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A Sum of Squares Approach to Guaranteed Cost Control of Polynomial Discrete Fuzzy Systems *

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Abstract: This paper presents a sum of squares (SOS) approach to guaranteed cost control of polynomial discrete fuzzy systems. First, we present a polynomial discrete fuzzy model that is more general representation of the well-known discrete Takagi-Sugeno (T-S) fuzzy model. Secondly, we derive a design condition based on polynomial Lyapunov functions that contain quadratic Lyapunov functions as a special case. Hence, the design approach discussed in this paper is more general than that based on the existing LMI approaches to discrete T-S fuzzy control system designs. The design condition realizes guaranteed cost control by minimizing the upper bound of a given performance function. In addition, the design condition can be represented in terms of SOS and is numerically (partially symbolically) solved via the recent developed SOSTOOLS. A design example is provided to illustrate the validity of the design approach.

1. INTRODUCTION

The Takagi-Sugeno (T-S) fuzzy model-based control methodology (Tanaka [2001]) has received a great deal of attention over the last decade. There is no loss of generality in adopting the T-S fuzzy model based control design framework as it has been established that any smooth nonlinear control systems can be approximated by the T-S fuzzy models (with liner model consequence) (Wang [2000]). In addition, it provides a natural, simple and effective design approach to complement other nonlinear control techniques (e.g., Sepulcher [1997]) that require special and rather involved knowledge. Within the general framework of T-S fuzzy model-based control systems, there has been, in particular, a flurry of research activities in the analysis and design of fuzzy control systems based on linear matrix inequalities (LMIs) (e.g., Tanaka [2001], Feng [2006]).

Recently, we presented a sum of squares (SOS) approach (Tanaka [2007], Tanaka, Yoshida et al. [2007], Tanaka, Yamauchi et al. [2007], Tanaka [2008]) to control system designs for polynomial fuzzy systems. This is a completely different approach from the existing LMI approaches. To the best of our knowledge, the paper (Tanaka [2007]) presented the first attempt at applying an SOS to fuzzy systems. Our SOS approach (Tanaka [2007], Tanaka, Yoshida et al. [2007], Tanaka, Yamauchi et al. [2007], Tanaka [2008]) provided more extensive results for the existing LMI approaches to T-S fuzzy model and control.

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This paper presents a sum of squares approach to guaranteed cost control of polynomial *discrete* fuzzy systems. Guaranteed cost control of polynomial *continuous* fuzzy systems via an SOS approach has been discussed in (Tanaka, Yamauchi et al. [2007]). However, there have been no papers dealing with guaranteed cost control of polynomial *discrete* fuzzy systems via an SOS approach First, we present a polynomial fuzzy model and controller that are more general representation of the well-known Takagi-Sugeno (T-S) fuzzy model and controller. Secondly, we derive a design condition based on polynomial Lyapunov functions that contain quadratic Lyapunov functions as a special case. Hence, the design approach discussed in this paper is more general than that based on the existing LMI approaches (Tanaka [2001]) to T-S fuzzy control system designs. The design condition realizes guaranteed cost control by minimizing the upper bound of a given performance function. In addition, the design condition can be represented in terms of SOS and is numerically (partially symbolically) solved via the recent developed SOSTOOLS (Prajna [2004]). A design example is provided to illustrate the validity of the design approach.

2. POLYNOMIAL FUZZY MODEL AND STABLE CONTROLLER DESIGN

We proposed a new type of fuzzy model with polynomial model consequence (Tanaka [2007]), i.e., fuzzy model whose consequent parts are represented by polynomials. Stability conditions for the T-S fuzzy system and the quadratic Lyapunov function reduce to LMIs, e.g., (Tanaka [2001]). Hence, the stability conditions can be solved numerically and efficiently by interior point algorithms, e.g., by the Robust Control Toolbox of MATLAB [1]. On the other hand, design conditions for polynomial fuzzy systems and polynomial Lyapunov functions reduce to SOS problems (Tanaka [2007], Tanaka, Yoshida et al. [2007], Tanaka, Yamauchi et al. [2007], Tanaka [2008]). Clearly, the problem is never solved by LMI solvers and can be solved via SOSTOOLS (Prajna [2004]).

SOSTOOLS is a free, third party MATLAB toolbox for solving sum of squares problems. The techniques behind it are based on the sum of squares decomposition for multivariate polynomials, which can be efficiently computed using semidefinite programming. SOSTOOLS is developed as a consequence of the recent interest in sum of squares polynomials, partly due to the fact that these techniques provide convex relaxations for many hard problems such as global, constrained, and boolean optimization. For more details, see the manual of SOSTOOLS (Prajna [2004]).

2.1 Polynomial fuzzy model

Consider the following nonlinear system:

$$\boldsymbol{x}(t+1) = \boldsymbol{f}(\boldsymbol{x}(t), \boldsymbol{u}(t)), \tag{1}$$

where f is a nonlinear function.

$$\boldsymbol{x}(t) = [x_1(t)\ x_2(t)\ \cdots\ x_n(t)]^T$$

is the state vector and

$$\boldsymbol{u}(t) = [u_1(t)\ u_2(t)\ \cdots\ u_m(t)]^T$$

is the input vector. Based on the sector nonlinearity concept (Tanaka [2001]), we can exactly represent (1) with a Takagi-Sugeno fuzzy model (globally or at least semi-globally). The Takagi-Sugeno fuzzy model (Takagi [1985]) is described by fuzzy IF-THEN rules which represent local linear input-output relations of a nonlinear system. The main feature of this model is to express the local dynamics of each fuzzy implication (rule) by a linear system model. The overall fuzzy model of the system is achieved by fuzzy blending of the linear system models.

A polynomial *discrete* fuzzy model has been proposed in (Tanaka [2008]). Using the sector nonlinearity concept, we exactly represent (1) with the following polynomial fuzzy model (2). The main difference between the T-S fuzzy model and the polynomial fuzzy model is consequent part representation. The fuzzy model (2) has a polynomial model consequence.

