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A Sampled-Data Scheme for Disturbance Rejection of Nonlinear Systems in Output Feedback Form ⋆

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Abstract: This paper presents a sampled-data control scheme for disturbance rejection of nonlinear systems in output feedback form. The continuous-time controller is designed first using a filtered transformation and the internal model technique. Obtained on an emulation-based approach, the proposed sampled-data control uses the sampled output and a discrete-time implementation of the filter and the internal model is involved. The proposed control is shown to render the overall system stable in a spirit of fast sampling. Specifically, the ultimate bound of the output is allowed to be arbitrarily small by choosing appropriate gain parameters.

1. INTRODUCTION

Nowadays digital computers have been widely used for implementation of control strategies. For linear systems, computer implementation can be carried out based on the discrete-time version of the control design, as there is always discrete time version which has the same structure as the original continuous-time model. Sampled-data control of linear systems has been thoroughly studied in literature, see Chen and Francis [1995], for instance.

In fact most of the engineering systems are nonlinear, for which a number of control design methods and control strategies have been proposed during the past decades. To implement control laws for nonlinear systems in discrete time, we need to investigate the performance of nonlinear sampled-data systems. Unfortunately the results on sampled-data control of linear systems can not be applied directly to nonlinear sampled-data systems, due to the fact that unlike in the linear context, an exact, discrete-time model of a general nonlinear system is hard to obtain and then not available for controller design. Most of the continuous-time nonlinear theories are not applicable to the sampled-data case either, since they rely on particular structures of nonlinear systems, which are usually destroyed by sampling, for example, feedback linearizability [Grizzle and Kokotovic, 1988]. These facts strongly motivate the research on nonlinear sampled-data control systems, to which a great deal of attention has been drawn recently, see Hou et al. [1997], Teel et al. [1998], Dabroom and Khalil [2001], Khalil [2004] and Nesic and Teel [2004].

Results on the problem of output feedback sampled-data control of nonlinear systems have been reported in Dabroom and Khalil [2001] and Khalil [2004]. Both of them employed the emulation method to design sampled-data controllers for the same class of disturbance-free systems, while continuous-time controllers are actually state-feedback controllers using high-gain observers. The results in Dabroom and Khalil [2001] and Khalil [2004] showed that, for some class of nonlinear systems, the obtained sampled-data controller can recover the performance of the continuous-time state feedback controller, provided that the sampling period $T \to 0$.

This paper concentrates on sampled-data control of disturbance rejection for a class of nonlinear systems in the output feedback form. The sampled-data controller is designed using emulation approach to make full use of existing methods for designing continuous-time controllers. A rigorous analysis is presented in the paper to show that under fast sampling condition, the stability of the overall sampled-data control system can be established. Particularly, given any compact sets of initial state and any $\epsilon > 0$, there exists a T^* such that for all $T \in (0, T^*)$, $|y(t)|$ is ultimately bounded by ϵ.

2. PROBLEM STATEMENT

Considered in the paper is a class of nonlinear systems in the form:

$$\begin{aligned} \dot{x} &= Ax + \phi(y) + bu + Ew, \\ y &= Cx, \end{aligned} \tag{1}$$

with

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ b_\rho \\ \vdots \\ b_n \end{bmatrix}$$

$$C = [1 \quad 0 \quad \cdots \quad 0]$$

where $x \in R^n$ is the state vector, $u \in R$ the control, $y \in R$ the output, $\phi(y)$ a known nonlinear smooth vector field

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in R^n and $\phi(0) = 0$, and $w \in R^n$ the disturbance and generated from the exosystem $\dot{w} = Sw$.

Assumption 1 The system is of minimum phase, i.e., the polynomial $\mathcal{B}(s) = \sum_{i=\rho}^{n} b_i s^{n-i}$ is Hurwitz.

Assumption 2 The eigenvalues of S are with zero real parts and are distinct.

The problem concerned is to investigate the conditions, particularly those on the sampling period and gain parameters, under which the sampled-data version of a continuous-time controller that stabilises the system to be controlled with the property $\lim_{t\to\infty} y(t) = 0$, will still stabilise the close-loop sampled-data system and recover disturbance rejection property to a certain degree.

3. PRELIMINARY RESULTS

This section outlines design of the continuous-time controller which stabilises the whole system with the property $\lim_{t\to\infty} y(t) = 0$.

