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Linear Quadratic Regulation for Continuous-Time Systems with Time-Varying Delay ⋆

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Abstract: In this paper we investigate the finite horizon linear quadratic regulation (LQR) problem for a linear continuous time system with time-varying delay in control input and a quadratic criterion. We assume that the time-varying delay is of a known upper bound, then the LQR problem is transformed into the optimal control problem for systems with multiple input channels, each of which has single constant delay. The purpose of this paper is to obtain an explicit solution to the addressed LQR problem via establishing a duality principle, which is applied to the optimal smoothing for an associated continuous time system with a multiple delayed measurement.

1. INTRODUCTION

This paper considers the finite horizon LQR problem for continuous time systems with time-varying delay in control input, which is perfectly meaningful in both theory and application. The potential applications of the proposed LQR problem for linear systems with time-varying delay in control input are related to a large number of remote control problems in networked control systems and wireless sensor networks, where the input-state link, as well as state output one, is intrinsically subject to communication delays Nilsson et al. [1998], Sinopoli et al. [2005].

linear system states has been well studied, and their solutions and properties have been well documented, as well as the filtering one, since 1960s, such as Kucera [1979]. While the solution to linear quadratic regulation for continuous time systems has been well known, the same problem for systems with delays, especially with time-varying delays remains difficult. For continuous time systems, the time delay problems can in principle be treated by the infinite-dimension system theory Delfour [1986], Keulen [1993]. However it leads to a solution in terms of operator Riccati equations which difficult to be understood and implemented. In Kharatashvili [1967], a maximum principle was used to discuss systems with delays while the dynamic programming method was applied to a specific time-delay case in Oguztoreli [1966]. It should be noted that no explicit formula for optimal control law was given in these works. In Liu [2006], the LQR optimal controller was implemented for single input and single output system by applying a special conversion from the transfer function to the state space expression. In very recent years, the optimal control problems for systems with multiple input delays under both H_2 and H_∞ performance criterion have received much attention and several important progresses have been made, such as Moelja et al. [2005], Zhang et al. [2006], Kojima et al. [2006]. In Moelja et al. [2005], the H_2 optimal control problem of systems with multiple i/o delays was proposed and discussed by considering the regulator problem in time-domain as a linear quadratic regulator problem with multiple input delays. Kojima et al. [2006] addressed the H_∞ preview control for systems with multiple delay and provided an explicit LQ control law based an operator Riccati equation.

In this paper we first establish a duality between the LQR problem for systems with multiple input delays and a smoothing problem for a backward stochastic delay free system. In doing so, the complicated LQR problem for systems with time-varying delay is transformed into a smoothing one, and the obtained optimal controller gain matrix is constructed as dual transpose to the optimal smoother gain one and the optimal controller gain equation is obtained as dual to the variance equation in the optimal smoother.

In the interest of space, proofs of lemmas and theorems in this paper are omitted.

2. PROBLEM STATEMENT

This paper studies linear continuous time systems with single input delay described by

$$\dot{x}(t) = \Phi_t x(t) + \Gamma_t u(t - h(t)) \tag{1}$$

where $h(t)$ is time-varying delay, $x(t) \in R^n$ and $u(t) \in R^m$ are measurable state and input vectors at time t; Φ_t and Γ_t are the state transition and input distribution matrices, respectively. Our aim is to solve linear time-varying quadratic regulation problem with time-varying

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input delay by finding control input $u(t-h(t))(h(t) \le t \le t_f)$ of system (1) that minimizing the finite horizon cost

$$J_{t_f} = x'_{t_f} P_{t_f} x_{t_f} + \int_0^{t_f} u'(t) R_t u(t) dt + \int_0^{t_f} x'(t) Q_t x(t) dt \quad (2)$$

where x_{t_f} is the terminal state, $P_{t_f} = P'_{t_f} \ge 0$ imposes a penalty weighting matrix on the terminal state. $R_t > 0$ and $Q_t \ge 0$ are bounded matrix functions. We assume that all control inputs have a finite maximum delay, that is to say the time-varying input delay $h(t)$ can be chosen in an interval $[h_0, \cdots, h_N]$, where $0 = h_0 < h_1 < \cdots < h_N$.

Note that the control input is single and input delay is time-varying and can be chosen in a known interval $[h_0, \cdots, h_N]$ which is bounded, the finite horizon LQR problem with time-varying input delay can be transformed into the one with multiple input channels by defining the following variable to model the control input,

$$\gamma_{t,i} \triangleq \begin{cases} 1, \text{ the input delay equal to } h_i \text{ at time } t; \\ 0, \text{ otherwise.} \end{cases} \quad (3)$$

In consideration of the property of h_i that it can only be chosen one value from the interval $[h_0, \cdots, h_N]$ at time t, we can know apparently that

$$\gamma_{t,i} \times \gamma_{t,j} = \begin{cases} 1, \text{ if } i = j; \\ 0, \text{ if } i \ne j. \end{cases} \quad (4)$$

In consequence, the system (1) can be rewritten as

$$\dot{x}(t) = \Phi_t x(t) + \sum_{i=0}^{N} \gamma_{t,i} \Gamma_t u(t - h_i) \quad (5)$$

and the associated quadratic cost function as

$$J_{t_f} = x'_{t_f} P_{t_f} x_{t_f} + \sum_{i=0}^{N} \int_0^{t_f - h_i} u'(t) \gamma_{t,i} R_t u(t) dt + \int_0^{t_f} x'(t) Q_t x(t) dt \quad (6)$$