Model Rule i:

$$If\ z_1(t)\ is\ M_{i1}\ and \cdots\ and\ \ z_p(t)\ is\ M_{ip}$$

$$then\ \boldsymbol{x}(t+1) = \boldsymbol{A}_i(\boldsymbol{x}(t))\hat{\boldsymbol{x}}(\boldsymbol{x}(t)) + \boldsymbol{B}_i(\boldsymbol{x}(t))\boldsymbol{u}(t), \tag{2}$$

where $i = 1, 2, \cdots, r$. $z_j(t)$ $(j = 1, 2, \cdots, p)$ is the premise variable. The membership function associated with the ith *Model Rule* and jth premise variable component is denoted by M_{ij}. r denotes the number of *Model Rules*. Each $z_j(t)$ is a measurable time-varying quantity that may be states, measurable external variables and/or time. $\boldsymbol{A}_i(\boldsymbol{x}(t))$ and $\boldsymbol{B}_i(\boldsymbol{x}(t))$ are polynomial matrices in $\boldsymbol{x}(t)$. $\hat{\boldsymbol{x}}(\boldsymbol{x}(t))$ is a column vector whose entries are all monomials in $\boldsymbol{x}(t)$. That is, $\hat{\boldsymbol{x}}(\boldsymbol{x}(t)) \in \boldsymbol{R}^N$ is an $N \times 1$ vector of monomials in $\boldsymbol{x}(t)$. A monomial in $\boldsymbol{x}(t)$ is a function of the form $x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$, where $\alpha_1, \alpha_2, \cdots, \alpha_n$ are nonnegative integers. Therefore, $\boldsymbol{A}_i(\boldsymbol{x}(t))\hat{\boldsymbol{x}}(\boldsymbol{x}(t)) + \boldsymbol{B}_i(\boldsymbol{x}(t))\boldsymbol{u}(t)$ is a polynomial vector. Thus, the polynomial fuzzy model (2) has a polynomial in each consequent part. In Sections 3 and 4, we will consider $\boldsymbol{T}(\tilde{\boldsymbol{x}}(t)) \in R^{N\times m}$ that is a polynomial matrix defined by $\hat{\boldsymbol{x}}(\boldsymbol{x}(t)) = \boldsymbol{T}(\tilde{\boldsymbol{x}}(t))\boldsymbol{x}(t)$. The details of $\hat{\boldsymbol{x}}(\boldsymbol{x}(t))$ will be given in Proposition 1. The definition of $\tilde{\boldsymbol{x}}(t)$ will be also presented later.

[1] A registered trademark of MathWorks, Inc.

We assume that

$$\hat{\boldsymbol{x}}(\boldsymbol{x}(t)) = 0 \text{ iff } \boldsymbol{x}(t) = 0$$

throughout this paper.

The defuzzification process of the model (2) can be represented as

$$\boldsymbol{x}(t+1) = \sum_{i=1}^{r} h_i(\boldsymbol{z}(t))\{\boldsymbol{A}_i(\boldsymbol{x}(t))\hat{\boldsymbol{x}}(\boldsymbol{x}(t)) + \boldsymbol{B}_i(\boldsymbol{x}(t))\boldsymbol{u}(t)\}, \tag{3}$$

where

$$h_i(\boldsymbol{z}(t)) = \frac{\prod_{j=1}^{p} M_{ij}(z_j(t))}{\sum_{k=1}^{r}\prod_{j=1}^{p} M_{kj}(z_j(t))}.$$

It should be noted from the properties of membership functions that $h_i(\boldsymbol{z}(t)) \geq 0$ for all i and $\sum_{i=1}^{r} h_i(\boldsymbol{z}(t)) = 1$.

Thus, the overall fuzzy model is achieved by fuzzy blending of the polynomial system models. A stability condition for the polynomial discrete fuzzy systems was derived in (Tanaka [2008]).

If $\hat{\boldsymbol{x}}(\boldsymbol{x}(t)) = \boldsymbol{x}(t)$ and $\boldsymbol{A}_i(\boldsymbol{x}(t))$ and $\boldsymbol{B}_i(\boldsymbol{x}(t))$ are constant matrices for all i, then $\boldsymbol{A}_i(\boldsymbol{x}(t))\hat{\boldsymbol{x}}(\boldsymbol{x}(t)) + \boldsymbol{B}_i(\boldsymbol{x}(t))\boldsymbol{u}(t)$ reduces to $\boldsymbol{A}_i\boldsymbol{x}(t) + \boldsymbol{B}_i\boldsymbol{u}(t)$, that is, then (3) reduces to the ordinary Takagi-Sugeno fuzzy model. Therefore, (3) is a more general representation.

Remark 1. As shown in (Tanaka [2007], Tanaka, Yoshida et al. [2007], Tanaka, Yamauchi et al. [2007], Tanaka [2008]), the number of rules in polynomial fuzzy model generally becomes fewer than that in T-S fuzzy model. In addition, our SOS approach to polynomial fuzzy model and control provides much more relaxed stability and stabilization results than the existing LMI approaches to T-S fuzzy model and control.

3. STABLE CONTROLLER DESIGN

3.1 Sum of Squares

The computational method used in this paper relies on the sum of squares decomposition of multivariate polynomials. A multivariate polynomial $f(\boldsymbol{x}(t))$ (where $\boldsymbol{x}(t) \in R^n$) is a sum of squares (SOS, for brevity) if there exist polynomials $f_1(\boldsymbol{x}(t)), \cdots, f_m(\boldsymbol{x}(t))$ such that $f(\boldsymbol{x}(t)) = \sum_{i=1}^{m} f_i^2(\boldsymbol{x}(t))$. It is clear that $f(\boldsymbol{x}(t))$ being an SOS naturally implies $f(\boldsymbol{x}(t)) > 0$ for all $\boldsymbol{x}(t) \in R^n$. This can be shown equivalent to the existence of a special quadric form stated in the following proposition.