3.1 State transformation

For system (1) with relative degree $\rho \geq 2$, we can introduce the following $(\rho-1)$th order filter

$$\begin{aligned}\dot{\xi}_1 &= -\lambda_1\xi_1 + \xi_2\\ \dot{\xi}_2 &= -\lambda_2\xi_2 + \xi_3\\ &\vdots\\ \dot{\xi}_{\rho-1} &= -\lambda_{\rho-1}\xi_{\rho-1} + u \end{aligned} \tag{2}$$

where $\lambda_i > 0$, $i = 1, \cdots, \rho-1$, are the design parameters. With the vectors $\bar{d}_i \in R^n$ for $i = 1, \cdots, \rho-1$, defined recursively by $\bar{d}_{\rho-1} = b$ and $\bar{d}_i = A_c\bar{d}_{i+1} + \lambda_{i+1}\bar{d}_{i+1}$ for $i = \rho-2, \cdots, 1$, the following filtered transformation

$$\zeta = x - \sum_{i=1}^{\rho-1} \bar{d}_i\xi_i \tag{3}$$

can transform system (1) into

$$\begin{aligned}\dot{\zeta} &= A\zeta + \phi(y) + d\xi_1 + Ew\\ y &= C\zeta\end{aligned} \tag{4}$$

where $d = [A_c\bar{d}_1 + \lambda_1\bar{d}_1]$. It can be shown that $d_1 = b_\rho$ and

$$\sum_{i=1}^{n} d_i s^{n-i} = \mathcal{B}(s)\prod_{i=1}^{\rho-1}(s+\lambda_i) \tag{5}$$

Since all λ_i are positive, d is a Hurwitz vector with $d_1 = b_\rho = 1$ (here we assume $b_\rho = 1$ without loss of generality). Therefore with ξ_1 being the input, system (4) is of minimum phase and relative degree one. To extract the internal dynamics of (4), introduce the following state transform

$$\begin{aligned} z_1 &= \zeta_2 - d_2\zeta_1\\ &\vdots\\ z_{n-1} &= \zeta_n - d_n\zeta_1\\ y &= \zeta_1\end{aligned} \tag{6}$$

where $z \in R^{n-1}$. In the new coordinates, system (4) can be written as

$$\begin{aligned}\dot{z} &= Dz + \phi_z(y) + \bar{E}w\\ \dot{y} &= z_1 + \phi_y(y) + \xi_1 + E_1 w\end{aligned} \tag{7}$$

where D is the companion matrix of $d[1]$ and given by

$$D = \begin{bmatrix} -d_2 & 1 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ -d_n & 0 & \cdots & 0\end{bmatrix}$$

and

$$\phi_z(y) = y\begin{bmatrix} d_3 - d_2^2\\ d_4 - d_3d_2\\ \vdots\\ d_n - d_{n-1}d_2\\ -d_nd_2\end{bmatrix} + \begin{bmatrix}\phi_2 - \phi_1 d_2\\ \phi_3 - \phi_1 d_3\\ \vdots\\ \phi_{n-1} - \phi_1 d_{n-1}\\ \phi_n - \phi_1 d_n\end{bmatrix}$$

$$\phi_y(y) = \phi_1 + yd_2$$

$$\bar{E} = \begin{bmatrix} E_2 - E_1 d_2\\ E_3 - E_1 d_3\\ \vdots\\ E_{n-1} - E_1 d_{n-1}\\ E_n - E_1 d_n\end{bmatrix}$$

where ϕ_i is the ith component of the vector ϕ and E_i the ith row vector of E, respectively.

The continuous-time control design is based on the internal model principle with the following assumption:

Assumption 3 There exists an invariant manifold $\pi(w) \in R^{n-1}$ satisfying [Ding, 2003]

$$\frac{\partial\pi(w)}{\partial w}Sw = D\pi(w) + \bar{E}w \tag{8}$$

It follows from Assumption 3 that $\pi_1(w) + \alpha + E_1 w = 0$. With the technique of system immersion

$$\begin{aligned}\frac{\partial\theta(w)}{\partial w} &= \Phi\theta(w)\\ \alpha(w) &= \Theta\theta(w)\end{aligned}$$

and choosing a controllable pair $\{F, G\}$ with appropriate dimensions, the internal model can be reparametrized as the following

$$\begin{aligned}\dot{\eta} &= F\eta + Gl^T\eta\\ \alpha &= l^T\eta\end{aligned} \tag{9}$$

where $\eta = M\theta$, $l = \Theta M^{-1}$ with M being the solution of

$$M\Phi - FM = G\Theta \tag{10}$$

Finally applying the transform $\tilde{z} := z - \pi$ to system (7) produces the model for control design

$$\dot{\tilde{z}} = D\tilde{z} + \phi_z(y) \tag{11}$$

$$\dot{y} = \tilde{z}_1 + (\xi_1 - l^T\eta) + \phi_y(y) \tag{12}$$

together with the filter shown in (2).

3.2 Continuous-time control design

The following is a non-adaptive version of that in Ding [2003]. For presentation convenience, the continuous-time control design is demonstrated for two cases.

