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Stability and Passivity of Complex Spatio-Temporal Switching Networks with Coupling Delays ⋆

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Abstract: In this paper, a new model of complex dynamical network is established. It is called a complex spatio-temporal switching network. That is, the subsystem of each node at different time interval corresponds to different switching mode. Based on passivity property, the stability and passivity of this kind of network with time-delay interconnections are addressed. An example and simulation results are included.

1. INTRODUCTION

Complex networks are currently being studied across many fields of science and engineering Strogatz [2001], stimulated by the fact that many real systems, such as social, biological, and communication systems can be described by models of complex networks. Random networks, small-world networks and scale-free networks are most noticeable among all kinds of complex networks Bollobás [1985], Strogatz [2001], Wang and Chen [2003], whose nodes represent individuals and links represent the interactions among them. In view of the wide occurrence of complex networks in nature, it is important to study the effects of topological structures and their dynamical processes.

Many complex networks are hybrid in the sense that they have both discrete-state and continuous-state dynamics. In addition, these two aspects of the system behaviors often interact to a significant extent that they cannot be decoupled and must be analyzed simultaneously. Our previous work on complex dynamical networks with switching topology has been studied in Guan, Yang, and Yao [2007], Yao er al. [2006]. Note that dynamical behaviours of complex networks not only depends on their structures but also dynamics of their nodes. If each subsystem is a switching system, that is, switching signal of complex network depends on both instant time t and the index of node i, then we call it as a complex spatio-temporal switching network. In this paper, it has been formulated for the first time. Furthermore, it is well known that the information flow in complex networks is not instantaneous in general. Hence, the finite speed of signal transmission over a distance gives rise to a finite time delay. The dynamics of networks with delayed coupling have been extensively studied in recent years. For example, in Earl, and Strogatz [2003], Li, Xu et al. [2004], Yeung and Strogatz [1999], the effects of time delay on a specific coupled oscillators network was discussed. The work in Angeli, and Bliman [2005], Li and Chen [2004a,b], Li et al. [2004], Lee and Spong [2006], Yang [2002] focuses on dynamical behaviors in delayed networks.

In this paper, we address the stability and passivity of complex dynamical networks, including the presence of communication time-delays. Based on the Lyapunov functional method, corresponding results are derived in terms of solutions of appropriate algebraic inequalities. The paper is organized as follows: a class of complex spatio-temporal switching networks with time-delay interconnections is presented in Section 2. Its stability and passivity property is addressed in Section 3. Section 4 contains a numerical example. Concluding remarks are collected in Section 5.

2. PROBLEM FORMULATION

Let $R_+ = [0, +\infty)$, $J = [t_0, +\infty)\,(t_0 \geq 0)$, and R^n denote the n-dimensional Euclidean space. For $x = (x_1, \ldots, x_n)^\top \in R^n$, the norm of x is $\|x\| := \Big(\sum_{i=1}^{n} x_i^2\Big)^{\frac{1}{2}}$. Correspondingly, for $A = (a_{ij})_{n\times n} \in R^{n\times n}$, $\|A\| := \lambda_{\max}^{\frac{1}{2}}\big(A^T A\big)$. The identity matrix of order n is denoted as I_n (or simply I if no confusion arises).

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Consider a complex spatio-temporal switching network with time-delay interconnections, consisting of N identical coupled nodes, with each node being an $n-$dimensional dynamical system. This dynamical network is described by

$$\begin{cases} \dot{x}_i(t) = A_\sigma x_i(t) + f_i(t, x_i(t)) + E_i w_i(t) \\ + \sum_{j=1}^{N} D_{ij} x_j(t - \tau_i), \qquad i = 1, 2, \cdots N \\ z_i(t) = C_i x_i(t) \end{cases} \tag{1}$$

where $t \in J\ (t_0 \geq 0), f_i : R^n \to R^n$ is a nonlinear vector-valued function with $f_i(t, 0) \equiv 0, t \in J$. $x_i = (x_{i1}, x_{i2}, \cdots, x_{in})^\top \in R^n$ are the state variables of node i. $w_i \in R^q$ and $z_i \in R^q$ denote the exogenous input vector and the output vector of each subsystem, respectively. $\tau_i > 0$ represents the delay for each node, and C_i and E_i are known matrices with appropriate dimensions. $D = (D_{ij})_{N \times N}$ is the coupling configuration matrix. If there is a connection between node i and node j $(i \neq j)$, then $D_{ij} = D_{ji} > 0$; otherwise, $D_{ij} = D_{ji} = 0$ $(i \neq j)$, and the diagonal elements of matrix D are defined by

$$D_{ii} = -\sum_{\substack{j=1 \\ j \neq i}}^{N} D_{ij} = -\sum_{\substack{j=1 \\ j \neq i}}^{N} D_{ji} \qquad i = 1, 2, \cdots N.$$

Switching signal $\sigma = \sigma(i, t)$ is a piecewise constant function. Note that switching signal is dependent on instant time t and the index of node i. So it is called a complex spatio-temporal switching network. In general, the subsystem of each node at different time interval corresponds to different switching mode. That is, A_σ takes a constant value on every interval between two consecutive switching times, which gives rise to a finite set $A_\sigma = \{A_1, \cdots, A_m\}$.