Proposition 1. (Parrilo [2000]) Let $f(\boldsymbol{x}(t))$ be a polynomial in $\boldsymbol{x}(t) \in R^n$ of degree $2d$. In addition, let $\hat{\boldsymbol{x}}(\boldsymbol{x}(t))$ be a column vector whose entries are all monomials in $\boldsymbol{x}(t)$ with degree no greater than d. Then $f(\boldsymbol{x}(t))$ is a sum of squares iff there exists a positive semidefinite matrix $\boldsymbol{P}$ such that

$$f(\boldsymbol{x}(t)) = \hat{\boldsymbol{x}}^T(\boldsymbol{x}(t))\boldsymbol{P}\hat{\boldsymbol{x}}(\boldsymbol{x}(t)). \quad (4)$$

Expressing an SOS polynomial using a quadratic form as in (4) has also been referred to as the Gram matrix method.

As mentioned before, a monomial in $\boldsymbol{x}(t)$ is a function of the form $x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_n^{\alpha_n}$, where $\alpha_1, \alpha_2, \cdots, \alpha_n$ are nonnegative integers. In this case, the degree of the monomial is given by $\alpha_1 + \alpha_2 + \cdots + \alpha_n$.

A sum of squares decomposition for $f(\boldsymbol{x}(t))$ can be computed using semidefinite programming, since it amounts to searching for an element $\boldsymbol{P}$ in the intersection of the cone of positive semidefinite matrices and a set defined by some affine constraints that arise from (4). Note in particular that the polynomial $f(\boldsymbol{x}(t))$ is globally nonnegative if it can be decomposed as a sum of squares. Hence the sum of squares decomposition in conjunction with semidefinite programming provides a polynomial-time computational relaxation for proving global nonnegativity of multivariate polynomials (Parrilo [2000], Skor [1987]), which belongs to the class of NP-hard problems. Even though the sum of squares condition is not necessary for nonnegativity, numerical experiments seem to indicate that the gap between sum of squares and nonnegativity is small (Prajna et al. [2004]).

3.2 Polynomial Fuzzy Controller

Since the parallel distributed compensation (PDC) mirrors the structure of the fuzzy model of a system, a fuzzy controller with polynomial rule consequence can be constructed from the given polynomial fuzzy model (2).

Control Rule i:

$$\begin{aligned} &If\ z_1(t)\ is\ M_{i1}\ and \cdots\ and\ \ z_p(t)\ is\ M_{ip} \\ &then\ \boldsymbol{u}(t) = -\boldsymbol{F}_i(\boldsymbol{x}(t))\hat{\boldsymbol{x}}(\boldsymbol{x}(t)) \quad i = 1, 2, \cdots, r \end{aligned} \quad (5)$$

The overall fuzzy controller can be calculated by

$$\boldsymbol{u}(t) = -\sum_{i=1}^{r} h_i(\boldsymbol{z}(t))\boldsymbol{F}_i(\boldsymbol{x}(t))\hat{\boldsymbol{x}}(\boldsymbol{x}(t)). \quad (6)$$

From (3) and (6), the closed-loop system can be represented as

$$\begin{aligned} \boldsymbol{x}(t+1) = &\sum_{i=1}^{r}\sum_{j=1}^{r} h_i(\boldsymbol{z}(t))h_j(\boldsymbol{z}(t)) \\ &\times \{\boldsymbol{A}_i(\boldsymbol{x}(t)) - \boldsymbol{B}_i(\boldsymbol{x}(t))\boldsymbol{F}_j(\boldsymbol{x}(t))\}\hat{\boldsymbol{x}}(\boldsymbol{x}(t)). \end{aligned} \quad (7)$$

If $\hat{\boldsymbol{x}}(\boldsymbol{x}(t)) = \boldsymbol{x}(t)$ and $\boldsymbol{A}_i(\boldsymbol{x}(t))$, $\boldsymbol{B}_i(\boldsymbol{x}(t))$ and $\boldsymbol{F}_j(\boldsymbol{x}(t))$ are constant matrices for all i and j, then (3) and (6) reduce to the Takagi-Sugeno fuzzy model and controller, respectively. Therefore, (3) and (6) are more general representation.

3.3 Stable Controller Design Conditions

To obtain more relaxed stability results, we employ a polynomial Lyapunov function (Tanaka [2007], Tanaka, Yoshida et al. [2007], Tanaka, Yamauchi et al. [2007], Tanaka [2008]) represented by

$$\hat{\boldsymbol{x}}^T(\boldsymbol{x}(t))\boldsymbol{P}(\tilde{\boldsymbol{x}}(t))\hat{\boldsymbol{x}}(\boldsymbol{x}(t)), \quad (8)$$

where $\boldsymbol{P}(\tilde{\boldsymbol{x}}(t))$ is a polynomial matrix in $\boldsymbol{x}(t)$. If $\hat{\boldsymbol{x}}(t) = \boldsymbol{x}(t)$ and $\boldsymbol{P}(\tilde{\boldsymbol{x}}(t))$ is a constant matrix, then (8) reduces to the quadratic Lyapunov function $\boldsymbol{x}^T(t)\boldsymbol{P}\boldsymbol{x}(t)$. Therefore, (8) is a more general representation.

We provide another important proposition with respect to the relaxation.

Proposition 2. (Prajna et al. [2004]) Let $\boldsymbol{L}(\boldsymbol{x}(t))$ be an $N \times N$ symmetric polynomial matrix of degree $2d$ in $\boldsymbol{x}(t) \in R^n$. Furthermore, let $\hat{\boldsymbol{x}}(\boldsymbol{x}(t))$ be a column vector whose entries are all monomials in $\boldsymbol{x}(t)$ with degree no greater than d, and consider the following conditions.