In case of $\rho = 1$, the control is designed as

$$u_{c1} = l^T\hat{\eta} - ky - \phi_y - y^{-1}\gamma_1^2\phi_y^T G^T P_\eta G\phi_y - y^{-1}\gamma_2^2\phi_z^T D^T P_z D\phi_z \quad (13)$$

$$\dot{\hat{\eta}} = F\hat{\eta} + Gu_{c1} \quad (14)$$

where positive reals k, γ_1 and γ_2 are design parameters, and P_z, P_η satisfy

$$D^T P_z + P_z D = -I, \quad F^T P_\eta + P_\eta F = -I$$

Define $\tilde{\eta} := \eta - \hat{\eta} + Gy$. The stability of the overall system can be shown via the Lyapunov function

$$V_{c1} = \gamma_1\tilde{\eta}^T P_\eta\tilde{\eta} + \gamma_2\tilde{z}^T P_z\tilde{z} + \frac{1}{2}y^2$$

It can be shown that there exist sufficiently large γ_1 and γ_2 and finally sufficiently large k such that its time derivative

$$\dot{V}_{c1} \le -c_1||\tilde{\eta}||^2 - c_2||\tilde{z}||^2 - c_3 y^2$$

where $c_1 = \gamma_1 - 4$, $c_2 = \gamma_2 - 2 - \gamma_1^2|P_\eta G|^2$, $c_3 = k - 1 - |Gl^T| - |l|^2 - \gamma_1^2|P_\eta FG|^2$.

If $\rho \ge 2$, backstepping technique will be applied to obtain the final control. The outline of control design is shown below.

$$\hat{\xi}_1 = u_{c1}$$

$$\hat{\xi}_2 = -y + \lambda_1\hat{\xi}_1 + \frac{\partial\hat{\xi}_1}{\partial\hat{\eta}}\dot{\hat{\eta}} + \frac{\partial\hat{\xi}_1}{\partial y}\left(\phi_y + \xi_1 - l^T\hat{\eta}\right) - 2\tilde{\xi}_1\left(\frac{\partial\hat{\xi}_1}{\partial y}\right)^2$$

$$\hat{\xi}_i = -\tilde{\xi}_{i-2} - \lambda_{i-1}\hat{\xi}_{i-1} + \frac{\partial\hat{\xi}_1}{\partial\hat{\eta}}\dot{\hat{\eta}} + \frac{\partial\hat{\xi}_1}{\partial y}\left(\phi_y + \xi_1 - l^T\hat{\eta}\right) - 2\tilde{\xi}_{i-1}\left(\frac{\partial\hat{\xi}_1}{\partial y}\right)^2 + \sum_{j=1}^{i-2}\frac{\partial\hat{\xi}_{i-1}}{\partial\xi_j}\dot{\xi}_j$$

where $\tilde{\xi}_i := \hat{\xi}_i - \xi_i$, $i = 1, \cdots, \rho - 1$. The estimate of the internal model is given by

$$\dot{\hat{\eta}} = F\hat{\eta} + G\xi_1 \quad (15)$$

In the end the continuous-time control is set as $u_{c2} = \hat{\xi}_\rho$.

Similarly, the stability analysis can be carried out using the Lyapunov function

$$V_{c2} = \gamma_1\tilde{\eta}^T P_\eta\tilde{\eta} + \gamma_2\tilde{z}^T P_z\tilde{z} + \frac{1}{2}y^2 + \sum_{i=1}^{\rho}\tilde{\xi}_i^2$$

with its time derivative

$$\dot{V}_{c2} \le -c_4||z||^2 - c_5||\tilde{\eta}||^2 - c_6 y^2 - \sum_{i=1}^{\rho}\lambda_i\tilde{\xi}_i^2$$

where $c_4 = c_1 - \frac{1}{2}|l|^2$, $c_5 = c_2 - \frac{1}{2}$ and $c_6 = c_3 - |l|^2 - Gl^T$. Notice that the above design involves a sequences of intermediate functions $\hat{\xi}_i$ with the following property.

$$\hat{\xi}_1 = \hat{\xi}_1(y, \hat{\eta}), \ \hat{\xi}_1(0,0) = 0,$$
$$\hat{\xi}_i = \hat{\xi}_i(y, \hat{\eta}, \xi_1, \cdots, \xi_{i-1}), \ \ \hat{\xi}_i(0, \cdots, 0) = 0, \quad i = 2, \cdots, \rho - 1. \quad (16)$$

4. MAIN RESULTS

The following lemma is needed.