For convenience, we define

$$\xi_l(i, t) = \begin{cases} 1, & \text{the switching mode } A_l \text{ is active at time} \\ & t_k \text{ of node } i, \\ 0, & \text{others,} \end{cases}$$

with discontinuity points

$$t_1 < t_2 < \cdots < t_k < \cdots, \quad \lim_{k \to \infty} t_k = \infty$$

where $t_1 > t_0$, $i = 1, 2, \cdots N$, $k = 1, 2, \cdots$, $l = 1, 2, \cdots m$.

Let $A(\xi(i, t)) = \sum_{l=1}^{m} \xi_l(i, t) A_l$, then the isolate node of the spatio-temporal switching network can be written as

$$\begin{aligned} \dot{x}_i(t) &= \sum_{l=1}^{m} \xi_l(i, t) A_l x_i(t) + f_i(t, x_i(t)), \\ &= A(\xi(i, t)) x_i(t) + f_i(t, x_i(t)), \qquad t \in (t_{k-1}, t_k], \\ &\qquad i = 1, 2, \cdots N, \qquad k = 1, 2, \cdots \end{aligned}$$

Accordingly, the complex spatio-temporal switching network with time-delay interconnections (1) becomes

$$\begin{cases} \dot{x}_i(t) = A(\xi(i, t)) x_i(t) + f_i(t, x_i(t)) + E_i w_i(t) \\ + \sum_{j=1}^{N} D_{ij} x_j(t - \tau_i), \qquad t \in (t_{k-1}, t_k], \quad k = 1, 2, \cdots \\ z_i(t) = C_i x_i(t), \qquad i = 1, 2, \cdots N \end{cases} \tag{2}$$

In what follows, the stability and passivity of the spatio-temporal switching network (2) is first studied, and then, an example is investigated.

3. STABILITY AND PASSIVITY OF SWITCHING NETWORKS WITH DELAY INTERCONNECTIONS

In the subsequent discussion, the following definition will be needed.

Definition 3.1. (Hill and Moylan [1980], Lozano et al. [2000]) A system with input w and output z where $w(t), z(t) \in R^q$ is said to be passive if there is a constant β such that

$$\int_0^T w^\top(s) z(s) ds \geq -\beta^2 \tag{3}$$

for all $T \geq 0$. If in addition, there are constants $\varepsilon_1 \geq 0$ and $\varepsilon_2 \geq 0$ such that

$$\begin{aligned} \int_0^T w^\top(s) z(s) ds \geq -\beta^2 + \varepsilon_1 \int_0^T w^\top(s) w(s) ds \\ + \varepsilon_2 \int_0^T z^\top(s) z(s) ds \end{aligned} \tag{4}$$

for all $T \geq 0$, then the system is input strictly passive if $\varepsilon_1 > 0$, output strictly passive if $\varepsilon_2 > 0$.

Let us note that (2) will be (strictly) passive if it satisfies the above inequalities (3) or (4). Several sufficient conditions for stability and passivity of the spatio-temporal switching network (2) can be stated as follows.

For system (2), assume that, for $t \in J, x_i \in R^n$, there exist continuous functions $\varphi_i(t) \geq 0$ and positive definite matrices P_i, such that

$$f_i^\top(t, x_i) P_i x_i \leq \varphi_i(t) x_i^\top P_i x_i, \quad i = 1, 2, \cdots, N. \tag{5}$$

Remark 3.1. For nonlinear function $f_i(t, x_i)$, the inequality (5) is less conservative than the Lipschitz condition which is usually assumed in literature, see Remark 3.1 in Guan, Hill, and Shen [2005].

Theorem 3.1. If there exist continuous functions $\varphi_i(t) \geq 0$, positive-definite matrices $P_i > 0$, $Q_i > 0$ and a scalar $\lambda \leq 0$ such that

$$\begin{aligned} \Omega(t, P_i) :=& A_l^\top P_i + P_i A_l + 2\varphi_i(t) P_i + N Q_i \\ &+ \sum_{j=1}^{N} P_i D_{ij} Q_j^{-1} D_{ij} P_i \leq \lambda C_i^\top C_i, \\ &\qquad i = 1, 2, \cdots N, l = 1, 2, \cdots m \end{aligned} \tag{6}$$

where $C_i = E_i^\top P_i$, then the complex spatio-temporal switching network with time-delay interconnections (2) is stable and passive under arbitrary switching.