(1) $\boldsymbol{L}(\boldsymbol{x}(t)) \geq 0$ for all $\boldsymbol{x}(t) \in R^n$.
(2) $\boldsymbol{v}^T(t)\boldsymbol{L}(\boldsymbol{x}(t))\boldsymbol{v}(t)$ is a sum of squares, where $\boldsymbol{v}(t) \in R^N$.
(3) There exists a positive semidefinite matrix $\boldsymbol{Q}$ such that $\boldsymbol{v}^T(t)\boldsymbol{L}(\boldsymbol{x}(t))\boldsymbol{v}(t) = (\boldsymbol{v}(t) \otimes \hat{\boldsymbol{x}}(\boldsymbol{x}(t)))^T\boldsymbol{Q}(\boldsymbol{v} \otimes \hat{\boldsymbol{x}}(\boldsymbol{x}(t)))$, where $\otimes$ denotes the Kronecker product.

Then (1) $\Leftarrow$ (2) and (2) $\Longleftrightarrow$ (3).

This subsection gives a stable control design condition whose feasibility can be checked via SOSTOOLS. Hence the stable fuzzy controller design with polynomial rule consequence is numerically a feasibility problem.

From now, to lighten the notation, we will drop the notation with respect to time t. For instance, we will employ $\boldsymbol{x}$, $\hat{\boldsymbol{x}}(\boldsymbol{x})$ instead of $\boldsymbol{x}(t)$, $\hat{\boldsymbol{x}}(\boldsymbol{x}(t))$, respectively. Thus, we drop the notation with respect to time t, but it should be kept in mind that $\boldsymbol{x}$ means $\boldsymbol{x}(t)$. However, to distinguish between $\boldsymbol{x}(t)$ and $\boldsymbol{x}(t+1)$, we will remain the notation with respect to time $t+1$. Hence we will employ $\boldsymbol{x}(t+1)$, $\hat{\boldsymbol{x}}(t+1)$, etc

$\boldsymbol{K} = \{k_1, k_2, \cdots k_m\}$ denote the row indices of $\boldsymbol{B}_i(\boldsymbol{x})$ whose corresponding row is equal to zero and the row indices of $\boldsymbol{A}_i(\boldsymbol{x})$ whose corresponding row does not contain non-polynomial nonlinear terms (e.g. trigonometric functions). Using k_1, k_2, $\cdots$, k_{m-1} and k_m, we define $\tilde{\boldsymbol{x}} = (x_{k_1}, x_{k_2}, \cdots x_{k_m})$. From the definition, a partial system of (3) can be represented as

$$\tilde{\boldsymbol{x}}(t+1) = \tilde{\boldsymbol{A}}(\boldsymbol{x})\boldsymbol{x}, \quad (9)$$

where $\tilde{\boldsymbol{A}}(\boldsymbol{x})$ is a polynomial matrix. The equation (9) will play an important role in Theorems 3 and 4.

Theorem 3. (Tanaka [2008]) The control system consisting of (3) and (6) is stable if there exist a symmetric polynomial matrix $\boldsymbol{X}(\tilde{\boldsymbol{x}}) \in \boldsymbol{R}^{N\times N}$ and a polynomial matrix $\boldsymbol{M}_i(\boldsymbol{x}) \in \boldsymbol{R}^{m\times N}$ such that (10), (11) and (12) are satisfied, where $\epsilon_1(\boldsymbol{x})$ and $\epsilon_{2ij}(\boldsymbol{x})$ are non negative polynomials such that $\epsilon_1(\boldsymbol{x}) > 0$ ($\boldsymbol{x} \neq 0$) and $\epsilon_{2ij}(\boldsymbol{x}) \geq 0$ for all $\boldsymbol{x}$.

$$\boldsymbol{v}_1^T(\boldsymbol{X}(\tilde{\boldsymbol{x}}) - \epsilon_1(\boldsymbol{x})\boldsymbol{I})\boldsymbol{v}_1\ is\ SOS \quad (10)$$

$$\boldsymbol{v}_2^T\begin{bmatrix} \boldsymbol{X}(\tilde{\boldsymbol{x}}) - \epsilon_{2ii}(\boldsymbol{x})\boldsymbol{I} & * \\ \boldsymbol{T}(\tilde{\boldsymbol{A}}(\boldsymbol{x})\boldsymbol{x})\Omega_{ii}(\boldsymbol{x}) & \boldsymbol{X}(\tilde{\boldsymbol{A}}(\boldsymbol{x})\boldsymbol{x}) \end{bmatrix}\boldsymbol{v}_2 \quad is\ SOS, \quad (11)$$

$$\boldsymbol{v}_3^T \begin{bmatrix} \boldsymbol{X}(\tilde{\boldsymbol{x}}) - \epsilon_{2ij}(\boldsymbol{x})\boldsymbol{I} & * \\ \frac{1}{2}\boldsymbol{T}(\tilde{\boldsymbol{A}}(\boldsymbol{x})\boldsymbol{x})\left(\Omega_{ij}(\boldsymbol{x}) + \Omega_{ji}(\boldsymbol{x})\right) & \boldsymbol{X}(\tilde{\boldsymbol{A}}(\boldsymbol{x})\boldsymbol{x}) \end{bmatrix} \boldsymbol{v}_3 \ is\ SOS, \quad i < j, \tag{12}$$

where $\Omega_{ij}(\boldsymbol{x}) = \boldsymbol{A}_i(\boldsymbol{x})\boldsymbol{X}(\tilde{\boldsymbol{x}}) - \boldsymbol{B}_i(\boldsymbol{x})\boldsymbol{M}_j(\boldsymbol{x})$. * denotes the transposed elements (matrices) for symmetric positions. $\boldsymbol{v}_1 \in R^N$, $\boldsymbol{v}_2$, $\boldsymbol{v}_3 \in R^{2N}$ are vectors that are independent of $\boldsymbol{x}$. In addition, if (11) and (12) hold with $\epsilon_{2ij}(\boldsymbol{x}) > 0$ for $\boldsymbol{x} \neq 0$, then the zero equilibrium is asymptotically stable. If $\boldsymbol{X}(\tilde{\boldsymbol{x}})$ is a constant matrix, then the stability holds globally. A stabilizing feedback gain $\boldsymbol{F}_i(\boldsymbol{x})$ can be obtained from $\boldsymbol{X}(\tilde{\boldsymbol{x}})$ and $\boldsymbol{M}_i(\boldsymbol{x})$ as

$$\boldsymbol{F}_i(\boldsymbol{x}) = \boldsymbol{M}_i(\boldsymbol{x})\boldsymbol{X}^{-1}(\tilde{\boldsymbol{x}}). \tag{13}$$