Lemma 1. Let $V : R^n \to R$ be a continuously differentiable, radically unbounded, positive definite function. Define $\mathcal{D} := \{\chi \in R^n | V(\chi) \le r\}$ with $r > 0$. Suppose the following holds

$$\dot{V} \le -\alpha V + \beta V_m + \delta, \quad \forall t \in [mT, (m+1)T) \quad (17)$$

where $V_m := V(\chi(mT))$, $m = 0, 1, 2, \cdots,$, α, β are any given positive reals with $\alpha > \beta$, $\delta > 0$ is a constant and T is the sampling period. If $\chi(0) \in \mathcal{D}$, then the following holds for all $T > 0$:

$$V(\chi(t)) \le V_0 + \frac{\delta}{\alpha - \beta} + \frac{\delta}{\alpha} \triangleq \bar{V}, \forall t \ge 0 \quad (18)$$

and

$$V(\chi(t)) \to \frac{\delta}{\alpha - \beta} + \frac{\delta}{\alpha}, t \to \infty \quad (19)$$

Proof. Using comparison lemma [Khalil, 2002] it is easy to get from (17) that

$$V(\chi(t)) \le e^{-\alpha(t-mT)}V_m + \frac{1 - e^{-\alpha(t-mT)}}{\alpha}(\beta V_m + \delta) = q(t - mT)V_n + p(t - mT)\delta \quad (20)$$

where

$$q(t) := \left(e^{-\alpha t} + \frac{\beta}{\alpha}(1 - e^{-\alpha t})\right)$$

$$p(t) := \frac{1 - e^{-\alpha t}}{\alpha}$$

Letting $t = (m+1)T$ in (20) leads to

$$V_{m+1} \le q_T V_m + p_T\delta \quad (21)$$

where $q_T := q(T)$ and $p_T := p(T)$. Since $\alpha > \beta > 0$, then $q_T \in (0, 1)$. First we shall show that (18) holds. Assume (18) holds for $i = 0, 1, \cdots, j$, then

$$V_{j+1} \le q(T)V_j + p_T\delta \le q(T)\big(V_0 + \frac{\delta}{\alpha - \beta} + \frac{\delta}{\alpha}\big) + p_T\delta \le V_0 + \frac{\delta}{\alpha - \beta} + \frac{\delta}{\alpha} \quad (22)$$

As for $t \in (iT, (i+1)T)$, direct calculations using (20) give

$$V(\chi(t)) \le V_0 + p_T\delta \le \bar{V}, \quad t \in (0, T)$$
$$V(\chi(t)) \le V_1 + p_T\delta \le \bar{V}, \quad t \in (T, 2T)$$
$$\vdots$$
$$V(\chi(t)) \le V_j + p_T\delta \le q_T^j V_0 + \frac{p_T\delta}{1 - q_T} + p_T\delta \le \bar{V}, \ t \in (jT, (j+1)T)$$

which, together with (22), claims by induction that

$$V(\chi(t)) \le V_0 + \frac{\delta}{\alpha - \beta} + \frac{\delta}{\alpha}, \forall t \ge 0$$

Furthermore we have

$$V_m \le q_T^m V_0 + p_T\delta(1 + q_T + \cdots + q_T^{m-1}) = q_T^m V_0 + p_T\delta\frac{1 - q_T^m}{1 - q_T} \le q_T^m V_0 + \frac{p_T\delta}{1 - q_T} \quad (23)$$

Thus

$$\lim_{m\to\infty} V_m = \frac{\delta}{1-q_T} = \frac{\delta}{\alpha-\beta}$$

and consequently, from (20), the ultimate bound of $V(\chi(t))$ can be obtained as $\frac{\delta}{\alpha-\beta}+\frac{\delta}{\alpha}$.

4.1 Results for the case $\rho = 1$

In this case, the sampled-data controller can be simply implemented as follows

$$u_{d1}(mT) = u_{c1}(y(mT), \hat{\eta}(mT)) \quad (24)$$

$$\hat{\eta}(mT) = e^{FT}\hat{\eta}((m-1)T) + F^{-1}(1-e^{FT})u_{d1}((m-1)T) \quad (25)$$

In order to accomplish the analysis, we first investigate $|u_{d1} - u_{c1}|$ for the interval $t \in [mT, mT+T)$. Indeed,

$$|u_{d1}(y(mT), \hat{\eta}(mT)) - u_{c1}(y(t), \hat{\eta}(t))| \le L_u|y - y(mT)| + L_u||\hat{\eta} - \hat{\eta}(mT)|| \quad (26)$$

where L_u is the Lipschitz constant of u_{c1} with respect of the domain specified later.

