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Discontinuous Output Regulation of the Pendubot ⋆

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Abstract: In this work we address the problem of nonlinear output regulation of an underactuated system by means of discontinuous control actions, in particular, the sliding mode output regulator problem in the case of error feedback is considered for the Pendubot system. The theory is revisited for nonlinear systems presented in the so-called Regular form. Simulations are carried out to verify the effectiveness of the discontinuous method.

Keywords: Regulation; Nonlinear system control.

1. INTRODUCTION

The Pendubot Spong et al. (1989) is a two link planar underactuated robotic mechanism, whose first link is actuated while the second one is not. The main purpose of the Pendubot is research and education within the nonlinear control systems framework. Concepts like nonlinear dynamics, linearization, robotics and control systems design can be achieved with the Pendubot. Common control problems for the Pendubot are swing up, stabilization and tracking. The interest of this work is the tracking of a sinusoidal shape signal for the second link angle. Since trajectory tracking is the central role of output regulation theory Isidori et al. (1990), results of interest to design output regulators for the Pendubot. In the works of Ramos et al. (1997) and Samposh et al. (2002), classical output regulators has been designed for the Pendubot system where perturbations are not considered. In the work of Loukianov et al. (1999) a full state feedback sliding mode output regulator has been designed and then applied to the Pendubot as well, where sliding mode control introduces robustness to matched perturbations. In this work, the application of an error feedback sliding mode output regulation technique Loukianov et al. (2004) is applied to the Pendubot and compared with the classical output regulator.

The paper is organized as follows. In Section 2, the regulator theory concepts are briefly revisited for the classical and discontinuous settings. Section 3, consist of the application of the sliding mode regulation to the Pendubot. Simulation results are presented in Section 4. Final comments conclude the paper in Section 5.

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2. REGULATOR THEORY CONCEPTS

In this section, the classical output regulator concepts are briefly revisited as well for the sliding mode output regulator.

2.1 Error feedback regulator

Consider a nonlinear system

$$\dot{x} = f(x) + g(x)u + d(x)w \tag{1}$$

$$y = h(x) \tag{2}$$

with state x, defined on a neighborhood X of the origin of $\Re^n$, and $u \in \Re^m$, $y \in \Re^p$. The vector $f(x)$, the columns of $g(x)$ and $d(x)$ are smooth vector fields of class $C^{\infty}_{[t,\infty)}$, and in addition, it is assumed that $f(0) = 0$ and $h(0) = 0$. The output tracking error is defined as the difference between the output of the system, y, and a reference signal, $q(w)$, i.e.

$$e = y - q(w) \tag{3}$$

where the reference signal, $q(w)$, is generated by a given exosystem described by

$$\dot{w} = s(w),\ s(0) = 0 \tag{4}$$

with state w, defined on a neighborhood W of the origin of $\Re^s$. This system is characterized by the following assumption :

H1. *The Jacobian matrix* $S = \left[\frac{\partial s}{\partial w}\right]_{(0)}$ *at the equilibrium point* $w = 0$ *has all eigenvalues on the imaginary axis.*

It is assumed also that only the components of the error e are available for measurement. In Isidori et al. (1990) it has been shown that the control action to (1) can be provided by an *error feedback* dynamic system

$$\dot{\xi} = \eta(\xi, e)$$
$$u = \theta(\xi). \quad (5)$$

where $\xi \in \Xi \subset \Re^r$. The solvability of the *Error Feedback Regulator Problem* (EFRP), under assumption H1, can be stated in terms of the existence of a pair of mappings $x = \pi(w)$ and $\xi = \rho(w)$, with $\pi(0) = 0$ and $\rho(0) = 0$, which solve the partial differential equation (FIB equations)

$$f(\pi(w)) + g(\pi(w))\theta(\rho(w)) + d(\pi(w))w = \frac{\partial\pi(w)}{\partial w}s(w)$$
$$\eta(\rho(w), 0) = \frac{\partial\rho(w)}{\partial w}s(w) \quad (6)$$
$$h(\pi(w)) - q(w) = 0.$$

The linear solution may be derived by considering the linear approximation of the system (1) - (4) at the equilibrium point $(x, w) = (0, 0)$, namely

$$\dot{x} = Ax + Bu + Dw$$
$$\dot{w} = Sw \quad (7)$$
$$e = Cx - Qw$$
$$(8)$$

where $A = \left[\frac{\partial f(x)}{\partial x}\right]_{(0)}$, $B = g(0)$, $D = d(0)$, $S = \left[\frac{\partial s(w)}{\partial w}\right]_{(0)}$, $C = \left[\frac{\partial h}{\partial x}\right]_{(0)}$, and $Q = \left[\frac{\partial q}{\partial w}\right]_{(0)}$. In this case, equations (6) take the form of the *Sylvester* matrix equation

$$A\Pi + BH\Sigma + D = \Pi S \quad (9)$$
$$F\Sigma = \Sigma S \quad (10)$$
$$C\Pi - Q = 0 \quad (11)$$

where $\Sigma = \left[\frac{\partial\theta(\xi)}{\partial w}\right]_{(0)}$; $F = \left[\frac{\partial\eta(\xi,0))}{\partial w}\right]_{(0)}$ and $\Pi = \left[\frac{\partial\pi(w)}{\partial w}\right]_{(0)}$. It can be shown that the existence of the previous equations are implied, under some additional conditions, by the existence of a solution $x = \Pi w$ and $u = \Gamma w$ of the following equations Isidori et al. (1990):

$$A\Pi + B\Gamma + D = \Pi S \quad (12)$$
$$C\Pi - Q = 0 \quad (13)$$

and thus, to guarantee the solvability of the EFRP, conditions (12) and (13) are assumed, together with the following necessary conditions:

H2. *The pair* $\{A, B\}$ *is stabilizable and*

H3. *The pair* $\left\{[\,C\ Q\,], \begin{bmatrix} A & D \\ 0 & S \end{bmatrix}\right\}$ *is detectable.*

2.2 Error feedback sliding mode control problem

Analogously to EFRP, the *Error Feedback Sliding Mode Regulation Problem* (EFSMRP) is defined as the problem of finding a function $\sigma(\xi) = (\sigma_1, \ldots, \sigma_m)^T$ and a dynamic discontinuous controller

$$\dot{\xi} = \eta(\xi, u, e) \quad (14)$$
$$u_i(\xi) = \begin{cases} u_i^+(\xi) & if \quad \sigma_i(\xi) > 0 \\ u_i^-(\xi) & if \quad \sigma_i(\xi) < 0 \end{cases}, \quad i = 1, \ldots, m \quad (15)$$

where $u_i^+(\xi)$, $u_i^-(\xi)$ are chosen to induce asymptotic convergence of the state vector to the manifold

$$\sigma(\xi) = 0 \quad (16)$$

such that the following conditions hold:

- (SMS_{ef}) (Sliding Mode Stability). The state of the closed-loop system (1)-(2), with the controller (14)-(15), converges to the manifold (16) in finite time;
- (S_{ef}). The equilibrium $(x, \xi) = (0, 0)$ of the sliding mode dynamics
$$\dot{x} = [f(x) + g(x), u_{eq}]\,|_{\sigma(\xi)=0}$$
$$\dot{\xi} = \eta(\xi, u_{eq}, 0)$$
is asymptotically stable, where u_{eq} is the equivalent control derived from the condition $\dot{\sigma} = 0$;
- (R_{ef}). There exists a neighborhood $V \subset X \times \Xi \times W$ of $(0, 0, 0)$ such that, for each initial condition $(x_0, \xi_0, w_0) \in V$, the output tracking error (3) goes asymptotically to zero, i.e. $lim_{t\to\infty}e(t) = 0$.

In the following, the case of nonlinear systems presented in the regular form will be presented.

2.3 Sliding regulator for nonlinear systems in regular form

Let us now consider that the nonlinear system (1) is transformed by a diffeomorphism $x' = \varphi(x)$ to the Regular form Loukianov et al. (1981):

$$\dot{x}_1 = f_1(x_1, x_2) + d_1(x_1, x_2)w \quad (17)$$
$$\dot{x}_2 = f_2(x') + g_2(x')u + d_2(x')w$$
$$\dot{w} = s(w) \quad (18)$$
$$e = h(x_1, x_2) - q(w) \quad (19)$$

where $x' = (x_1, x_2)^T$, $x_1 \in X_1 \subset \Re^{n-m}$, $x_2 \in X_2 \subset \Re^m$ and $\text{rank}[g_2(x')] = m\ \forall x' \in X \subset \Re^n$.

Let us now introduce the steady state for x_1 and x_2 as $\pi_1(w)$ and $\pi_2(w)$, respectively. Then, defining the steady state error

$$z = x' - \pi(w) = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \begin{pmatrix} \pi_1(w) \\ \pi_2(w) \end{pmatrix} \quad (20)$$

the dynamic equation for (20) with tracking error e can be obtained from (17) - (19) as

$$\dot{z}_1 = f_1(z_1 + \pi_1(w), z_2 + \pi_2(w)) \quad (21)$$
$$+d_1(z_1 + \pi_1(w), z_2 + \pi_2(w))w - \frac{\partial\pi_1(w)}{\partial w}s(w)$$
$$\dot{z}_2 = f_2(z + \pi(w)) + g_2(z + \pi(w))u + d_2'(z, w) \quad (22)$$
$$e = h(z_1 + \pi_1(w), z_2 + \pi_2(w)) - q(w) \quad (23)$$

where $d_2'(z, w) = d_2(z_1+\pi_1(w), z_2+\pi_2(w))w - \frac{\partial\pi_2(w)}{\partial w}s(w)$. The proposed sliding manifold is expressed as

$$\sigma_2 = z_2 - \sigma_1(z_1) = 0,\ \sigma_1(0) = 0,\ \left[\frac{\partial\sigma_1}{\partial z_1}\right]_{(0)} = \Sigma_1 \quad (24)$$

and the $(n - m)th$ order sliding mode equation describing the motion on (20), is given by

$$\begin{aligned}\dot{z}_1 = & f_1(z_1+\pi_1(w), \sigma_1(z_1)+\pi_2(w)) \\ & +d_1(z_1+\pi_1(w), \sigma_1(z_1)+\pi_2(w))w \\ & -\frac{\partial \pi_1(w)}{\partial w}s(w)\end{aligned} \tag{25}$$