3. PRELIMINARIES

We first introduce the following notations:

$$\bar{u}(t) \triangleq \begin{cases} \begin{bmatrix} u(t-h_0) \\ \vdots \\ u(t-h_i) \end{bmatrix}, \ h_i \le t < h_{i+1}; \\ \begin{bmatrix} u(t-h_0) \\ \vdots \\ u(t-h_N) \end{bmatrix}, \ t \ge t_N, \end{cases} \quad (7)$$

$$\tilde{u}(t) \triangleq \begin{cases} \sum_{j=i+1}^{N} \Gamma_{t,j} u(t-h_j), \ h_i \le t < h_{i+1}; \\ 0, \quad t \ge h_N, \end{cases} \quad (8)$$

$$\bar{\Gamma}_t \triangleq \begin{cases} [\Gamma_{t,0}, \Gamma_{t,1}, \cdots, \Gamma_{t,i}], \ h_i \le t < h_{i+1}; \\ [\Gamma_{t,0}, \Gamma_{t,1}, \cdots, \Gamma_{t,N}], \ t \ge h_N, \end{cases} \quad (9)$$

$$\bar{R}_t \triangleq \begin{cases} diag\{R_{t-h_0,0}, R_{t-h_1,1}, \cdots, R_{t-h_i,i}\}, \\ \qquad h_i \le t < h_{i+1}; \\ diag\{R_{t-h_0,0}, R_{t-h_1,1}, \cdots, R_{t-h_N,N}\}, \\ \qquad t \ge h_N, \end{cases} \quad (10)$$

where $\Gamma_{t,i} = \gamma_{t,i}\Gamma_t$ and $R_{t-h_i,i} = \gamma_{t,i} R_{t-h_i}$, respectively. By using the above notations, the system (5) can be rewritten as

$$\dot{x}(t) = \begin{cases} \Phi_t x(t) + \bar{\Gamma}_t \bar{u}(t) + \tilde{u}(t), \ h_i \le t < h_{i+1}; \\ \Phi_t x(t) + \bar{\Gamma}_t \bar{u}(t), \quad t \ge h_N. \end{cases} \quad (11)$$

and the associated cost function (6) as

$$J_{t_f} = x'_{t_f} P_{t_f} x_{t_f} + \int_0^{t_f} \bar{u}'(t) \bar{R}_t \bar{u}(t) dt + \int_0^{t_f} x'(t) Q_t x(t) dt. \quad (12)$$

Now we introduce the following backward dual state space model (the details can be seen in Zhang et al. [2006]):

$$-\dot{\mathbf{x}}(t) = \Phi'_t \mathbf{x}(t) + \mathbf{w}(t), \quad (13)$$

$$\mathbf{y}(t) = \bar{\Gamma}'_t \mathbf{x}(t) + \mathbf{v}(t) \quad (14)$$

with terminal state $\mathbf{x}(t_f) = \mathbf{x}_{t_f}$ and

$$\left\langle \begin{bmatrix} \mathbf{x}_{t_f} \\ \mathbf{w}(t) \\ \mathbf{v}(t) \end{bmatrix}, \begin{bmatrix} \mathbf{x}_{t_f} \\ \mathbf{w}(\tau) \\ \mathbf{v}(\tau) \end{bmatrix} \right\rangle = \begin{bmatrix} P_{t_f} & 0 & 0 \\ 0 & Q_t \delta(t-\tau) & 0 \\ 0 & 0 & \bar{R}_t \delta(t-\tau) \end{bmatrix}.$$

In view of (9)and (14), it is obvious that $\mathbf{y}(t)$ has the dimension

$$dim\{\mathbf{y}(t)\} = \begin{cases} (i+1)m \times 1, \ h_i \le t < h_{i+1}; \\ (N+1) \times 1, \ t \ge h_N. \end{cases} \quad (15)$$

Next, we shall denote the Gramian operator corresponding to the continuously indexed collections $\mathbf{y} = \{\mathbf{y}(t), 0 \le t \le t_f\}$ by $R_{\mathbf{y}}$, which is determined by its kernels $R_{\mathbf{y}}(t,\tau) = \langle \mathbf{y}(t), \mathbf{y}(\tau) \rangle$; and denote the cross-Gramian operator corresponding to the continuously indexed collections $\mathbf{y} = \{\mathbf{y}(t), 0 \le t \le t_f\}$ and $\mathbf{x} = \{\mathbf{x}(s), 0 \le s \le t_N\}$ by $R_{\mathbf{yx}}$, which is determined by its kernels $R_{\mathbf{yx}}(t,s) = \langle \mathbf{y}(t), \mathbf{x}(s) \rangle$. Then take equation (13)-(14) into account, the cost function (12) can be further rewritten as

$$J_{t_f} = \begin{bmatrix} x(0) \\ \tilde{u} \\ \bar{u} \end{bmatrix}' \left\langle \begin{bmatrix} \mathbf{x}(0) \\ \mathbf{x} \\ \mathbf{y} \end{bmatrix}, \begin{bmatrix} \mathbf{x}(0) \\ \mathbf{x} \\ \mathbf{y} \end{bmatrix} \right\rangle \begin{bmatrix} x(0) \\ \tilde{u} \\ \bar{u} \end{bmatrix} = \begin{bmatrix} \xi \\ \bar{u} \end{bmatrix}' \begin{bmatrix} R_{\mathbf{x}_0} & R_{\mathbf{x}_0\mathbf{y}} \\ R_{\mathbf{y}\mathbf{x}_0} & R_{\mathbf{y}} \end{bmatrix} \begin{bmatrix} \xi \\ \bar{u} \end{bmatrix}, \quad (16)$$