As for $||\hat{\eta} - \hat{\eta}(mT)||$, we have from $\dot{\hat{\eta}} = F\hat{\eta} + Gu_{c1}$ that

$$||\hat{\eta} - \hat{\eta}(mT)|| \le (1 - e^{F(t-mT)})||\hat{\eta}(mT)|| + |u_{d1}(y(mT), \hat{\eta}(mT))| \int_{mT}^{t} Ge^{F(t-\tau)}d\tau.$$

Furthermore, the disturbance η is bounded, that is $|\eta(t)| \le c_0$. We then have from the definition $\tilde{\eta} := \eta - \hat{\eta} + Gy$ that

$$||\hat{\eta}(mT)|| \le ||\tilde{\eta}(mT)|| + ||G|||y(mT)| + c_0 \quad (27)$$

It can be verified that for $t \in [mT, mT+T)$

$$\|\hat{\eta} - \hat{\eta}(mT)\| \le \delta_1(T)|y(mT)| + \delta_2(T)(||\tilde{\eta}(mT)|| + c_0) \quad (28)$$

where $\delta_1(T) = 2F^{-1}L_uG\big(1-e^{FT}\big) + G(1-e^{FT})$, $\delta_2(T) = (F^{-1}L_u+1)\big(1-e^{FT}\big)$.

As for $|y - y(mT)|$, we start from the dynamics of y, which implies

$$y(t) = y(mT) + \int_{mT}^{t} \tilde{z}_1(\tau)d\tau + \int_{mT}^{t} (\phi_y - \phi_y(y(mT)))d\tau + \int_{mT}^{t} \phi_y(y(mT))d\tau + \int_{mT}^{t} |u_{d1} - l^T\eta|d\tau$$

which further produces

$$|y - y(mT)| \le \int_{mT}^{t} ||\tilde{z}(\tau)||d\tau + \int_{mT}^{t} L_1|y - y(mT)|d\tau + L_u\int_{mT}^{t} (|y(mT)| + ||\hat{\eta}(mT)||)d\tau + L_1\int_{mT}^{t} |y(mT)|d\tau + \int_{mT}^{t} ||l^T\eta||d\tau \quad (29)$$

where L_1 is a Lipschitz constant of ϕ_y with respect to y. Since D is a Hurwitz matrix, there exist positive reals k_2, σ such that $\|e^{D(t-mT)}\| \le k_2e^{-\sigma(t-mT)}$. Thus,

$$\|\tilde{z}(t)\| \le k_2e^{-\sigma(t-mT)}\|\tilde{z}(mT)\| + L_2|y(mT)|\int_{mT}^{t} k_2e^{-\sigma(t-\tau)}d\tau + L_2\int_{mT}^{t} k_2e^{-\sigma(t-\tau)}|y(\tau) - y(mT)|d\tau \quad (30)$$

where L_2 is a Lipschitz constant of ϕ_z with respect to y. Then the following inequality holds

$$\int_{mT}^{t} \|\tilde{z}(\tau)\|d\tau \le \frac{k_2\|\tilde{z}(mT)\|}{\sigma}\big(1-e^{-\sigma(t-mT)}\big) + \frac{k_2L_2}{\sigma}|y(mT)|(t-mT) + \frac{k_2L_2}{\sigma}\int_{mT}^{t} |y(\tau) - y(mT)|d\tau \quad (31)$$

Therefore from (29) and (31) the estimate of $|y - y(mT)|$ shown in (29) is computed as

$$|y(t) - y(mT)| \le A_1\big(1-e^{-\sigma(t-mT)}\big) + B_1(t-mT) + H\int_{mT}^{t} |y(\tau) - y(mT)|d\tau \quad (32)$$

where $A_1 = \sigma^{-1}k_2\|z(mT)\|$, $B_1 = L_{u1}|y(mT)| + L_1|y(mT)| + \sigma^{-1}k_2L_2|y(mT)| + L_u||\hat{\eta}(mT)|| + l^Tc_0$ and $H = \sigma^{-1}k_2L_2 + L_1$. Invoking Gronwall-Bellman inequality [Khalil, 2002] we then have

$$|y - y(mT)| \le \delta_3(T)|y(mT)| + \delta_4(T)||\tilde{z}(mT)|| + \delta_5(T)||\tilde{\eta}(mT)|| + \delta_6(T)c_0 \quad (33)$$

where

$$\delta_3(T) = H^{-1}(L_{u1} + L_1 + \sigma^{-1}k_2L_2)\big(e^{HT}-1\big)$$
$$\delta_4(T) = \sigma^{-1}k_2\big(\sigma e^{HT} + He^{-\sigma T} - (H+\sigma)\big)(H+\sigma)^{-1} + \sigma^{-1}k_2\big(1-e^{-\sigma T}\big)$$
$$\delta_5(T) = H^{-1}L_{u1}\big(e^{HT}-1\big)$$
$$\delta_6(T) = H^{-1}|l|\big(e^{HT}-1\big)$$

For the case of $\rho = 1$, the results can be summarised in the following theorem.

