamic equation in terms of e_i and z_i as follows

$$\begin{aligned} \dot{e}_1 &= -\gamma e_1 + z_1 \cos(e_3 + \theta^*) \\ \dot{e}_2 &= -\gamma e_1 \tan(e_3 + \theta^*) + u_1^* \sin e_3 / \cos(e_3 + \theta^*) \\ &\quad + z_1 \sin(e_3 + \theta^*) \\ \dot{e}_3 &= -a e_2 - b e_3 + z_2 \\ \dot{z}_1 &= -\mu u_1 + \psi_1 + v_1 \\ \dot{z}_2 &= -\mu u_2 + \psi_2 + v_2. \end{aligned} \tag{32}$$

Let the scalar function

$$V_c(e, z) = V(e) + \frac{1}{2} z_1^2 + \frac{1}{2} z_2^2 \tag{33}$$

where $z = [z_1, z_2]^T$ and $V(e)$ is given by (7), be a *Lyapunov candidate function* for (32) . Recall (6) and (8), the derivative of $V_c(e)$ along the trajectories of (32) is given by (see (7))

$$\begin{aligned} \dot{V}_c(e) &= \delta e_1 \dot{e}_1 + \delta_1 e_2 \dot{e}_2 + \dot{e}_2 e_3 + e_2 \dot{e}_3 + e_3 \dot{e}_3 \\ &\quad + z_1 \dot{z}_1 + z_2 \dot{z}_2 \\ &= \dot{V}(e) + z_1 (\delta e_1 \cos(e_3 + \theta^*) \\ &\quad + (\delta_1 e_2 + e_3) \sin(e_3 + \theta^*)) \\ &\quad + z_2 (e_2 + e_3) + z_1(-\mu u_1 + \psi_1 + v_1) \\ &\quad + z_2 (-\mu u_2 + \psi_2 + v_2) \\ &\le -W(e, t) + z_1 g_1(e, t) + z_2 g_2(e, t) \\ &\quad + z_1 v_1 + z_2 v_2 \end{aligned} \tag{34}$$

where (see (8)) $-W(e, t) < 0$ and

$$\begin{aligned} g_1(e, t) &\doteq \delta e_1 \cos(e_3 + \theta^*) \\ &\quad + (\delta_1 e_2 + e_3) \sin(e_3 + \theta^*) - \mu u_1 + \psi_1 \\ g_2(e, t) &\doteq e_2 + e_3 - \mu u_2 + \psi_2. \end{aligned} \tag{35}$$

Let the control law in (32) be given by

$$\begin{aligned} v_1 &= -g_1(e, t) - z_1 \\ v_2 &= -g_2(e, t) - z_2. \end{aligned} \tag{36}$$

Then, observing (30), (35), and (36) it is clear that the state vector $\xi = [e_1, e_2, e_3, z_1, z_2]^T = 0$ is an equilibrium point of the resulting closed-loop system (32). Furthermore, using (36) we have in (34)

$$\dot{V}_c(e) \le -W(e, t) - z_1^2 - z_2^2 < 0. \tag{37}$$

Hence, the origin of (32) is asymptotically stable.

Remark. To remove some constraints let x and y be the coordinates of a given vector in the original frame, say $\{\}_0$, in Fig. 1, and x_1 and y_1 be the coordinates of the same point in a frame $\{\}_1$. If p and p_1 are the representations of the same point in $\{\}_0$ and $\{\}_1$ respectively, we have $p = R_\eta p_1 + d$ where η is the angle of rotation between the two frames, R_η is the rotation (orthogonal) matrix and $d = [d_x, d_y]^T$ is the vector from the origin of $\{\}_0$ to the origin of $\{\}_1$, expressed in the coordinate system $\{\}_0$. This allows us to extend the approach and to apply the controllers for motion in the entire xy plane.

Example. The example demonstrates the performance of a controller for tracking a time parameterized path in a mobile robot in which the velocity signals are generated by a first order linear models. The results are demonstrated in Figs. 3 and 4.

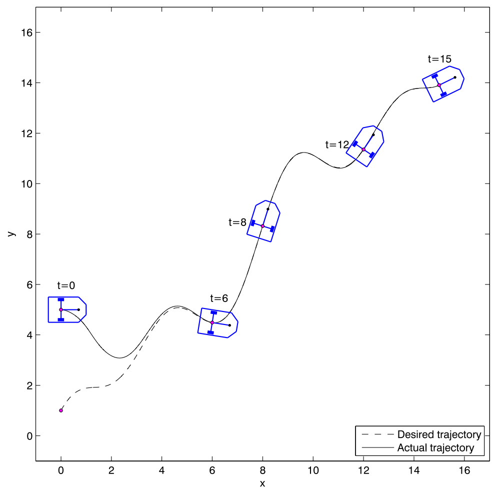

Fig. 3. Tracking a time parameterized path in a mobile robot with model that includes the actuator dynamics.

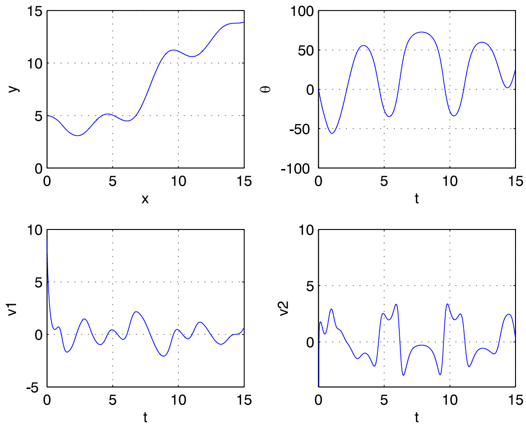

Fig. 4. Time histories of the state and input variables.

7. CONCLUSION

This paper presents a simple approach for motion planning and control of a unicycle-type vehicle with nonholonomic constraints. The established control algorithms are based on the concept of flatness. Open and closed-loop control schemes have been studied. Simple computational tools support the derivation of the required control strategies for regulating the vehicle motion while it converges to a prescribed trajectory and when it follows a leader of a group of vehicles moving in convoy.

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