To estimate the states of system (21)-(22) and (18), the proposed nonlinear observer is designed as

$$\dot{\xi} = \begin{bmatrix} f_1(\hat{z}_1+\pi_1(\hat{w}), \hat{z}_2+\pi_2(\hat{w})) \\ +d_1(\hat{z}_1+\pi_1(\hat{w}), \hat{z}_2+\pi_2(\hat{w}))\hat{w} \\ -\frac{\partial \pi_1(\hat{w})}{\partial \hat{w}}s(\hat{w}) \\ f_2(\hat{z}+\pi(\hat{w}))+g_2(\hat{z}+\pi(\hat{w}))u \\ +d_2'(\hat{z},\hat{w}) \\ s(\hat{w}) \end{bmatrix} + L'(e-\hat{e}) \tag{26}$$

with $\xi = (\hat{z}_1, \hat{z}_2, \hat{w})^T$ the estimate of $\zeta = (z_1, z_2, w)^T$, and $\hat{e} = h(\hat{z}_1+\pi_1(\hat{w}), \hat{z}_2+\pi_2(\hat{w})) - q(w)$. To analyze the stability of the sliding dynamics (25) and observer (26), the systems (21) - (23) and (18) are represented in the form:

$$\begin{aligned}\begin{pmatrix}\dot{z}_1\\ \dot{z}_2\end{pmatrix} = & \begin{pmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{pmatrix}\begin{pmatrix}z_1\\ z_2\end{pmatrix} + \begin{pmatrix}0\\ B_2\end{pmatrix}u \\ & +\begin{pmatrix}R_1\\ R_2\end{pmatrix}w + \begin{pmatrix}\phi_1(\zeta)\\ \phi_2(\zeta)\end{pmatrix} \\ \dot{w} = & Sw+\phi_w(w) \\ e = & C_1z_1+C_2z_2+(C_1\Pi_1+C_2\Pi_2-Q)w \\ & +\phi_e(\zeta).\end{aligned} \tag{27}$$

Then, the sliding mode equation (25) can be rewritten as

$$\dot{z}_1 = (A_{11}-A_{12}\Sigma_1)z_1+R_1w+\phi_{1s}(z_1,w)$$

where $R_1 = A_{11}\Pi_1+A_{12}\Pi_2-\Pi_1S+D_1$ and $R_2 = A_{21}\Pi_1+A_{22}\Pi_2-\Pi_2S+D_2$, with $A_{ij} = \left[\frac{\partial f_i}{\partial x_i}\right]_{(0,0)}$, $B_2 = g_2(0)$, $C_i = \left[\frac{\partial h}{\partial x_j}\right]_{(0,0)}$, $D_i = d_i(0,0)$, $\Pi_i = \left[\frac{\partial \pi_i}{\partial w}\right]_{(0)}$; the functions $\phi_i(z,w)$, $\phi_w(w)$, $\phi_e(z,w)$ and ϕ_{1s} vanish at the origin with its first derivatives; $\forall\, i,j = \{1,2\}$, and the constant matrices S and Q are already defined in assumption H1 and equation (7) respectively. Then using (26) - (27), the observer error dynamics becomes

$$\dot{\varepsilon} = (A'-L'C')\varepsilon+\Phi'(\zeta,\varepsilon) \tag{28}$$

where $\varepsilon = \zeta-\xi = (\varepsilon_1,\varepsilon_2,\varepsilon_3)^T$, $A' = \begin{pmatrix}A_{11} & A_{12} & R_1\\ A_{21} & A_{22} & R_2\\ 0 & 0 & S\end{pmatrix}$, $B' = \begin{pmatrix}0\\ B_2\\ 0\end{pmatrix}$, $L' = \begin{pmatrix}L_1\\ L_2\\ L_3\end{pmatrix}$,

$$C' = (\,C_1,\ C_2,\ (C_1\Pi_1+C_2\Pi_2-Q)\,)$$

and $\Phi'(\zeta,\varepsilon) = \begin{pmatrix}\phi_1(\zeta)-\phi_1(\xi)+L_1(\phi_e(\zeta)-\phi_e(\xi))\\ \phi_2(\zeta,u)-\phi_2(\xi,u)+L_2(\phi_e(\zeta)-\phi_e(\xi))\\ \phi_w(w)-\phi_w(\hat{w})+L_3(\phi_e(\zeta)-\phi_e(\xi))\end{pmatrix}$.

An assumption similar to H4) is at this point introduced to guarantee the stability of the system (28).

H5. *The pair $\{C', A'\}$ is detectable.*

Before defining the estimated sliding manifold and control, the solvability conditions of the EFSMRP for the nonlinear system in Regular form will be derived.