Theorem 2. When the controller shown in (24) and (25) is applied to system (1), the following holds for any given compact set of initial state, $X_0 \subset R^n$, and any $\epsilon > 0$: there exist a $T^* > 0$ and sufficiently large gain parameters, γ_1, γ_2 and k such that for all $T \in (0, T^*)$, all the signals of the overall system remain bounded with the property $\sup|y(t)| \le \epsilon$, as $t \to \infty$.

Proof. First specify some concerned sets. Let Z_0 and Y_0 be corresponding mapped sets for any given set X_0. let

$\tilde{z}_0 \in Z_0$ and $y_0 \in Y_0$. Set $\hat{\eta}_0 = 0$. Then with the definition $\tilde{\eta} = \eta - \hat{\eta} + Gy$, then an appropriate set Π_0 can be found such that $\tilde{\eta}_0 \in \Pi_0$ whenever $\tilde{z}_0 \in Z_0$ and $y_0 \in Y_0$. Define

$$\bar{V}_1 := \max_{\tilde{\eta}_0 \in \Pi_0, \tilde{z} \in Z_0, y_0 \in Y_0} V_{c1}(\tilde{\eta}, \tilde{z}, y),$$

and $\bar{V}_2 = \bar{V}_1 + 2c_0^2$. It then can be concluded that there exist Z_Ω, Y_Ω and Π_Ω such that when $V(\tilde{\eta}, \tilde{z}, y) \leq \bar{V}_2$, we have

$$\tilde{\eta} \in \Pi_\Omega, \tilde{z} \in Z_\Omega, y \in Y_\Omega.$$

It follows that all the Lipschitz constants including L_u, L_1 and L_2 are chosen based on the sets Z_Ω, Y_Ω and Π_Ω.

Now investigate the derivative in the interval $[mT, mT+T)$ of V_{c1} for the sampled-data system. When the sampled-data controller is applied to the continuous-time plant, we have

$$\dot{V}_{c1} \leq -c_1||\tilde{\eta}||^2 - c_2||\tilde{z}||^2 - c_3 y^2 + |y||u_{d1} - u_{c1}| \quad (34)$$

Combining (26), (28) and (33) produces

$$\begin{aligned} |y||u_{d1} - u_{c1}| \leq & \frac{L_u}{2}(\delta_2 + \delta_5)\|\tilde{\eta}(mT)\|^2 + \\ & \frac{L_u}{2}\delta_4\|\tilde{z}(mT)\|^2 + \frac{L_u}{2}(\delta_1 + \delta_3)|y(mT)|^2 \\ & + \frac{L_u}{2}|y|^2 \sum_{i=}^{6} \delta_i + \frac{L_u}{2}(\delta_2 + \delta_6)c_0^2 \\ \triangleq & \varepsilon_1||\tilde{\eta}(mT)||^2 + \varepsilon_2||\tilde{z}(mT)||^2 + \varepsilon_3|y(mT)|^2 \\ & + \varepsilon_4|y|^2 + \varepsilon_5 c_0^2 \end{aligned} \quad (35)$$

Substituting (35) into (34) we have

$$\begin{aligned} \dot{V}_{c1} \leq & -c_1||\tilde{\eta}||^2 - c_2||z||^2 - (c_3 - \varepsilon_4)y^2 \\ & + \varepsilon_1||\tilde{\eta}(mT)||^2 + + \varepsilon_2||\tilde{z}(mT)||^2 + \varepsilon_3|y(mT)|^2 + \varepsilon_5 c_0^2 \\ \triangleq & -\alpha_1 V + \beta_1 V_n + \varepsilon_5 c_0^2 \end{aligned} \quad (36)$$

where

$$\begin{aligned} \alpha_1 &= \min\left\{\frac{c_1}{\lambda_{max}(P_\eta)}, \frac{c_2}{\lambda_{max}(P_z)}, 2(c_3 - \varepsilon_4)\right\} \\ \beta_1 &= \max\left\{\frac{\varepsilon_1}{\lambda_{min}(P_\eta)}, \frac{\varepsilon_2}{\lambda_{min}(P_z)}, 2\varepsilon_3\right\} \end{aligned} \quad (37)$$

Define T_1 such that if $T \in (0, T_1)$, $\alpha_1 > \beta_1$. Define T_2 such that $\varepsilon_5(T) < 1$ for all $T \in (0, T_2)$. The existence of such T_1, T_2 is assured by the continuity of $\delta_i(T)$ and the fact that $\delta_i(0) = 0$. Next define $T^* = \min(T_1, T_2)$ and choose γ_1, γ_2 and k such that

$$\frac{c_0^2}{\alpha_1 - \beta_1} + \frac{c_0^2}{\alpha_1} \leq \min\{2c_0^2, \frac{1}{2}\epsilon^2\}.$$