where $\xi = \begin{bmatrix} x(0) \\ \tilde{u} \end{bmatrix}$, $\mathbf{x}_0 = \begin{bmatrix} \mathbf{x}(0) \\ \mathbf{x} \end{bmatrix}$ and $R_{\mathbf{x}_0} = \langle \mathbf{x}_0, \mathbf{x}_0 \rangle$, $R_{\mathbf{x}_0\mathbf{y}} = \langle \mathbf{x}_0, \mathbf{y} \rangle = R'_{\mathbf{y}\mathbf{x}_0}$ and $R_{\mathbf{y}} = \langle \mathbf{y}, \mathbf{y} \rangle$; $\bar{u}$ and $\tilde{u}$ denote the continuously indexed collections $\{\bar{u}(t), 0 \le t \le t_f\}$ and $\{\tilde{u}(s), 0 \le s \le h_N\}$, respectively.

Now recall the Krein space stochastic model (13) and (14) and in view of $\bar{\Gamma}_t$ defined in (9), the measurement $\mathbf{y}(t)$ and the noise $\mathbf{v}(t)$ can be decomposed as follows:

$$\mathbf{y}(t) = \begin{cases} col\{\mathbf{y}_0(t), \cdots, \mathbf{y}_i(t)\}, \ h_i \le t < h_{i+1}; \\ col\{\mathbf{y}_0(t), \cdots, \mathbf{y}_N(t)\}, \ t \ge h_N, \end{cases} \quad (17)$$

$$\mathbf{v}(t) = \begin{cases} col\{\mathbf{v}_0(t), \cdots, \mathbf{v}_i(t)\}, \ h_i \le t < h_{i+1}; \\ col\{\mathbf{v}_0(t), \cdots, \mathbf{v}_N(t)\}, \ t \ge h_N, \end{cases} \quad (18)$$

where $\mathbf{y}_i(t)$ and $\mathbf{v}_i(t)$ with dimension $m \times 1$ satisfy the following equation

$$\mathbf{y}_i(t) = \gamma_{t,i}\Gamma_t'\mathbf{x}(t) + \gamma_{t,i}\mathbf{v}_i(t), \quad (19)$$

and the inner product of $\mathbf{v}_i(t)$ in Krein space is

$$\langle \mathbf{v}_i(t), \mathbf{v}_i(s)\rangle = R_{t-h_i}\delta(t-s).$$

Next, we reorganize the decomposed measurements $\mathbf{y}_i(t)$ and noise $\mathbf{v}_i(t)$ $(i = N, \cdots, 1)$ as follows

$$\check{\mathbf{y}}(t) = \begin{cases} col\{\gamma_{t,0}\mathbf{y}_0(t+h_0), \cdots, \gamma_{t,N}\mathbf{y}_N(t+h_N)\}, \\ \qquad 0 \le t \le t_f - h_N; \\ col\{\gamma_{t,0}\mathbf{y}_0(t+h_0), \cdots, \gamma_{t,i-1}\mathbf{y}_{i-1}(t+h_{i-1})\}, \\ \qquad t_f - h_i < t \le t_f - h_{i-1} \end{cases} \quad (20)$$

$$\check{\mathbf{v}}(t) = \begin{cases} col\{\gamma_{t,0}\mathbf{v}_0(t+h_0), \cdots, \gamma_{t,N}\mathbf{v}_N(t+h_N)\}, \\ \qquad 0 \le t \le t_f - h_N; \\ col\{\gamma_{t,0}\mathbf{v}_0(t+h_0), \cdots, \gamma_{t,i-1}\mathbf{v}_{i-1}(t+h_{i-1})\}, \\ \qquad t_f - h_i < t \le t_f - h_{i-1} \end{cases} \quad (21)$$

and they satisfy

$$\check{\mathbf{y}}(t) = \begin{cases} \begin{bmatrix} \gamma_{t,0}\Gamma'_{t+h_0}\mathbf{x}(t+h_0) \\ \cdots \\ \gamma_{t,N}\Gamma'_{t+h_N}\mathbf{x}(t+h_N) \end{bmatrix} + \check{\mathbf{v}}(t), \\ \qquad 0 \le t \le t_f - h_N; \\ \begin{bmatrix} \gamma_{t,0}\Gamma'_{t+h_0}\mathbf{x}(t+h_0) \\ \cdots \\ \gamma_{t,i-1}\Gamma'_{t+h_{i-1}}\mathbf{x}(t+h_{i-1}) \end{bmatrix} + \check{\mathbf{v}}(t), \\ \qquad t_f - h_i < t \le t_f - h_{i-1}. \end{cases} \quad (22)$$

In the same manner, we can further reorganize the control input $\bar{u}(t)$ defined in (7) as

$$\check{u}(t) = \begin{cases} col\{\overbrace{u(t), \cdots, u(t)}^{N+1\ blocks}\}, 0 \le t \le t_f - h_N; \\ col\{\overbrace{u(t), \cdots, u(t)}^{i\ blocks}\}, t_f - h_i < t \le t_f - h_{i-1}. \end{cases} \quad (23)$$

In virtue of the cost function (16) and the above descriptions, we can obtain the following result.