4. GUARANTEED COST CONTROL

Recall the polynomial fuzzy model and controller, respectively.

$$\boldsymbol{x}(t+1) = \sum_{i=1}^{r} h_i(\boldsymbol{z})\{\boldsymbol{A}_i(\boldsymbol{x})\hat{\boldsymbol{x}} + \boldsymbol{B}_i(\boldsymbol{x})\boldsymbol{u}\}. \tag{14}$$

$$\boldsymbol{u} = -\sum_{i=1}^{r} h_i(\boldsymbol{z})\boldsymbol{F}_i(\boldsymbol{x})\hat{\boldsymbol{x}}. \tag{15}$$

Next, we define the outputs as

$$\boldsymbol{y} = \sum_{i=1}^{r} h_i(\boldsymbol{z})\boldsymbol{C}_i(\boldsymbol{x})\hat{\boldsymbol{x}}, \tag{16}$$

where $\boldsymbol{C}_i(\boldsymbol{x})$ is also a polynomial matrix.

Consider the following performance function to be minimized.

$$J = \sum_{t=0}^{\infty}\{\boldsymbol{y}^T\boldsymbol{Q}\boldsymbol{y} + \boldsymbol{u}^T\boldsymbol{R}\boldsymbol{u}\} = \sum_{t=0}^{\infty} \hat{\boldsymbol{y}}^T \begin{bmatrix} \boldsymbol{Q} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{R} \end{bmatrix} \hat{\boldsymbol{y}}, \tag{17}$$

where $\boldsymbol{Q}$ and $\boldsymbol{R}$ are positive definite matrices, and $\hat{\boldsymbol{y}}$ is defined as

$$\hat{\boldsymbol{y}} = \begin{bmatrix} \boldsymbol{y} \\ \boldsymbol{u} \end{bmatrix} = \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(\boldsymbol{z})h_j(\boldsymbol{z}) \begin{bmatrix} \boldsymbol{C}_i(\boldsymbol{x}) \\ -\boldsymbol{F}_j(\boldsymbol{x}) \end{bmatrix} \hat{\boldsymbol{x}}. \tag{18}$$

Theorem 4 provides the SOS design condition that minimizes the upper bound of the given performance function (17).

Theorem 4. If there exist a symmetric polynomial matrix $\boldsymbol{X}(\tilde{\boldsymbol{x}}) \in \boldsymbol{R}^{N \times N}$ and a polynomial matrix $\boldsymbol{M}_i(\boldsymbol{x}) \in \boldsymbol{R}^{m \times N}$ such that (19), (20), (21) and (22) hold, the guaranteed cost controller that minimizes the upper bound of the given performance function (17) can be designed by solving the following design problem.

$$\underset{X(\tilde{x}), M_i(x)}{minimize} \quad \lambda$$

subject to

$$\boldsymbol{v}_1^T(\boldsymbol{X}(\tilde{\boldsymbol{x}}) - \epsilon_1(\boldsymbol{x})\boldsymbol{I})\boldsymbol{v}_1 \ is\ SOS \tag{19}$$

$$\boldsymbol{v}_2^T \begin{bmatrix} \lambda & \hat{\boldsymbol{x}}^T(\boldsymbol{0}) \\ \hat{\boldsymbol{x}}(\boldsymbol{0}) & \boldsymbol{X}(\tilde{\boldsymbol{x}}) \end{bmatrix} \boldsymbol{v}_2 \ is\ SOS \tag{20}$$

$$\boldsymbol{v}_3^T \boldsymbol{W}_{ii} \boldsymbol{v}_3 \ is\ SOS, \tag{21}$$

$$\boldsymbol{v}_4^T\left(\boldsymbol{W}_{ij} + \boldsymbol{W}_{ji}\right)\boldsymbol{v}_4 \ is\ SOS, \quad i < j \tag{22}$$

where

$$\boldsymbol{W}_{ij} = \begin{bmatrix} \boldsymbol{X}(\tilde{\boldsymbol{x}}) - \epsilon_{2ij}(\boldsymbol{x})\boldsymbol{I} & * & * & * \\ \boldsymbol{T}(\tilde{\boldsymbol{A}}(\boldsymbol{x})\boldsymbol{x})\boldsymbol{\Omega}_{ij}(\boldsymbol{x}) & \boldsymbol{X}(\tilde{\boldsymbol{A}}(\boldsymbol{x})\boldsymbol{x}) & * & * \\ \boldsymbol{C}_i(\boldsymbol{x})\boldsymbol{X}(\tilde{\boldsymbol{x}}(t)) & \boldsymbol{0} & \boldsymbol{Q}^{-1} & * \\ -\boldsymbol{M}_j(\boldsymbol{x}) & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{R}^{-1} \end{bmatrix}, \tag{23}$$

and

$$\boldsymbol{\Omega}_{ij}(\boldsymbol{x}) = \boldsymbol{A}_i(\boldsymbol{x})\boldsymbol{X}(\tilde{\boldsymbol{x}}) - \boldsymbol{B}_i(\boldsymbol{x})\boldsymbol{M}_j(\boldsymbol{x}).$$