Proposition 1. Under assumptions H1, H2 and H5, if there exists C^k $(k \geq 2)$ mappings $x_1 = \pi_1(w)$ and $x_2 = \pi_2(w)$, with $\pi_1(0) = 0$ and $\pi_2(0) = 0$, defined in neighborhood W of 0 , that satisfy the following conditions:

$$\begin{aligned}\frac{\partial \pi_1(w)}{\partial w}s(w) = & f_1(\pi_1(w),\pi_2(w)) \\ & +d_1(\pi_1(w),\pi_2(w))w \end{aligned} \tag{29}$$

$$0 = h(\pi_1(w),\pi_2(w))-q(w) \tag{30}$$

then, the EFSMRP for nonlinear systems in Regular form is solvable.

Proof 1. We define the estimated sliding manifold and control as

$$u = -kB_2^{-1}sign(\hat{\sigma}),\ \hat{\sigma}_2 = \hat{z}_2+\hat{\sigma}_1(\hat{z}_1) = 0.$$

If the control gain k is chosen such such that $k > \| g_2(\hat{z},\hat{w})u_{eq}(\hat{z},\hat{w}) \|$ where $u_{eq}(\hat{z},\hat{w})$ is a solution of $\dot{\hat{\sigma}} = 0$, then the condition (SMS_{ef}) holds. After sliding mode occurs, we have $\hat{z}_2 = \hat{\sigma}_1(\hat{z}_1)$ and $z_2 = \sigma_1(z_1-\varepsilon_1)-\varepsilon_2$, and the motion of the closed-loop system will be governed by

$$\begin{aligned}\dot{z}_1 = & (A_{11}-A_{12}\Sigma_1)z_1+R_1w+\hat{\phi}_{1s}(z,w,\varepsilon) \\ \dot{w} = & Sw+\phi_w(w) \\ \dot{\varepsilon} = & (A'-L'C')\varepsilon+\Phi'(\zeta,\varepsilon) \\ e = & h(z_1+\pi_1(w),\sigma_1(z_1-\varepsilon_1)-\varepsilon_2+\pi_2(w))-q(w)\end{aligned}$$

were $\hat{\phi}_{1s}(z,w,\epsilon)$ vanishes at the origin with it first derivative, and $\hat{\phi}_{1s}(z,w,0) = \phi_{1s}(z,w)$. The matrices $(A_{11}-A_{12}\Sigma_1)$ and $(A'-L'C')$ are Hurwitz by a proper choice of Σ_1 Utkin et al. (1978) and L', respectively, and, if condition (29) holds, then $R_1w+\hat{\phi}_{1s}(z,w,0) = f_1(\pi_1(w),\pi_2(w))+d_1(\pi_1(w),\pi_2(w))w-\frac{\partial\pi_1(w)}{\partial w}s(w) = 0$. Hence, under the property of center manifolds, we have $z_1(t)\to 0 \Rightarrow x_1(t)\to\pi_1(w(t))$, and $z_2(t)\to 0\Rightarrow x_2(t)\to\pi_2(w(t))$ with $t\to\infty$. Thus, the requirement (S_{ef}) is fulfilled. So, by continuity, if condition (30) holds, then the output tracking error (19) converges to zero and condition (R_{ef}) holds too.

3. APPLICATION TO THE PENDUBOT

In this section an sliding mode regulator for the Pendubot is designed. The Pendubot is a planar motion underactuated system schematically shown in Figure 1.

The equation of motion for the Pendubot can be described by the following general equation Spong et al. (1989):

$$D(q)\ddot{q}+C(q,\dot{q})+G(q)+F(\dot{q}) = \tau \tag{31}$$

where $q = [q_1,\ q_2]^T \in \Re^n$ is the vector of joint variables (generalized coordinates), $q_1 \in \Re^m$ represents the actuated joints, and $q_2 \in \Re^{(n-m)}$ represents the unactuated ones. $D(q)$ is the $n\times n$ inertia matrix, $C(q,\dot{q})$ is the vector of Coriolis and centripetal torques, $G(q)$ contains the gravitational terms, $F(\dot{q})$ is the vector of viscous frictional terms,

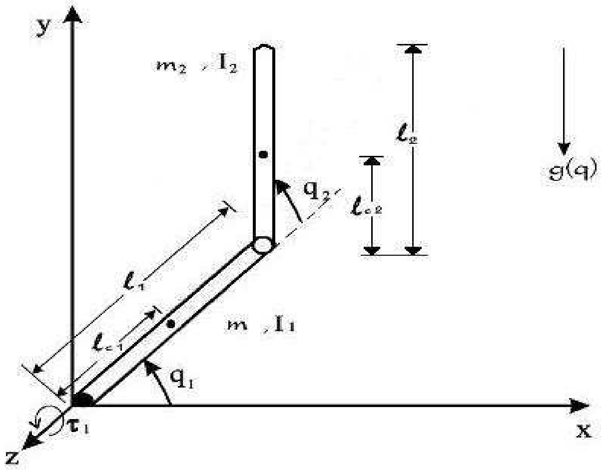

Fig. 1. Schematic diagram of the Pendubot.