Lemma 1. The cost function (16) can be reformulated equivalently in the following quadratic form:

$$J_{t_f} = \begin{bmatrix} \xi \\ \check{u} \end{bmatrix}' \check{\Pi} \begin{bmatrix} \xi \\ \check{u} \end{bmatrix}, \quad (24)$$

where

$$\check{\Pi} = \left\langle \begin{bmatrix} \mathbf{x}_0 \\ \check{\mathbf{y}} \end{bmatrix}, \begin{bmatrix} \mathbf{x}_0 \\ \check{\mathbf{y}} \end{bmatrix} \right\rangle = \begin{bmatrix} R_{\mathbf{x}_0} & R_{\mathbf{x}_0\check{\mathbf{y}}} \\ R_{\check{\mathbf{y}}\mathbf{x}_0} & R_{\check{\mathbf{y}}} \end{bmatrix},$$

ξ and $\mathbf{x}_0$ are as defined in (16), $\check{u}$ and $\check{\mathbf{y}}$ denote the continuously indexed collections $\{\check{u}(t), 0 \le t \le t_f\}$ and $\{\check{y}(t), 0 \le t \le t_f\}$, respectively.

In order to obtain a more simpler form of cost function (24), we introduce the following notation:

$$\mathbf{z}(t) = \begin{cases} \sum_{j=0}^{N} \gamma_{t,j}\mathbf{y}_j(t+h_j), 0 \le t \le t_f - h_N; \\ \sum_{j=1}^{i} \gamma_{t,j-1}\mathbf{y}_{j-1}(t+h_{j-1}), \\ \qquad t_f - h_i < t \le t_f - h_{i-1}, \end{cases} \quad (25)$$

and it satisfies

$$\mathbf{z}(t) = \begin{cases} \sum_{j=0}^{N} \gamma_{t,j}\Gamma'_{t+h_j}\mathbf{x}(t+h_j) + \mathbf{v}_z(t), \\ \qquad 0 \le t \le t_f - h_N; \\ \sum_{j=1}^{i} \gamma_{t,j-1}\Gamma'_{t+h_{j-1},j-1}\mathbf{x}(t+h_{j-1}) + \mathbf{v}_z(t), \\ \qquad t_f - h_i < t \le t_f - h_{i-1}, \end{cases} \quad (26)$$

where

$$\mathbf{v}_z(t) = \begin{cases} \sum_{j=0}^{N} \gamma_{t,j}\mathbf{v}_j(t+h_j), 0 \le t \le t_f - h_N; \\ \sum_{j=1}^{i} \gamma_{t,j-1}\mathbf{v}_{j-1}(t+h_{j-1}), \\ \qquad t_f - h_i < t \le t_f - h_{i-1}, \end{cases} \quad (27)$$

with zero means and covariance

$$Q_t^{\mathbf{v}_z} = \begin{cases} \sum_{j=0}^{N} \gamma_{t,j}R_t, \quad 0 \le t \le t_f - h_N; \\ \sum_{j=1}^{i} \gamma_{t,j-1}R_t,\ t_f - h_i < t \le t_f - h_{i-1}. \end{cases}$$

On the basis of the above description, we can get a new performance index equivalent to (24) according to the following lemma.

Lemma 2. The cost function (24) is equivalent to the following quadratic form

$$J_{t_f} = \begin{bmatrix} \xi \\ u \end{bmatrix}' \mathbf{M} \begin{bmatrix} \xi \\ u \end{bmatrix}, \quad (28)$$

where

$$\mathbf{M} = \left\langle \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{z} \end{bmatrix}, \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{z} \end{bmatrix} \right\rangle = \begin{bmatrix} R_{\mathbf{x}_0} & R_{\mathbf{x}_0\mathbf{z}} \\ R_{\mathbf{z}\mathbf{x}_0} & R_{\mathbf{z}} \end{bmatrix},$$

and $\mathbf{x}_0$ is as in (16), u and $\mathbf{z}$ are the continuously indexed collections, i.e.,

$$u = \{u(t), 0 \le t \le t_f\},$$
$$\mathbf{z} = \{\mathbf{z}(t), 0 \le t \le t_f\}.$$

The cost function (28) can be further rewritten as

$$J_{t_f} = \xi'\mathcal{P}\xi + (u - u^*)'R_{\mathbf{z}}(u - u^*), \quad (29)$$

where

$$u^* = \{u^*(t), 0 \le t \le t_f\} = -R_{\mathbf{z}}^{-1}R_{\mathbf{z}\mathbf{x}_0}\xi, \quad (30)$$
$$\mathcal{P} = \langle \mathbf{x}_0 - \hat{\mathbf{x}}_0,\ \mathbf{x}_0 - \hat{\mathbf{x}}_0\rangle, \quad (31)$$
$$\hat{\mathbf{x}}_0 = \{\hat{\mathbf{x}}(t),\ 0 \le t \le t_f\}, \quad (32)$$

and u^* is the minimizing solution of the cost function (28) and $\hat{\mathbf{x}}(t)$ is the projection of state $\mathbf{x}(t)$ onto the linear space $\mathcal{L}\{\mathbf{z}(t),\ 0 \le t \le t_f\}$. In that

$$R_{\mathbf{z}\mathbf{x}_0} = \left\langle \mathbf{z},\ \begin{bmatrix} \mathbf{x}(0) \\ \mathbf{x} \end{bmatrix} \right\rangle = [R_{\mathbf{z}\mathbf{x}(0)},\ R_{\mathbf{z}\mathbf{x}}],$$

the minimizing solution (30) can be redescribed by the following form

$$u^* = -R_{\mathbf{z}}^{-1} R_{\mathbf{z}\mathbf{x}(0)} x(0) - R_{\mathbf{z}}^{-1} R_{\mathbf{z}\mathbf{x}} \tilde{u}. \tag{33}$$