$\boldsymbol{v}_1$, $\boldsymbol{v}_2$, $\boldsymbol{v}_3$ and $\boldsymbol{v}_4$ are vectors that are independent of $\boldsymbol{x}$. $\epsilon_1(\boldsymbol{x})$ and $\epsilon_{2ij}(\boldsymbol{x})$ are non negative polynomials such that $\epsilon_1(\boldsymbol{x}) > 0$ and $\epsilon_{2ij}(\boldsymbol{x}) > 0$ at $\boldsymbol{x} \neq \boldsymbol{0}$, and $\epsilon_1(\boldsymbol{x}) = 0$ and $\epsilon_{2ij}(\boldsymbol{x}) = 0$ at $\boldsymbol{x} = \boldsymbol{0}$. The polynomial feedback gains $\boldsymbol{F}_i(\boldsymbol{x})$ can be obtained from $\boldsymbol{X}(\tilde{\boldsymbol{x}})$ and $\boldsymbol{M}_i(\boldsymbol{x})$ as

$$\boldsymbol{F}_i(\boldsymbol{x}) = \boldsymbol{M}_i(\boldsymbol{x})\boldsymbol{X}^{-1}(\tilde{\boldsymbol{x}}). \tag{24}$$

(Proof) Consider

$$V(\boldsymbol{x}) = \hat{\boldsymbol{x}}^T\boldsymbol{P}(\tilde{\boldsymbol{x}})\hat{\boldsymbol{x}} \tag{25}$$

as a candidate Lyapunov function, where $\boldsymbol{P}(\tilde{\boldsymbol{x}})$ is a positive definite matrix. From (14) and (15), the time difference of (25), i.e., $\Delta V(\boldsymbol{x}) = V(\boldsymbol{x}(t+1)) - V(\boldsymbol{x})$, is obtained as

$$\begin{aligned} &\Delta V(\boldsymbol{x}) \\ =&\hat{\boldsymbol{x}}^T(t+1)\boldsymbol{P}(\tilde{\boldsymbol{A}}(\boldsymbol{x})\boldsymbol{x})\hat{\boldsymbol{x}}(t+1) - \hat{\boldsymbol{x}}^T\boldsymbol{P}(\tilde{\boldsymbol{x}})\hat{\boldsymbol{x}} \\ =&\sum_{i=1}^{r}\sum_{j=1}^{r} h_i(\boldsymbol{z})h_j(\boldsymbol{z})\hat{\boldsymbol{x}}^T \\ &\times \{\boldsymbol{\Pi}_{ij}^T(\boldsymbol{x})\boldsymbol{T}^T(\tilde{\boldsymbol{A}}(\boldsymbol{x})\boldsymbol{x})\boldsymbol{P}(\tilde{\boldsymbol{A}}(\boldsymbol{x})\boldsymbol{x})\boldsymbol{T}(\tilde{\boldsymbol{A}}(\boldsymbol{x})\boldsymbol{x})\boldsymbol{\Pi}_{ij}(\boldsymbol{x}) \\ &- \boldsymbol{P}(\tilde{\boldsymbol{x}}(t))\}\hat{\boldsymbol{x}}, \end{aligned} \tag{26}$$

where

$$\boldsymbol{\Pi}_{ij}(\boldsymbol{x}) = \boldsymbol{A}_i(\boldsymbol{x}) - \boldsymbol{B}_i(\boldsymbol{x})\boldsymbol{F}_j(\boldsymbol{x}).$$

Now, we assume that there is a positive definite matrix $\boldsymbol{P}(\tilde{\boldsymbol{x}})$ such that the following condition holds.

$$\begin{aligned} &\sum_{i=1}^{r}\sum_{j=1}^{r} h_i(\boldsymbol{z})h_j(\boldsymbol{z})\hat{\boldsymbol{x}}^T \\ &\times \Big(\{\boldsymbol{\Pi}_{ij}^T(\boldsymbol{x})\boldsymbol{T}^T(\tilde{\boldsymbol{A}}(\boldsymbol{x})\boldsymbol{x})\boldsymbol{P}(\tilde{\boldsymbol{A}}(\boldsymbol{x})\boldsymbol{x}) \\ &\times \boldsymbol{T}(\tilde{\boldsymbol{A}}(\boldsymbol{x})\boldsymbol{x})\boldsymbol{\Pi}_{ij}(\boldsymbol{x}) - \boldsymbol{P}(\tilde{\boldsymbol{x}}(t))\} \\ &+ \left[\boldsymbol{C}_i^T(\boldsymbol{x}) \ -\boldsymbol{F}_j^T(\boldsymbol{x})\right] \begin{bmatrix} \boldsymbol{Q} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{R} \end{bmatrix} \begin{bmatrix} \boldsymbol{C}_i(\boldsymbol{x}) \\ -\boldsymbol{F}_j(\boldsymbol{x}) \end{bmatrix}\Big)\hat{\boldsymbol{x}} \\ &< 0 \end{aligned} \tag{27}$$

Note that the closed-loop system consisting of (14) and (15) is stable if the assumption is satisfied.