Then consider the case that $\tilde{\eta}_0 \in \Pi_0$, $\tilde{z}_0 \in Z_0$ and $y_0 \in Y_0$ for all $T \in (0, T^*)$. It follows from Lemma 1 that

$$V_{c1} \leq V_0 + \frac{c_0^2}{\alpha_1 - \beta_1} + \frac{c_0^2}{\alpha_1} \leq \bar{V}_2, \text{for all } t \geq 0,$$

which implies that all the signals in the close-loop sampled-data system are bounded. Also from Lemma 1, we have

$$V_{c1} \to \frac{c_0^2}{\alpha_1 - \beta_1} + \frac{c_0^2}{\alpha_1}, \text{ as } t \to \infty \quad (38)$$

Accordingly we have $\sup|y(t)| \leq \epsilon, \text{as } t \to \infty$, which completes the proof.

4.2 Results for the case $\rho \geq 2$

For $\rho \geq 2$, the sampled-data controller will be of the form

$$\begin{aligned} u_{d2} &= u_{c2}(y(mT), \hat{\eta}(mT), \xi(mT)) & (39) \\ \hat{\eta}(mT) &= e^{FT}\hat{\eta}((m-1)T) + F^{-1}(1 - e^{FT})\xi_1(mT) & (40) \\ \xi(mT) &= e^{\Lambda T}\xi((m-1)T) \\ & + \Lambda^{-1}(1 - e^{\Lambda T})u_{d2}((m-1)T) & (41) \end{aligned}$$

where $\xi := [\,\xi_1, \cdots, \xi_{\rho-1}\,]'$. The following theorem summarizes the primary result for this case.

Theorem 3. If the controller shown in (39)–(41) is applied to system (1), the following holds for any given compact set of initial state, $X_0 \subset R^n$, and any $\epsilon > 0$: there exist a $T^* > 0$ and sufficiently large gain parameters, γ_1, γ_2, $k > 0$ such that for all $T \in (0, T^*)$, all the signals of the overall system remain bounded with the property $\sup|y(t)| \leq \epsilon, \text{as } t \to \infty$.

Proof. The stability analysis for the case $\rho \geq 2$ is more involved due to the inclusion of the filter (41), yet can still be established in a similar way to the case $\rho = 1$. Only a sketch of proof is given here due to the page restriction.

We aim to formulate the derivative of V_{c2} for the sampled-data system in the interval $[mT, mT + T)$ into a form similar to (36). Following the same procedures as for the case $\rho = 1$, the bound of $|u_{d2} - u_{c2}|$ during the interval $[mT, (m + 1)T)$ needs to be computed first. Only this time ξ comes into play besides y, $\tilde{z}$ and $\tilde{\eta}$. Since ξ is governed by a linear equation (2), then ξ can be treated in the same way as $\hat{\eta}$. Subsequently, the bound of $|u_{d2} - u_{c2}|$ is generally expected to contain terms dependent on $||\xi(mT)||$. Therefore, the conclusion can be established in the same was as for the case $\rho = 1$ if $||\xi(mT)||$ can be expressed by the terms exclusively using $||\tilde{\xi}(mT)||$, $||\hat{\eta}(mT)||$, $|y(mT)|$, which is shown below.

In fact, due to the special structure of equation (2) and the properties of stabilising functions $\hat{\xi}_i$ shown in (16) we have

$$\begin{aligned} |\xi_1(mT)| \leq & |\tilde{\xi}_1(mT)| \leq |\tilde{\xi}_1(mT)| \\ & + \mathcal{L}_1(|y(mT)| + ||\hat{\eta}(mT)||) \\ |\xi_2(mT)| \leq & |\tilde{\xi}_2(mT)| + |\hat{\xi}_2(mT)| \leq |\tilde{\xi}_2(mT)| \\ & + \mathcal{L}_2|y(mT)| + \mathcal{L}_2||\hat{\eta}_n|| + \mathcal{L}_2|\xi_1(mT)| \\ & \vdots \\ |\xi_{\rho-1}(mT)| \leq & |\tilde{\xi}_{\rho-1}(mT)| + |\hat{\xi}_{\rho-1}(mT)| \leq |\tilde{\xi}_{\rho-1}(mT)| \\ & + \mathcal{L}_{\rho-1}(|y(mT)| + ||\hat{\eta}(mT)|| + \sum_{i=1}^{\rho-1}|\xi_i(mT)|) \end{aligned}$$

where with a bit abuse of notation, $\mathcal{L}_i$ is the Lipschitz constant of $\hat{\xi}_i$. Note that $\mathcal{L}_i$ is constant in the domain concerned. Thus, a constant $\mathcal{L}_0$ can be found such that

$$||\xi(mT)|| \leq \mathcal{L}_0(||\hat{\eta}(mT)|| + ||\tilde{\xi}(mT)|| + |y(mT)|).$$

Remark 1. Both Theorem 2 and Theorem 3 only declare the existence of a certain upper limit of sampling period, but states no information of the affects of control parameters and initial sets of the system on the upper limit. The way the initial set and the controller parameters affect the upper limit is still subject to further investigations.