Now, we define a new measurement

$$\mathbf{z}_i(t) \triangleq \gamma_{t,i} \Gamma'_{t+h_i} \mathbf{x}(t+h_i) + \gamma_{t,i} \mathbf{v}_i(t+h_i) \tag{34}$$

and denote

$$\tilde{\mathbf{z}}(t) = \begin{cases} \begin{bmatrix} \mathbf{z}_0(t-h_0) \\ \mathbf{z}_1(t-h_1) \\ \vdots \\ \mathbf{z}_N(t-h_N) \end{bmatrix}, \; t \geq h_N; \\ \begin{bmatrix} \mathbf{z}_0(t-h_0) \\ \mathbf{z}_1(t-h_1) \\ \vdots \\ \mathbf{z}_i(t-h_i) \end{bmatrix}, \quad h_i \leq t < h_{i+1}, \end{cases} \tag{35}$$

and it satisfies that

$$\tilde{\mathbf{z}}(t) = \tilde{\Gamma}'_t \mathbf{x}(t) + \tilde{\mathbf{v}}(t), \tag{36}$$

where

$$\tilde{\Gamma}'_t = \begin{cases} col\{\gamma_{t,0}\Gamma'_t, \gamma_{t-h_1,1}\Gamma'_t, \cdots, \gamma_{t-h_N,N}\Gamma'_t\}, t \geq h_N; \\ col\{\gamma_{t,0}\Gamma'_t, \gamma_{t-h_1,1}\Gamma'_t, \cdots, \gamma_{t-h_i,i}\Gamma'_t\}, h_i \leq t < h_{i+1}, \end{cases}$$

$$\tilde{\mathbf{v}}(t) = \begin{cases} col\{\gamma_{t,0}\mathbf{v}_0(t), \gamma_{t-h_1,1}\mathbf{v}_1(t), \cdots, \gamma_{t-h_N,N}\mathbf{v}_N(t)\}, \\ \qquad t \geq h_N; \\ col\{\gamma_{t,0}\mathbf{v}_0(t), \gamma_{t-h_1,1}\mathbf{v}_1(t), \cdots, \gamma_{t-h_i,i}\mathbf{v}_i(t)\}, \\ \qquad h_i \leq t < h_{i+1}. \end{cases}$$

In view of $\mathbf{v}_z(t)$ defined in (27), it is readily to obtain that $\tilde{\mathbf{v}}(t)$ is white noise with zero mean and covariance matrix

$$Q_t^{\tilde{\mathbf{v}}} = \begin{cases} diag\{\gamma_{t,0}R_t, \gamma_{t-h_1,1}R_{t-h_1}, \cdots, \gamma_{t-h_N,N}R_{t-h_N}\}, \\ \qquad t \geq h_N; \\ diag\{\gamma_{t,0}R_t, \gamma_{t-h_1,1}R_{t-h_1}, \cdots, \gamma_{t-h_i,i}R_{t-h_i}\}, \\ \qquad h_i \leq t < h_{i+1}. \end{cases} \tag{37}$$

Seen from function (35), $\tilde{\mathbf{z}}(t)$ is composed of different measurement associated with the same state $\mathbf{x}(t)$. Obviously, there has no delay existed any more in measurement equation (36). Moreover, the following lemma is a truth.

Lemma 3. The linear space spanned by the reorganized measurements sequence (35) is equivalent to the one spanned by the measurements sequence (26), i.e.,

$$\mathcal{L}\{\tilde{\mathbf{z}}(t),\ 0 \leq t \leq t_f\} = \mathcal{L}\{\mathbf{z}(t),\ 0 \leq t \leq t_f\}. \tag{38}$$

Under the above lemma, the minimizing solution (30) can be further given as

$$u^* = -R_{\tilde{\mathbf{z}}}^{-1} R_{\tilde{\mathbf{z}}\mathbf{x}(0)} x(0) - R_{\tilde{\mathbf{z}}}^{-1} R_{\tilde{\mathbf{z}}\mathbf{x}} \tilde{u}. \tag{39}$$

Now that $\tilde{u}(t) = 0$ for $t \geq h_N$, (39) can also be represented as

$$u^* = -R_{\tilde{\mathbf{z}}}^{-1} R_{\tilde{\mathbf{z}}\mathbf{x}(0)} x(0) - \int_0^{h_N} R_{\tilde{\mathbf{z}}}^{-1} R_{\tilde{\mathbf{z}}\mathbf{x}(t)} \tilde{u}(t) dt \tag{40}$$

To sum up, the key sight to solve the optimal LQR problem with time-varying delay is to compute the filtering gain matrix $R_{\mathbf{x}(0)\tilde{\mathbf{z}}} R_{\tilde{\mathbf{z}}}^{-1}$ and the smoothing gain matrix $R_{\mathbf{x}(t)\tilde{\mathbf{z}}} R_{\tilde{\mathbf{z}}}^{-1}$ $(0 < t \leq h_N)$ of system (13) and (36).