Under the assumption, (26) can be rewritten as

$$\hat{\boldsymbol{y}}^T(\boldsymbol{x}) \begin{bmatrix} \boldsymbol{Q} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{R} \end{bmatrix} \hat{\boldsymbol{y}}(\boldsymbol{x}) < -\Delta V(\boldsymbol{x}). \quad (28)$$

Taking summation from 0 to ∞ on t, we can calculate J as follows:

$$\begin{aligned} J =& \sum_{t=0}^{\infty} \hat{\boldsymbol{y}}^T(\boldsymbol{x}) \begin{bmatrix} \boldsymbol{Q} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{R} \end{bmatrix} \hat{\boldsymbol{y}}(\boldsymbol{x}) \\ <& \sum_{i=0}^{\infty} \{\hat{\boldsymbol{x}}^T \boldsymbol{P}(\tilde{\boldsymbol{x}})\hat{\boldsymbol{x}} - \hat{\boldsymbol{x}}^T(t+1)\boldsymbol{P}(\tilde{\boldsymbol{x}}(t+1))\hat{\boldsymbol{x}}(t+1)\} \\ =& \hat{\boldsymbol{x}}^T(0)\boldsymbol{P}(\tilde{\boldsymbol{x}}(0))\hat{\boldsymbol{x}}(0) - \hat{\boldsymbol{x}}^T(\infty)\boldsymbol{P}(\tilde{\boldsymbol{x}}(\infty))\hat{\boldsymbol{x}}(\infty). \quad (29) \end{aligned}$$

Since the closed-loop system is stable under the assumption, $\hat{\boldsymbol{x}}(\infty) \to \boldsymbol{0}$ as $t \to \infty$. Hence, the above inequality becomes

$$J < \hat{\boldsymbol{x}}^T(\boldsymbol{0})\boldsymbol{P}(\tilde{\boldsymbol{x}})\hat{\boldsymbol{x}}(\boldsymbol{0}). \quad (30)$$

Next, we introduce λ satisfying

$$J < \hat{\boldsymbol{x}}^T(\boldsymbol{0})\boldsymbol{P}(\tilde{\boldsymbol{x}})\hat{\boldsymbol{x}}(\boldsymbol{0}) \le \lambda. \quad (31)$$

By minimizing λ as much as possible under the assumption, we realize the guaranteed cost control for the polynomial discrete fuzzy model. From Schur complement, if the condition (20) holds, then (31) is satisfied, where

$$\boldsymbol{X}(\tilde{\boldsymbol{x}}) = \boldsymbol{P}^{-1}(\tilde{\boldsymbol{x}}).$$

Since $\boldsymbol{X}(\tilde{\boldsymbol{x}}) = \boldsymbol{P}^{-1}(\tilde{\boldsymbol{x}})$, we can have the condition (19) satisfying the fact that $\boldsymbol{P}(\tilde{\boldsymbol{x}})$ is a positive definite matrix.

We can prove that the assumption (27) is satisfied if the conditions (21) and (22) hold. The detailed derivation of the conditions is omitted due to lack of space.

(Q.E.D.)

Remark 2. Note that $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ and $\boldsymbol{v}_4$ are vectors that are independent of $\boldsymbol{x}$, because $\boldsymbol{L}(\boldsymbol{x})$ is not always a positive semi-definite matrix for all $\boldsymbol{x}$ even if $\hat{\boldsymbol{x}}^T(\boldsymbol{x})\boldsymbol{L}(\boldsymbol{x})\hat{\boldsymbol{x}}(\boldsymbol{x})$ is an SOS, where $\boldsymbol{L}(\boldsymbol{x})$ is a symmetric polynomial matrix in $\boldsymbol{x}(t)$. However, it is guaranteed from Proposition 2 that if $\boldsymbol{v}^T\boldsymbol{L}(\boldsymbol{x})\boldsymbol{v}$ is an SOS, then $\boldsymbol{L}(\boldsymbol{x}) \ge 0$ for all $\boldsymbol{x}$.

Remark 3. Selection of the non-negative polynomials $\epsilon_1(\boldsymbol{x})$ and $\epsilon_{2ij}(\boldsymbol{x})$ influence the feasibility of the SOS problem. Hence, the polynomial structure of $\epsilon_1(\boldsymbol{x})$ and $\epsilon_{2ij}(\boldsymbol{x})$ is needed to select carefully.

Remark 4. To avoid introducing non-convex condition, we assume that $\boldsymbol{X}(\tilde{\boldsymbol{x}})$ only depends on states $\tilde{\boldsymbol{x}}$ whose corresponding row in $\boldsymbol{A}_i(\boldsymbol{x})$ does not contain non-polynomial nonlinear terms, and states whose dynamics is not directly affected by the control input, namely states whose corresponding rows in $\boldsymbol{B}_i(\boldsymbol{x})$ are zero.

Remark 5. When $\boldsymbol{A}_i(\boldsymbol{x})$, $\boldsymbol{B}_i(\boldsymbol{x})$, $\boldsymbol{F}_i(\boldsymbol{x})$ and $\boldsymbol{X}(\tilde{\boldsymbol{x}})$ are constant matrices and $\hat{\boldsymbol{x}}(\boldsymbol{x}) = \boldsymbol{x}$, the system representation is the same as the Takagi-Sugeno fuzzy model and control used in many of the references, e.g., (Tanaka [2001], Feng [2006]). Thus, our SOS approach to fuzzy model and control with polynomial rule consequence contains the existing LMI approaches to Takagi-Sugeno fuzzy model and control as a special case. Therefore, our SOS approach provides more relaxed results than the existing approaches to Takagi-Sugeno fuzzy model and control.

5. DESIGN EXAMPLE

To illustrate the validity of the design approach, this section provides a design example.

Consider the following nonlinear system:

$$\begin{cases} x_1(t+1) = x_1(t) + x_2^2(t) + u(t), \\ x_2(t+1) = -\tan x_1(t) + 2x_2(t), \end{cases} \quad (32)$$

where we assume that $-\pi/4 \le x_1(t) \le \pi/4$.