and τ is the vector of input torques. For the Pendubot system, the dynamic model (31) is particularized as

$$\begin{bmatrix} D_{11} & D_{12} \\ D_{12} & D_{22} \end{bmatrix} \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} + \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} + \begin{bmatrix} G_1 \\ G_2 \end{bmatrix} + \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = \begin{bmatrix} \tau_1 \\ 0 \end{bmatrix}$$

where $D_{11}(q_2) = m_1 l_{cl}^2 + m_2(l_1^2 + l_{c2}^2 + 2l_1 l_{c2} \cos q_2) + I_1 + I_2$, $D_{12}(q_2) = m_2(l_{c2}^2 + l_1 l_{c2} \cos q_2) + I_2$, $D_{22} = m_2 l_{c2}^2 + I_2$, $C_1(q_2, \dot{q}_1, \dot{q}_2) = -2m_2 l_1 l_{c2} \dot{q}_1 \dot{q}_2 \sin q_2 - m_2 l_1 l_{c2} \dot{q}_2^2 \sin q_2$, $C_2(q_2, \dot{q}_1) = m_2 l_1 l_{c2} \dot{q}_1^2 \sin q_2$, $G_1(q_1, q_2) = m_1 g l_{c1} \cos q_1 + m_2 g l_1 \cos q_1 + m_2 g l_{c2} \cos(q_1 + q_2)$, $G_2(q_1, q_2) = m_2 g l_{c2} \cos (q_1 + q_2)$, $F_1(\dot{q}_1) = \mu_1 \dot{q}_1$, $F_2(\dot{q}_2) = \mu_2 \dot{q}_2$, with m_1 and m_2 as the mass of the first and second link of the Pendubot respectively, l_1 is the length of the first link , l_{c1} and l_{c2} are the distance to the center of mass of link one and two respectively, g is the acceleration of gravity, I_1 and I_2 are the moment of inertia of the first and second link respectively about its centroids, and $\mu_1$and μ_2 are the viscous drag coefficients. The nominal values of the parameters are taken as follows: $m_1 = 0.8293$, $m_2 = 0.3402$, $l_1 = 0.2032$, $l_{c1} = 0.1551$, $l_{c2} = 0.1635125$, $g = 9.81$, $I_1 = 0.00595035$, $I_2 = 0.00043001254$, $\mu_1 = 0.00545$, $\mu_2 = 0.00047$. Choosing $x = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \end{pmatrix}^T = \begin{pmatrix} q_1 & q_2 & \dot{q}_1 & \dot{q}_2 \end{pmatrix}^T$ as the state vector, $u = \tau_1$ as the input, and x_2 as the output, the description of the system can be given in state space form as:

$$\dot{x}(t) = f(x) + g(x)u(t) \tag{32}$$

$$e(x, w) = x_2 - w_2$$

$$\dot{w} = s(w) \tag{33}$$

where $e(x, w)$ is output tracking error, $w = (w_1, w_2)^T$, and w_2 as the reference signal generated by the known exosystem (33),

$$f(x) = \begin{pmatrix} f_1(x_3) \\ f_2(x_4) \\ f_3(x) \\ f_4(x_1, x_2, x_3) \end{pmatrix} = \begin{pmatrix} x_3 \\ x_4 \\ b_3(x_2)p_1(x) \\ b_4(x_2)p_2(x) \end{pmatrix},$$

$$g(x) = \begin{pmatrix} b_1 \\ b_2 \\ b_3(x_2) \\ b_4(x_2) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \dfrac{D_{22}}{D_{11}(x_2)D_{22} - D_{12}^2(x_2)} \\ \dfrac{-D_{12}(x_2)}{D_{11}(x_2)D_{22} - D_{12}^2(x_2)} \end{pmatrix},$$

$$s(w) = \begin{pmatrix} \alpha w_2 \\ -\alpha w_1 \end{pmatrix}$$

$$p_1(x) = \frac{D_{12}(x_2)}{D_{22}} \left(C_2(x_2, x_3) + G_2(x_1, x_2) + F_2(x_4)\right) - C_1(x_2, x_3, x_4) - G_1(x_1, x_2) - F_1(x_3),$$

$$p_2(x) = \frac{D_{11}(x_2)}{D_{12}} \left(C_2(x_2, x_3) + G_2(x_1, x_2) + F_2(x_4)\right) - C_1(x_2, x_3, x_4) - G_1(x_1, x_2) - F_1(x_3).$$