4. SOLUTION TO THE PROPOSED PROBLEM

By applying the standard Kalman filtering formulations to the stochastic backward systems (13) and (36), the filtering estimate $\hat{\mathbf{x}}(t|t)$ can be calculated as

$$\begin{aligned} -\dot{\hat{\mathbf{x}}}(t|t) &= \Phi'_t \hat{\mathbf{x}}(t|t) + K_t[\tilde{\mathbf{z}}(t) - \tilde{\Gamma}'_t \hat{\mathbf{x}}(t|t)] \\ &= \bar{\Phi}_t \hat{\mathbf{x}}(t|t) + K_t \tilde{\mathbf{z}}(t), \end{aligned} \tag{41}$$

where

$$K_t Q_{\tilde{\mathbf{v}}}(t) = P(t)\tilde{\Gamma}_t, \tag{42}$$

$$\bar{\Phi}_t = \Phi'_t - K_t \tilde{\Gamma}'_t, \tag{43}$$

and the estimation error covariance matrix $P(t)$ obeys the following backward differential Riccati equation

$$-\dot{P}(t) = \Phi'_t P(t) + P(t)\Phi_t + Q_t - K_t Q_{\tilde{\mathbf{v}}}(t) K'_t \tag{44}$$

with the terminal condition $P(t_f) = P_{t_f}$. Let $\bar{\Psi}(t,\tau)$ be the transition matrix of $-\bar{\Phi}_t$, then we have

$$\hat{\mathbf{x}}(t|t) = \bar{\Psi}(t, t_f)\hat{\mathbf{x}}(t_f|t_f) - \int_{t_f}^{t} \bar{\Psi}(t,\tau) K_\tau \tilde{\mathbf{z}}(\tau) d\tau \tag{45}$$

with $\hat{\mathbf{x}}(t_f|t_f) = 0$, and for $t = 0$,

$$\hat{\mathbf{x}}(0|0) = \int_0^{t_f} \bar{\Psi}(t,\tau) K_\tau \tilde{\mathbf{z}}(\tau) d\tau. \tag{46}$$

Next, we give the smoother $\hat{\mathbf{x}}(t|0)$ according to the following lemma.

Lemma 4. The smoother $\hat{\mathbf{x}}(t|0)$ corresponding to the stochastic backward system (13) and (36) can be calculated as

$$\begin{aligned} \hat{\mathbf{x}}(t|0) = &\int_t^{t_f} \bar{\Psi}(t,\tau) K_\tau \tilde{\mathbf{z}}(\tau) d\tau + \int_0^t P(t)\bar{\Psi}'(\tau,t)\tilde{K}_\tau \tilde{\mathbf{z}}(\tau) d\tau \\ &- \int_0^t P(t)\bar{\Psi}'(\tau,t)\tilde{K}_\tau \tilde{\Gamma}'_\tau \left[\int_\tau^{t_f} \bar{\Psi}(\tau,s) K_s \tilde{\mathbf{z}}(s) ds \right] d\tau \end{aligned} \tag{47}$$

where

$$\tilde{K}_\tau Q_{\tilde{\mathbf{v}}}(\tau) = \tilde{\Gamma}_\tau. \tag{48}$$

In order to rewrite the smoother described in (47), we define the following notations,

$$I_0(\tau) \triangleq \begin{cases} [I, \overbrace{0, \cdots, 0}^{i\ blocks}]',\ h_i \leq \tau < h_{i+1}; \\ [I, \overbrace{0, \cdots, 0}^{N\ blocks}]',\ \tau \geq h_N. \end{cases} \tag{49}$$

$$I_i(\tau) \triangleq \begin{cases} [\underbrace{\overbrace{0, \cdots, 0, I}^{i+1\ blocks} 0, \cdots, 0}_{j\ blocks}]',\ h_{j-1} - h_i \leq \tau < h_j - h_i; \\ [\underbrace{\overbrace{0, \cdots, 0, I}^{i+1\ blocks} 0, \cdots, 0}_{N+1\ blocks}]',\ \tau \geq h_N - h_i. \end{cases} \tag{50}$$

By using the above notations, we can obtain the following corollary.

Corollary 5. The projection $\hat{\mathbf{x}}(t \mid 0)$ can be rewritten as

$$\hat{\mathbf{x}}(t \mid 0) = \int_0^{t_f} \sum_{i=0}^{N} \gamma_{\tau,i}\alpha_i(\tau,t_f)K(t,\tau+h_i)I_i(\tau)\mathbf{z}(\tau)d\tau \quad (51)$$

where $\alpha_i(\tau,t_f) = \varepsilon(\tau) - \varepsilon(\tau+h_i-t_f)$, $\varepsilon(\cdot)$ denotes the unit step function and

$$K(t,\tau+h_i) = \begin{cases} \{\bar{\Psi}(t,\tau+h_i) - P(t)[\int_0^t \bar{\Psi}'(r,t)\tilde{K}_r\tilde{\Gamma}'_r \\ \times\bar{\Psi}(r,\tau+h_i)dr]\}K_{\tau+h_i},\ t-h_i \le \tau \le t_f-h_i; \\ P(t)\{\bar{\Psi}'(\tau+h_i,t)\tilde{K}_{\tau+h_i} - [\int_0^{\tau+h_i} \bar{\Psi}'(r,t)\tilde{K}_r\tilde{\Gamma}'_r \\ \times\bar{\Psi}(r,\tau+h_i)dr]K_{\tau+h_i}\},\ 0 \le \tau < t-h_i. \end{cases}$$

Furthermore, by applying a similar discussion of *Corollary 5*, we can easily obtain

$$\hat{\mathbf{x}}(0|0) = \int_0^{t_f} \bar{\Psi}(t,\tau)K_\tau\tilde{\mathbf{z}}(\tau)d\tau$$
$$= \int_0^{t_f} \sum_{i=0}^{N} \gamma_{\tau,i}\alpha_i(\tau,t_f)\bar{\Psi}(t,\tau+h_i)K_{\tau+h_i}I_i(\tau)\mathbf{z}(\tau)d\tau \quad (52)$$

Now, in virtue of the filter expression (52) and the smoother expression (51) and the duality principle between the LQR problem and the smoothing problem for the backward stochastic delay-free system, we can obtain the optimal controller that minimizing the cost function (2) according to the following theorem.