5. SIMULATION

A simple example is included in the section to illustrate the main result of the paper. Consider the following system

$$\dot{x}_1 = x_2 + y^2 + u + w_1,$$
$$\dot{x}_2 = u + w_1;$$

where w is generated by

$$\dot{w} = \begin{bmatrix} 0 & 20 \\ -20 & 0 \end{bmatrix} w, \; w(0) = [10, \; 10]'.$$

Repeating the steps presented in Section 3.2 results in a continuous-time control shown in (13) with the following parameters:

$$F = \begin{bmatrix} -3 & 1 \\ -2 & 0 \end{bmatrix}, P_\eta = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -1 \end{bmatrix}, M = \begin{bmatrix} 0.1 & -0.3 \\ 0.6 & -0.8 \end{bmatrix},$$

and $G^T = [0 \quad 1]$, $l^T = [-8 \quad 3]$, $P_z = 1/2$, $\phi_z(y) = y^2 - y$, $\phi_y = y + y^2$. Set $\gamma_1 = 5$, $\gamma_2 = 40$, $k = 120$ for simulation studies.

Simulation results are shown in Fig.1–Fig.2. Results shown in Fig.3 and Fig.4 indicate that the sampled-data system is still stable when $T = 0.001s$ and $T = 0.002s$, and the disturbance is rejected to a certain extent where the output is kept very small. Particularly, a comparison of the results for two choices of T reveals that smaller T leads to better performance. In fact, further simulations show that the sampled-data system will be unstable if T increases to $0.01s$.

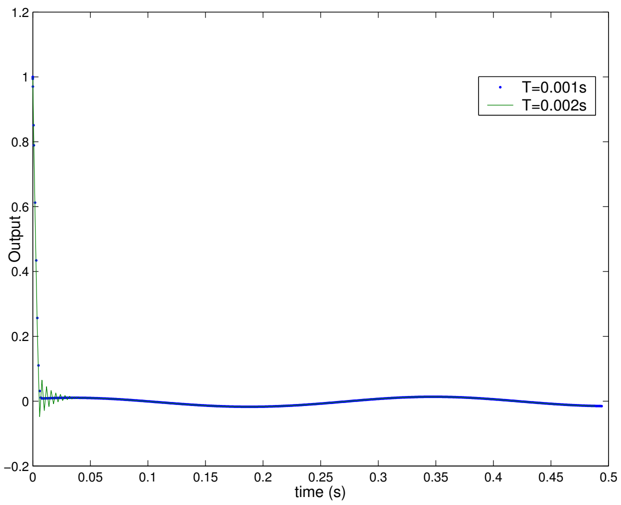

Fig. 1. Output of the sampled-data system

6. CONCLUSION

For disturbance rejection of a class of nonlinear systems in output feedback form, the paper has put forth a sampled-data scheme, which comprises the digital implementation

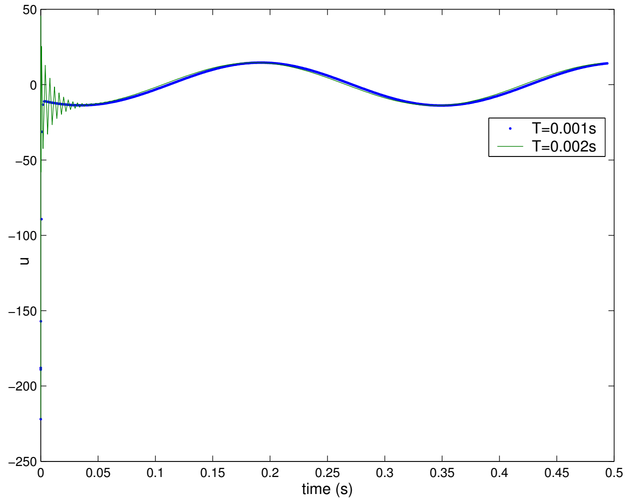

Fig. 2. Output of the sampled-data system

of the internal model and the filtered transformation. An analysis has been carried out to show that all the signals in the overall sample-data control system are bounded, provided that the sampling is fast enough, and specifically, the ultimate bound of the output can be made arbitrarily small by properly scheduling the gain parameters.

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