Using the concept of sector nonlinearity, we have the following polynomial fuzzy model that can exactly represent the dynamics under $-\pi/4 \le x_1(t) \le \pi/4$.

$$\boldsymbol{x}(t+1) = \sum_{i=1}^{2} h_i(z(t))\{\boldsymbol{A}_i(\boldsymbol{x}(t))\boldsymbol{x}(t) + \boldsymbol{B}_i(\boldsymbol{x}(t))u(t)\}, \quad (33)$$

where $\hat{\boldsymbol{x}}(t) = \boldsymbol{x}(t) = [x_1(t) \ x_2(t)]^T$, $z(t) = x_1(t)$ and

$$\boldsymbol{A}_1(\boldsymbol{x}(t)) = \begin{bmatrix} 1 & x_2(t) \\ -\dfrac{4}{\pi} & 2 \end{bmatrix},$$

$$\boldsymbol{A}_2(\boldsymbol{x}(t)) = \begin{bmatrix} 1 & x_2(t) \\ -1 & 2 \end{bmatrix},$$

$$\boldsymbol{B}_1(\boldsymbol{x}(t)) = \boldsymbol{B}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$\boldsymbol{B}_2(\boldsymbol{x}(t)) = \boldsymbol{B}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

The membership functions are obtained as

$$h_1(z(t)) = \frac{\tan x_1(t) - x_1(t)}{(\frac{4}{\pi} - 1)x_1(t)},$$

$$h_2(z(t)) = \frac{\frac{4}{\pi}x_1(t) - \tan x_1(t)}{(\frac{4}{\pi} - 1)x_1(t)}.$$

Note that $\boldsymbol{A}_1(\boldsymbol{x}(t))$ and $\boldsymbol{A}_2(\boldsymbol{x}(t))$ have polynomial elements.

As we stated in the previous paper (Tanaka [2008]), to obtain a T-S fuzzy model, the assumption that $-d \le x_2(t) \le d$ is needed, where d denotes a positive value. In addition, four rules are needed to represent the dynamics. On the other hand, we do not need the assumption that $-d \le x_2(t) \le d$ in the construction of the polynomial fuzzy models. Furthermore the number of rules in the polynomial fuzzy model is fewer than that in the T-S fuzzy model.

The SOS conditions in Theorem 3 are infeasible if $\boldsymbol{X}(\tilde{\boldsymbol{x}})$ and $\boldsymbol{M}_i(\boldsymbol{x})$ are constant matrices. However, the SOS conditions in Theorem 3 are feasible if we select $\boldsymbol{M}_i(\boldsymbol{x})$ as a polynomial matrix. Thus, the polynomial fuzzy controller (6) with the polynomial feedback vectors $\boldsymbol{F}_i(\boldsymbol{x})$ is more useful than the ordinary T-S fuzzy controller with the constant feedback vectors $\boldsymbol{F}_i$.

Furthermore, the SOS conditions in Theorem 4 are also feasible if we select $\boldsymbol{M}_i(\boldsymbol{x})$ as a polynomial matrix. The following polynomial feedback gains are found when the order of $\boldsymbol{M}_i(\boldsymbol{x})$ is one, i.e., when the order of $\boldsymbol{F}_i(\boldsymbol{x})$ is one.

$$\boldsymbol{F}_1(\boldsymbol{x}(t)) = \begin{bmatrix} 2.56 + 0.25e^{-6}x_1(t) - 0.03x_2(t) \\ -2.55 - 0.35e^{-6}x_1(t) + 0.86x_2(t) \end{bmatrix}^T$$

$$\boldsymbol{F}_2(\boldsymbol{x}(t)) = \begin{bmatrix} 2.24 + 0.79e^{-6}x_1(t) - 0.06x_2(t) \\ -2.51 - 0.16e^{-5}x_1(t) + 0.95x_2(t) \end{bmatrix}^T$$

Figure 1 shows the control result (using the above polynomial feedback gains) for the initial states $\boldsymbol{x}^T(0) = [0.5 \;\; 0]$. Figure 2 shows time transient of Lyapunov function during the control. The value of J in the guaranteed cost control is 1.37, where $\boldsymbol{Q} = \boldsymbol{I}$ and $\boldsymbol{R} = 1$. On the other hand, the value of J in the stable control is 2.46 for the same initial states. These results show the utility of our SOS-based design approach.

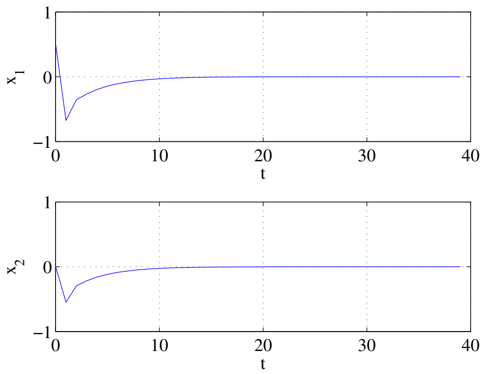

Fig. 1. Control result.

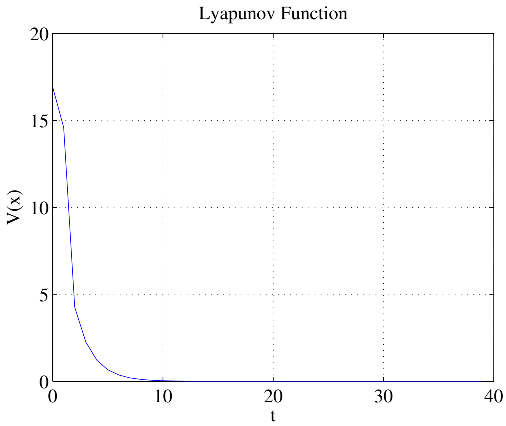

Fig. 2. Time transient of Lyapunov function.

6. CONCLUSIONS

This paper has presented a sum of squares (SOS) approach to guaranteed cost control of polynomial discrete fuzzy systems. First, we have presented a polynomial discrete fuzzy model that is more general representation of the well-known discrete Takagi-Sugeno (T-S) fuzzy model. Secondly, we have derived a design condition based on polynomial Lyapunov functions that contain quadratic Lyapunov functions as a special case. The design condition realizes guaranteed cost control by minimizing the upper bound of a given performance function. In addition, the design condition can be represented in terms of SOS and is numerically (partially symbolically) solved via the recent developed SOSTOOLS. A design example has been provided to illustrate the validity of the design approach.

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