Theorem 6. Consider the system (1), the minimizing solution of (2) can be given as follows:

$$u^*(t) = -\sum_{i=0}^{N} \gamma_{t,i}R_t^{-1}\Gamma'_{t+h_i}P(t+h_i)\bar{\Psi}'(0,t+h_i)x(0)$$
$$-\sum_{i=0}^{N} \gamma_{t,i}\int_0^{h_N} \alpha_i(t,h_N)\bar{K}(s,t+h_i)\tilde{u}(s)ds, \quad (53)$$

where

$$\bar{K}(s,t+h_i) = \begin{cases} R_t^{-1}\Gamma'_{t+h_i}P(t+h_i)\{\bar{\Psi}'(s,t+h_i) - G(t+h_i,s)P(s)\}, \\ \qquad 0 \le s \le t-h_i < h_N \text{ or } t-h_i \ge h_N; \\ R_t^{-1}\Gamma'_{t+h_i}\{\bar{\Psi}(t+h_i,s) - P(t+h_i) \\ \qquad \times G(t+h_i,s)\}P(s), \quad t-h_i \le s < h_N. \end{cases} \quad (54)$$

and

$$G(t+h_i,s) = \begin{cases} \int_0^s \bar{\Psi}'(s,t+h_i)\tilde{K}_r\tilde{\Gamma}'_r\bar{\Psi}(r,s)dr, \\ \qquad 0 \le s \le t-h_i < h_N \text{ or } t-h_i \ge h_N; \\ \int_0^{t+h_i} \bar{\Psi}'(s,t+h_i)\tilde{K}_r\tilde{\Gamma}'_r\bar{\Psi}(r,s)dr, t-h_i \le s < h_N. \end{cases} \quad (55)$$

However, the optimal control $u^*(t)$ derived in the previous theorem is given in terms of the initial state $x(0)$ rather than the current state $x(\tau)$. Next, we shall investigate this case by shifting the time interval $[0,h_N]$ to $[\tau,\tau+h_N]$.

Denote

$$\bar{u}^\tau(t) = \begin{cases} \begin{bmatrix} u(t+\tau-h_0) \\ \vdots \\ u(t+\tau-h_i) \end{bmatrix},\ h_i \le t < h_{i+1}; \\ \begin{bmatrix} u(t+\tau-h_0) \\ \vdots \\ u(t+\tau-h_N) \end{bmatrix},\ t \ge t_N, \end{cases} \quad (56)$$

$$\tilde{u}^\tau(t) = \begin{cases} \sum_{j=i+1}^{N} \Gamma_{t+\tau,j}u(t+\tau-h_j),\ h_i \le t < h_{i+1}; \\ 0, \qquad t \ge h_N, \end{cases} \quad (57)$$

$$\bar{\Gamma}_t^\tau = \begin{cases} [\Gamma_{t+\tau,0},\Gamma_{t+\tau,1},\cdots,\Gamma_{t+\tau,i}],\ \ h_i \le t < h_{i+1}; \\ [\Gamma_{t+\tau,0},\Gamma_{t+\tau,1},\cdots,\Gamma_{t+\tau,N}],\ t \ge h_N, \end{cases} \quad (58)$$

$$\bar{R}_t^\tau = \begin{cases} diag\{R_{t+\tau-h_0,0},R_{t+\tau-h_1,1},\cdots,R_{t+\tau-h_i,i}\}, \\ h_i \le t < h_{i+1}; \\ diag\{R_{t+\tau-h_0,0},R_{t+\tau-h_1,1},\cdots,R_{t+\tau-h_N,N}\}, \\ t \ge h_N, \end{cases} \quad (59)$$

where $\Gamma_{t+\tau,i}$ and $R_{t+\tau_h-i,i}$ denote $\gamma_{t+\tau,i}\Gamma_t$ and $\gamma_{t+\tau,i}R_{t-h_i}$, respectively. By using the above notations, the system (5) can be rewritten as

$$\dot{x}(t+\tau) = \begin{cases} \Phi_t^\tau x(t+\tau) + \bar{\Gamma}_t^\tau\bar{u}^\tau(t) + \tilde{u}^\tau(t),\ h_i \le t < h_{i+1}; \\ \Phi_t^\tau x(t+\tau) + \bar{\Gamma}_t^\tau\bar{u}^\tau(t), \qquad t \ge h_N. \end{cases} \quad (60)$$

the cost function (6) as

$$J_{t_f} = x(t_f)'P(t_f)x(t_f) + \int_0^{t_f-\tau} (\bar{u}^\tau(t))'\bar{R}_t^\tau\bar{u}^\tau(t)dt$$
$$+ \int_0^{t_f-\tau} x'(t+\tau)Q_{t+\tau}x(t+\tau)dt$$
$$+\sum_{i=0}^{N}\int_0^\tau u(t)\gamma_{t,i}R_tu(t)dt + \int_0^\tau x'(t)Q_tx(t)dt. \quad (61)$$

We also define the following Riccati equation:

$$-\frac{dP^\tau(t)}{dt} = \Phi_t^{\tau\prime}P^\tau(t) + P^\tau(t)\Phi_t^\tau + Q_{t+\tau} - K_t^\tau Q_{\tilde{\mathbf{v}}}^\tau(t)K_t^{\tau\prime}$$

with the terminal condition $P^\tau(t_f-\tau) = P(t_f)$, K_t^τ and $Q_{\tilde{\mathbf{v}}}^\tau(t)$ have the same form as (42) and (37), respectively.