Proof. Select a Lyapunov candidate function as

$$v(x_t) = \sum_{i=1}^{N} \Big\{ x_i^\top(t) P_i x_i(t) + \sum_{j=1}^{N} \int_{t-\tau_i}^{t} x_j^\top(s) Q_j x_j(s) ds \Big\}.$$

For $t \in (t_{k-1}, t_k]$, the differentiation of $v(x_t)$ along with the state trajectory of system (2) is

$$\begin{aligned} \dot{v}(x_t)\Big|_{(2)} = \sum_{i=1}^{N} \Big\{ & x_i^\top(t)(A^\top(\xi(i, t)) P_i + P_i A(\xi(i, t))) x_i(t) \\ & + 2 f_i^\top(t, x_i) P_i x_i + 2 x_i^\top(t) P_i E_i w_i(t) \end{aligned}$$

$$+\sum_{j=1}^{N}\Big(x_i^\top(t)P_iD_{ij}x_j(t-\tau_i)$$
$$+x_j^\top(t-\tau_i)D_{ij}P_ix_i(t)+x_j^\top(t)Q_jx_j(t)$$
$$-x_j^\top(t-\tau_i)Q_jx_j(t-\tau_i)\Big)\Big\}$$
$$\leq\sum_{i=1}^{N}\Big\{x_i^\top(t)\Big(A^\top(\xi(i,t))P_i+P_iA(\xi(i,t))+NQ_i$$
$$+2\varphi_i(t)P_i+\sum_{j=1}^{N}P_iD_{ij}Q_j^{-1}D_{ij}P_i\Big)x_i(t)-$$
$$\sum_{j=1}^{N}\varphi(x_i(t),x_j(t-\tau_i))+2x_i^\top(t)P_iE_iw_i(t)\Big\}, \qquad (7)$$

where

$$\varphi(x_i(t),x_j(t-\tau_i))=\Big[Q_j^{-1}D_{ij}P_ix_i(t)-x_j(t-\tau_i)\Big]^\top Q_j$$
$$\times\Big[Q_j^{-1}D_{ij}P_ix_i(t)-x_j(t-\tau_i)\Big].$$

Since only one subsystem of each node at every instant time is active, let $\xi_{i_k}(i,t)=1$, $t\in(t_{k-1},t_k]$, then from (7) we have

$$\dot v(x_t)\leq\sum_{i=1}^{N}\Big\{x_i^\top(t)\Omega(t,P_i)x_i(t)-\sum_{j=1}^{N}\varphi(x_i(t),x_j(t-\tau_i))$$
$$+2x_i^\top(t)P_iE_iw_i(t)\Big\}, \qquad (8)$$

Note that $\Omega(t,P_i)\leq\lambda C_i^\top C_i\leq 0$ and $\varphi(x_i(t),x_j(t-\tau_i))\geq 0$. Thus, from (6) and (8), it follows that $\dot v(x_t)\leq 0$, when $w_i(t)=0$. Then internal stable of the switching network (2) is verified.

Moreover, for any given $T\geq 0$, we have

$$2\int_0^T w^\top(s)z(s)ds$$
$$=2\int_0^T\sum_{i=1}^{N}w_i^\top(s)z_i(s)ds=2\sum_{i=1}^{N}\int_0^T w_i^\top(s)C_ix_i(s)ds$$
$$=\sum_{i=1}^{N}\Big(\int_0^T w_i^\top(s)E_i^\top P_ix_i(s)ds+\int_0^T x_i^\top(s)P_iE_iw_i(s)ds\Big)$$
$$=\sum_{i=1}^{N}\int_0^T\Big\{\Big(\frac{dx_i(s)}{ds}-A_{i_k}x_i(s)-f_i(s,x_i(s))$$
$$-\sum_{j=1}^{N}D_{ij}x_j(s-\tau_i)\Big)^\top P_ix_i(s)+x_i^\top(s)P_i\Big(\frac{dx_i(s)}{ds}$$
$$-A_{i_k}x_i(s)-f_i(s,x_i(s))-\sum_{j=1}^{N}D_{ij}x_j(s-\tau_i)\Big)\Big\}ds$$
$$\geq\int_0^T\frac{dv(x(s))}{ds}ds+\sum_{i=1}^{N}\int_0^T\Big(-x_i^\top(s)\Omega(s,P_i)x_i(s)$$
$$+\sum_{j=1}^{N}\varphi(x_i(s),x_j(s-\tau_i))\Big)ds$$
$$\geq v(x(T))-v(x(0))-\int_0^T\sum_{i=1}^{N}\lambda x_i^\top(s)C_i^\top C_ix_i(s)ds$$
$$\geq -v(x(0))-\lambda\int_0^T z^\top(s)z(s)ds.$$