Now, the model of the Pendubot (32) will be transformed to the regular form by means of a nonlinear transformation $x' = \begin{pmatrix} x_1' & x_2' & x_3' & x_4' \end{pmatrix}^T = \varphi(x)$. For, such transformation is proposed as follows:

$$x^1 = \begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} x_2 \\ x_1 \\ x_3 - b_3(x_2)b_4^{-1}(x_2)x_4 \end{pmatrix} \tag{34}$$

$$x^2 = x_4' = x_4$$

with

$$b_3(x_2)b_4^{-1}(x_2) = -D_{22}D_{12}^{-1}(x_2).$$

The Pendubot in the nonlinear regular form results as follows:

$$\begin{pmatrix} \dot{x}^1 \\ \dot{x}^2 \end{pmatrix} = \begin{pmatrix} f^1(x') \\ f^2(x') \end{pmatrix} + \begin{pmatrix} 0 \\ b_4(x_1') \end{pmatrix} u \tag{35}$$

$$e'(x', w) = x_1' - w_2$$

with

$$f^1(x') = \begin{pmatrix} x_4' \\ x_3' - D_{22}D_{12}^{-1}(x_1')x_4' \\ -D_{12}^{-1}(x_1') \begin{pmatrix} C_2(x') + G_2(x_1', x_2') \\ +F_2(x_4') \end{pmatrix} \\ +D_{22}D_{12}^{-2}(x_1')\dot{D}_{12}(x_1')x_4' \end{pmatrix},$$

$$f^2(x') = b_4(x_1')p_2(x').$$

Now, the steady-state zero output manifold $\pi'(w) = (\pi_1'(w), \pi_2'(w), \pi_3'(w), \pi_4'(w))^T$ is introduced. This manifold will be first calculated in the original coordinates with respect to system (32), making use of its respective regulator equations:

$$\frac{\partial \pi_1(w)}{\partial w} s(w) = \pi_3(w) \tag{36}$$

$$\frac{\partial \pi_2(w)}{\partial w} s(w) = \pi_4(w) \tag{37}$$

$$\frac{\partial \pi_3(w)}{\partial w} s(w) = b_3(\pi_2(w))p_1(\pi(w)) + b_3(\pi_2(w))c(w) \tag{38}$$

$$\frac{\partial \pi_4(w)}{\partial w} s(w) = b_4(\pi_2(w))p_2(\pi(w)) + b_4(\pi_2(w))c(w) \tag{39}$$

$$0 = \pi_2(w) - w_2 \tag{40}$$

$$\pi/2 = \pi_1(w) + \pi_2(w) \tag{41}$$

with $c(w)$ as the steady-state value for $u(t)$ that will be defined in the following lines, and

$$\pi(w) = (\pi_1(w), \pi_2(w), \pi_3(w), \pi_4(w))^T$$

as the steady-state zero output manifold. From equation (40) one directly obtains $\pi_2(w) = w_2$, then, replacing $\pi_2(w)$ in equation (37) yileds to $\pi_4(w) = -\alpha w_1$. For calculating $\pi_1(w)$ and $\pi_3(w)$, the solution of equations (36) and (38) are needed, but in general this is a difficult task, that it is commonly solved proposing an approximated solution as in Ramos et al. (1997) and Samposh et al. (2002). Thus, one proposes the following approximated solution for $\pi_1(w)$

$$\begin{aligned}\pi_1(w) = {}& a_0 + a_1w_1 + a_2w_2 + a_3w_1^2 + a_4w_1w_2 + a_5w_2^2 + a_6w_1^3 \\ & + a_7w_1^2w_2 + a_8w_1w_2^2 + a_9w_2^3 + O^4(\|w\|_1) \end{aligned} \tag{42}$$

replacing (42) in (36) and chosing $\alpha = 0.3$ yields the approximated solution for $\pi_3(w)$

$$\begin{aligned}\pi_3(w) = {}& 0.3a_1w_2 - 0.3a_2w_1 + 0.6a_3w_1w_2 + 0.3a_4w_2^2 \\ & -0.3a_4w_1^2 - 0.6a_5w_2w_1 + 0.9a_6w_1^2w_2 + 0.6a_7w_1w_2^2 \\ & -0.3a_7w_1^3 + 0.3a_8w_2^3 - 0.6a_8w_1^2w_2 \\ & -0.9a_9w_2^2w1 + O^4(\|w\|_1). \end{aligned} \tag{43}$$

Calculating from (39) $c(w) = -p_2(\pi(w)) - \alpha^2 w_2/b_4(\pi_2(w))$, and using it along with (43) in equation (38) and performing a series Taylor expansion of the right hand side of this equation around the equilibrium point $(\pi/2, 0, 0, 0)^T$, then, one can find the values a_i $(i = 0, \ldots, 9)$ if the coefficients of the same monomials appearing in both side of such equation are equalized. In this case, the coefficients results as follows: $a_0 = 1.570757$, $a_1 = -0.00025944$, $a_2 = -1.001871$, $a_3 = 0.0$, $a_4 = 0.0$, $a_5 = 0.0$, $a_6 = 0.0$, $a_7 = 0.001926$, $a_8 = 0.0$, $a_9 = -0.00001588$. It is worth to mention that there is a natural steady-state constraint (41) for the Pendubot (see Figure 1), i.e., the sum of the two angles, q_1 and q_2 equals $\pi/2$. Using such constraint one can easily calculate $\pi_{1a}(w) = \pi/2 - \pi_2(w)$, and replacing $\pi_{1a}(w)$ in equation (36) yields to $\pi_{3a}(w) = \alpha w_1$, where the sub-index a refers to an alternative manifold. This result was simulated yielding to the same results when using the approximated manifold, which is to be expected if the motion of the pendubot is forced only along the geometric constraints.