By using the above notations, the optimal controller $u^{\tau*}(t)$ can be given in terms of the current state $\mathbf{x}(\tau)$ according to the following theorem.

Theorem 7. Consider the system (60) and the performance index (61), the optimal controller $u^{\tau *}(t)$ associated with $u(t+\tau)$ can be given as follows:

$$u^{\tau *}(t) = -\sum_{i=0}^{N} \gamma_{t+\tau,i} R_{t+\tau}^{-1} \Gamma'_{t+\tau+h_i} P^{\tau}(t+h_i) \bar{\Psi}^{\tau\prime}(0, t+h_i) x(0) - \sum_{i=0}^{N} \gamma_{t+\tau,i} \int_0^{h_N} \alpha_i(t+\tau, h_N) \bar{K}^{\tau}(s, t+h_i) \tilde{u}^{\tau}(s) ds, \quad (62)$$

where

$$K^{\tau}(t+h_i, s) = \begin{cases} R_{t+\tau}^{-1} \Gamma'_{t+\tau+h_i} P^{\tau}(t+h_i) \{\bar{\Psi}^{\tau\prime}(s, t+h_i) - G^{\tau}(t+h_i, s) \\ \times P^{\tau}(s)\}, \quad 0 \leq s \leq t-h_i < h_N \text{ or } t-h_i \geq h_N; \\ R_{t+\tau}^{-1} \Gamma'_{t+\tau+h_i} \{\bar{\Psi}^{\tau}(t+h_i, s) - P^{\tau}(t+h_i) \\ \times G^{\tau}(t+h_i, s)\} P^{\tau}(s), \quad t-h_i < s \leq h_N, \end{cases}$$

and

$$G^{\tau}(t+h_i, s) = \begin{cases} \int_0^{s} \bar{\Psi}^{\tau\prime}(s, t+h_i) \tilde{K}_r^{\tau} \tilde{\Gamma}_r^{\tau\prime} \bar{\Psi}(r, s) dr, \\ \qquad 0 \leq s \leq t-h_i < h_N \text{ or } t-h_i \geq h_N; \\ \int_0^{t+h_i} \bar{\Psi}^{\tau\prime}(s, t+h_i) \tilde{K}_r^{\tau} \tilde{\Gamma}_r^{\tau\prime} \bar{\Psi}(r, s) dr, \\ \qquad t-h_i < s \leq h_N. \end{cases} \quad (63)$$

In terms of the above theorem, we can easily obtain the optimal controller $u^{\tau *}(0)$

$$u^{\tau *}(0) = -\sum_{i=0}^{N} \gamma_{\tau,i} R_{\tau}^{-1} \Gamma'_{\tau+h_i} P^{\tau}(h_i) \bar{\Psi}^{\tau\prime}(0, t+h_i) x(0) - \sum_{i=0}^{N} \gamma_{\tau,i} \int_0^{h_N} \alpha_i(\tau, h_N) \bar{K}^{\tau}(s, h_i) \tilde{u}^{\tau}(s) ds. \quad (64)$$

It is apparent that $u^*(\tau)$ is the optimal controller associated with the cost function (2) given in terms of the initial state $x(0)$ while $u^{\tau *}(0)$ is the optimal controller associated with the cost function (61) given in terms of the current state $x(\tau)$. Next, we express the optimal controller $u^*(\tau)$ minimizing the cost function (2) in terms of the current state $x(\tau)$ according to the following theorem.

Theorem 8. Consider the system (1) with time-varying delay, the optimal control $u(\tau)$ $(0 \leq \tau \leq t_f)$ that minimizing the cost function (2) is given as

$$u^*(\tau) = -\sum_{i=0}^{N} \gamma_{\tau,i} R_{\tau}^{-1} \Gamma'_{\tau+h_i} P^{\tau}(h_i) \bar{\Psi}^{\tau\prime}(0, t+h_i) x(\tau) - \sum_{i=0}^{N} \gamma_{\tau,i} \int_0^{h_N} \alpha_i(\tau, h_N) \bar{K}^{\tau}(s, h_i) \tilde{u}^{\tau *}(s) ds. \quad (65)$$

where $\tilde{u}^{\tau *}(\cdot)$ has the form as (57) with $u(\cdot)$ replaced by $u^*(\cdot)$ and $K^{\tau}(h_i, s)$ is as defined in (63).

5. CONCLUSION

In this paper, we have investigated the finite horizon optimal LQR problem for continuous time system with time-varying delay. We established a duality principle between the LQR problem for systems with multiple input delays and a smoothing problem for a backward stochastic delay free system. In doing so, the complicated LQR problem for systems with time-varying delay is transformed into a smoothing one. By applying the established duality principle, an intuitive and much simpler derivation and solution to the proposed problem is given.

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