That is,

$$\int_0^T w^\top(s)z(s)ds\geq-\frac{1}{2}v(x(0))-\frac{\lambda}{2}\int_0^T z^\top(s)z(s)ds.$$

If $\lambda=0$, then the system is passive. For $\lambda<0$, it gives strictly passive property of the spatio-temporal switching network (2). This immediately completes the proof. $\diamondsuit$

4. NUMERICAL EXAMPLE

In this section, we give an example to demonstrate the effectiveness of the proposed methods.

We consider a complex spatio-temporal switching network (2) with 10 coupled nodes in which the dynamics of each node is periodically switched between three modes. They are

$$A_1=\begin{bmatrix}-220 & -18 & -2\\ 0 & -77.2 & 0\\ 7 & 0 & -182.7\end{bmatrix},\ A_2=\begin{bmatrix}-175 & 10 & 2\\ 28 & -80 & 0\\ -8 & 0 & -96.6\end{bmatrix},$$
$$A_3=\begin{bmatrix}-180.9 & 0 & 2\\ 2 & -120.9 & 0\\ -7.9 & -30.99 & -80.5\end{bmatrix},\ f(t,x_i)=(2sin(t),0,0)^\top,$$

where $i=1,2,\cdots,N$. All the coupling strengths are chosen to be $D_{ij}=0.28$, $(i\neq j)$, and

$$D=0.28\times\begin{bmatrix}-4&1&1&0&0&0&0&0&1&1\\ 1&-5&1&1&1&0&0&0&0&1\\ 1&1&-5&1&1&0&0&0&0&1\\ 0&1&1&-5&1&1&1&0&0&0\\ 0&1&1&1&-5&1&1&0&0&0\\ 0&0&0&1&1&-5&1&1&1&0\\ 0&0&0&1&1&1&-5&1&1&0\\ 0&0&0&0&0&1&1&-4&1&1\\ 1&0&0&0&0&1&1&1&-5&1\\ 1&1&1&0&0&0&0&1&1&-5\end{bmatrix}.$$

Since different node maybe has different coupling delays, let $\tau_1=0.4$, $\tau_2=0.2$, $\tau_3=0.12$, $\tau_4=0.18$, $\tau_5=0.08$, $\tau_6=0.02$, $\tau_7=0.5$, $\tau_8=0.01$, $\tau_9=0.06$, $\tau_{10}=0.11$.

Thus, in (5), $\varphi_i(t)=0$. Let $\lambda=0$. There exist

$$P_i=\begin{bmatrix}0.0327 & 0.0094 & -0.0003\\ 0.0094 & 0.1270 & -0.0480\\ -0.0003 & -0.0480 & 0.0908\end{bmatrix},$$

and

$$Q_i=\begin{bmatrix}1.0114 & -0.0117 & 0.0082\\ -0.0117 & 0.9835 & 0.0226\\ 0.0082 & 0.0226 & 0.9818\end{bmatrix},$$

such that the inequality (6) holds, which implies, from Theorem 3.1, that the complex spatio-temporal switching

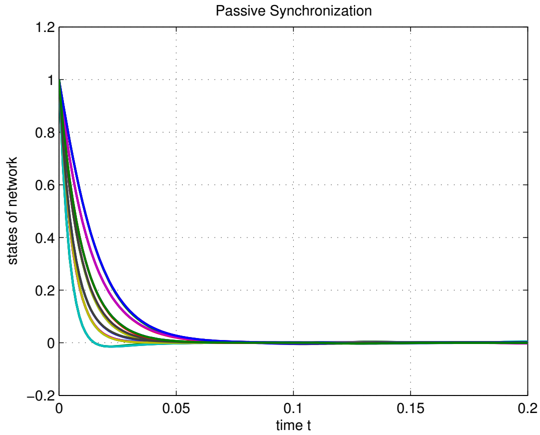

Fig. 1. The trajectories of the switched dynamical network

network with time-delay interconnections (2) is stable and passive under arbitrary switching. The trajectories of the switched dynamical network (2) are depicted in Fig.1.

5. CONCLUSIONS

This paper has formulated a class of complex spatio-temporal switching networks with coupling delays. The problem of stability and passivity for the dynamical networks are taken into account. Sufficient conditions have been derived in terms of solutions of some algebraic inequalities. An example is provided to verify the effectiveness of the proposed stability analysis.

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