Tansforming now $\pi(w)$ to regular form variables through the diffeomorphism (34)

$$\begin{aligned}\pi_1'(w) &= \pi_2(w) = w_2 \\ \pi_2'(w) &= \pi_1(w) \\ \pi_3'(w) &= \pi_3(w) + D_{22}D_{12}^{-1}(w_2)\pi_4(w) \\ \pi_4'(w) &= \pi_4(w) = -\alpha w_1.\end{aligned}$$

Now, the variable $z = x' - \pi'(w) = \left(\, z^1,\ z^2 \,\right)^T$ is introduced, where

$$\begin{aligned} z^1 &= (\, z_1,\ z_2,\ z_3 \,)^T = (\, x_1' - \pi_1',\ x_2' - \pi_2',\ x_3' - \pi_3' \,)^T \\ z^2 &= z_4 = x_4' - \pi_4'. \end{aligned} \tag{44}$$

Then, system (35) is represented in the new variables (44) as

$$\begin{aligned}\dot{z}_1 &= z_4 + \pi_4' - \frac{\partial \pi_1'}{\partial w}s(w) \\ \dot{z}_2 &= z_3 + \pi_3'(w) - D_{22}D_{12}^{-1}(z_1 + \pi_1')z_4 \\ & \quad -D_{22}D_{12}^{-1}(z_1 + \pi_1')\pi_4' - \frac{\partial \pi_2'}{\partial w}s(w) \\ \dot{z}_3 &= -D_{12}^{-1}(z_1 + \pi_1') \\ & \quad \begin{pmatrix} C_2(z + \pi') \\ +G_2(z_1 + \pi_1', z_2 + \pi_2') \\ +F_2(z_4 + \pi_4') \end{pmatrix} \\ & \quad +D_{22}D_{12}^{-2}(z_1 + \pi_1')\dot{D}_{12}(z_1 + \pi_1')(z_4 + \pi_4') \\ & \quad -\frac{\partial \pi_3'}{\partial w}s(w) \\ \dot{z}_4 &= b_4(z_1 + \pi_1')p_2(z + \pi') \\ & \quad +b_4(z_1 + \pi_1')c(w) - \frac{\partial \pi_4'}{\partial w}s(w) \\ e'(z, w) &= z_1 + \pi_1' - w_2 \\ \dot{w} &= s(w). \end{aligned} \tag{45}$$

In order to estimate the states of system (45) and introduce robustness we propose a reduced order nonlinear sliding mode observer Utkin (1992):

$$\dot{\xi} = \begin{pmatrix} \hat{z}_4 + \pi_4' - \frac{\partial \pi_1'}{\partial \hat{w}}s(w) + v_1 \\ \hat{z}_3 + \pi_3' - D_{22}D_{12}^{-1}(\hat{z}_1 + \pi_1')\hat{z}_4 \\ -D_{22}D_{12}^{-1}(\hat{z}_1 + \pi_1')\pi_4' \\ -\frac{\partial \pi_2'}{\partial \hat{w}}s(w) + v_2 \\ -D_{12}^{-1}(\hat{z}_1 + \pi_1') \\ \begin{pmatrix} C_2(\hat{z} + \pi') \\ +G_2(\hat{z}_1 + \pi_1', \hat{z}_2 + \pi_2') \\ +F_2(\hat{z}_4 + \pi_4') \end{pmatrix} \\ +D_{22}D_{12}^{-2}(\hat{z}_1 + \pi_1')\dot{D}_{12}(\hat{z}_1 + \pi_1')(\hat{z}_4 + \pi_4') \\ -\frac{\partial \pi_3'}{\partial \hat{w}}s(w) + v_3 \\ b_4(\hat{z}_1 + \pi_1')p_2(\hat{z} + \pi') + b_4(\hat{z}_1 + \pi_1')c(w) \\ -\frac{\partial \pi_4'}{\partial \hat{w}}s(w) + v_4 \end{pmatrix}$$

$$\hat{e}'(\hat{z}, w) = \hat{z}_1 + \pi_1 - w_2$$

where

$$\begin{aligned} v_1 &= l_1 sign(e'(z, w) - \widehat{e}'(\hat{z}, w)) = l_1 sign(e_1) \\ v_2 &= l_2 v_1, \quad v_3 = l_3 v_1, \quad v_4 = l_4 v_1, \end{aligned}$$

with $\xi = (\hat{z}_1, \hat{z}_2, \hat{z}_3, \hat{z}_4)^T$ the estimate of $\zeta = (z_1, z_2, z_3, z_4)^T$, $\hat{e}'$ as the estimate of e' and (l_1, l_2, l_3, l_4) are the observer gain matrix. The estimation error is defined as $e = (e_1, e_2, e_3, e_4)^T = \xi - \zeta$, and the linearized estimation error dynamics around the equilibrium point $\zeta_{ep} = (0, \pi/2, 0, 0)^T$ is:

$$\begin{pmatrix} \dot{e}_1 \\ \dot{e}_2 \\ \dot{e}_3 \\ \dot{e}_4 \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{pmatrix} - \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix}$$

with

$$A_{11} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 22.093 & 22.093 & 0 \end{pmatrix},\ A_{12} = \begin{pmatrix} 1 \\ -.542 \\ -.019 \end{pmatrix}$$
$$A_{21} = (\,68.655\ \ -48.969\ \ .330\,)\,,\ A_{22} = (-.266)$$

where the particular value of $\alpha = 0.3$ is chosen. If $l_1 > |e_4|$, then, sliding motion occurs along the surface $e_1 = 0$, and the resulting sliding mode dynamic is as follows:

$$\begin{pmatrix} \dot{e}_2 \\ \dot{e}_3 \\ \dot{e}_4 \end{pmatrix} = \begin{pmatrix} e_3 - .542e_4 - l_2 e_4 \\ 22.093e_2 - .019e_4 - l_3 e_4 \\ -48.969e_2 + .330e_3 - .266e_4 - l_4 e_4 \end{pmatrix}$$

This dynamic was obtained by replacing the equivalent value of v_1 as $v_{1eq} = e_4$ in the expressions for v_2, v_3, v_4. The observer gains are chosen as $l_1 = 10$, $l_2 = -5.79714$, $l_3 = -24.93026$ and $l_4 = 26.73314$, in order to place the sliding mode dynamic poles at $(-5, -9-, 13)$.

We now define the sliding manifold and control:

$$\sigma = \hat{z}_4 + \Sigma_1(\hat{z}_1 + \hat{z}_2 + \hat{z}_3)^T,\ \Sigma_1 = (\,k_1,\ k_2,\ k_3\,)$$
$$u = -kB_2^{-1} sign(\hat{\sigma});\ k > \|b_4(\hat{z}_1 + \pi_1(w))u_{eq}\|$$

where u_{eq} is a solution of $\dot{\sigma} = 0$, $k = 100$, $B_2^{-1} = -60.633$, and $\Sigma_1 = (\,39.88459,\ 45.54227,\ 9.68844\,)$ in order to have matrix $(A_{11} - A_{12}\Sigma_1)$ Hurwitz with pole locations at $(-5, -5, -5)$.

4. SIMULATIONS

In order to show the performance of the sliding mode regulator, simulations are carried out. The initial condition for the Pendubot is chosen near the equilibrium point as follows: $x_1(0) = 1.5$, $x_2(0) = 0.09$, and the initial conditions for the observer are set at the origin. For comparison purposes, a classical regulator as the one presented in Ramos et al. (1997) is simulated. Moreover, plant parameter variations are considered from time $t = 0$, due to possible measurement errors, therefore, the mass of the second link is considered as $m_2 = 0.5$, the moment of inertias of the first and second link are assumed to be $I_1 = 0.007$ and $I_2 = 0.0006$ respectively and the frictions of the first and second link are $\mu_1 = 0.01$ and $\mu_2 = 0.001$ respectively The results are given in Figure 2, where the robust performance of the sliding mode regulator versus the classical one is put in evidence.

5. CONCLUSIONS

The Error Feedback Sliding Mode Output Regulation Problem has been revisited. The sliding mode control technique allows straightforward solutions to be obtained, i.e., the steady state control needs not to be calculated, simplifying the control design, specially when compared to the classical solutions of the state or error feedback regulator problems. Additionally, the sliding mode based controller achieves robustness with respect to allowed uncertainties. The Pendubot system was presented in the regular form, where simulation results illustrates the robust performance of the sliding mode regulator when compared to the classical one.

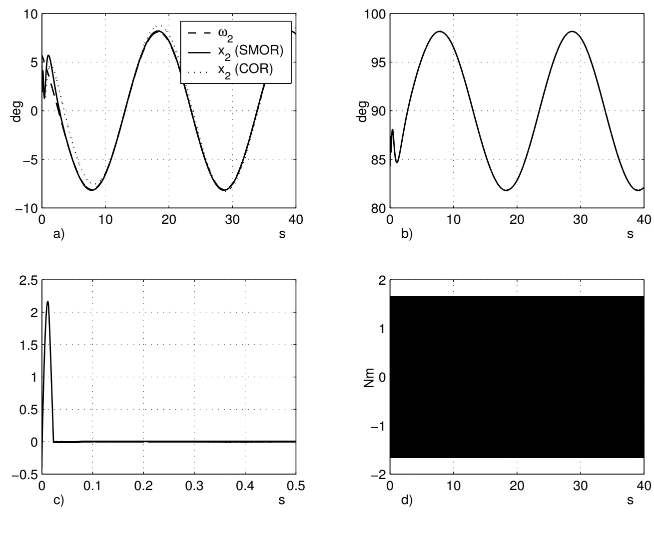

Fig. 2. a) Comparison of the output reference signal versus the output of the Pendubot controlled by the Sliding Mode Regulator and the Classical Output Regulator. b) The estimated angle of the first link. c) The estimated sliding surface. d) The sliding mode control signal.

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