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Almost Self-Bounded Controlled-Invariant Subspaces and Almost Disturbance Decoupling

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Abstract: The objective of this contribution is to characterize the so-called finite fixed poles of the Almost Disturbance Decoupling Problem by state feedback $(\mathbf{ADDP})'$. The most important step towards this result relies on the extension to almost invariant subspaces of the key notion of self-boundedness, as initially introduced by Basile and Marro for perfect controlled-invariants, namely, we introduce the Almost Self-Bounded Controlled-Invariant subspaces. We recall the pole placement flexibilities and constraints that both exist when using a particular almost invariant subspace as a support for the construction of specific (including high gain) feedbacks, and we show, when $(\mathbf{ADDP})'$ is solvable, what is the "best" almost invariant subspace to choose, in order to achieve $(\mathbf{ADDP})'$ and simultaneously place the "largest possible" set of finite poles for the closed loop solution. We finally characterize the set of fixed finite poles for $(\mathbf{ADDP})'$, in terms of some finite zero structures.

Keywords: Linear control systems; geometric approach; system structure; almost disturbance decoupling; pole assignment

1. INTRODUCTION

The exact version by state feedback of the Disturbance Decoupling Problem (**DDP**) has been widely considered for decades. For a recent survey about the structural aspects that are around, the reader can refer to M.Malabre (2006). A particular subspace plays a central role in the solution of (**DDP**) when, say, maximal pole placement is looked for. It is the supremal $(A, Im[B|D])$ controllability subspace included in the kernel of the output map, $\mathscr{R}_c^*$, where B and D are, respectively, the control input and the disturbance input matrix. When (**DDP**) is solvable, $\mathscr{R}_c^*$ is $(A, \mathscr{B})$ controlled-invariant and using feedback matrices which are so-called friends of $\mathscr{R}_c^*$ is indeed the way to place the maximal set of dynamics for the closed loop system, while rejecting the disturbance (see for instance M.Malabre et al. (1997)). This is a rather direct consequence of the fact that $\mathscr{R}_c^*$ is the smallest self-bounded $(A, \mathscr{B})$ controlled-invariant containing ImD (possibly modulo ImB), see G.Basile and G.Marro (1992).

We shall here prove that in the solution of the Almost Disturbance Decoupling Problem $(\mathbf{ADDP})'$, we have a similar conclusion, that is $\mathscr{R}_{ca}^*$, the supremal almost $(A, Im[B|D])$ controllability subspace included in the kernel of the output map, plays a central role, because it is an almost self-bounded $(A, \mathscr{B})$ controlled-invariant subspace that contains ImD.

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2. NOTATION AND BACKGROUND

2.1 Notation

We shall consider linear time-invariant strictly proper systems described by:

$$\Sigma : \begin{cases} \dot{x}(t) = Ax(t) + Bu(t) + Dq(t) \\ z(t) = Ex(t) \end{cases}$$

where x, u, q, and z are respectively the state, control input, disturbance input, and output to be controlled. These signals belong to the vector spaces $\mathscr{X}$, $\mathscr{U}$, $\mathscr{Q}$, and $\mathscr{Z}$, respectively.

In this paper, vectors will be denoted by lower case letters, matrices/maps by capitals and subspaces by script capitals. If A is a square matrix, then $\sigma(A)$ will denote its spectrum. If $A : \mathscr{X} \longmapsto \mathscr{Y}$ and $\mathscr{V} \subseteq \mathscr{X}$, the restriction of the map A to $\mathscr{V}$ is denoted by $A|\mathscr{V}$. If $\mathscr{V}_1$ and $\mathscr{V}_2$ are A-invariant subspaces and $\mathscr{V}_2 \subseteq \mathscr{V}_1$, the map induced by A in the quotient space $\mathscr{V}_1/\mathscr{V}_2$ is denoted by $A|\mathscr{V}_1/\mathscr{V}_2$. To simplify, we sometimes use $\mathscr{B}$ in place of ImB, the image of B; $\mathscr{K}$ in place of $KerE$, the kernel of E.

Let us denote $\Sigma_{(A,B)x} := \{x(t) : [0, \infty) \rightarrow \mathscr{X};\ x(t)$ is **a**.**c**.(absolutely continuous), and $\dot{x}(t) - Ax(t) \in ImB$ **a**.**e**.(almost everywhere)$\}$, and $\Sigma_{(A,[B|D])x} := \{x(t) : [0, \infty) \rightarrow \mathscr{X};\ x(t)$ is **a**.**c**., and $\dot{x}(t) - Ax(t) \in ImB + ImD$ **a**.**e**.$\}$.

If $\mathscr{X}$ is a normed vector space, with norm $\|.\|$, and $\mathscr{L}$ a subspace of $\mathscr{X}$, then for any $x \in \mathscr{X}$, its distance to $\mathscr{L}$ is denoted as: $d(x, \mathscr{L}) := \inf_{y \in \mathscr{L}} \|x - y\|$.

For any measurable function, say $W : [0, \infty) \to \mathscr{X}$, we say that $W \in L_p[0, \infty)$ if $\|W\|_{L_p} < +\infty$, where:

$$\|W\|_{L_p} := \begin{cases} \left(\int_0^\infty \|W\|^p \, dt \right)^{1/p} & \text{for } 1 \le p < \infty \\ \sup_{t \ge 0} \|W\| & \text{for } p = \infty \end{cases}$$

The reachable space of Σ (by the control u) will be denoted by $\langle A|\mathscr{B}\rangle := \mathscr{B} + A\mathscr{B} + A^2\mathscr{B} + \cdots + A^{n-1}\mathscr{B}$, where n is the dimension of $\mathscr{X}$.

A subspace $\mathscr{V} \subset \mathscr{X}$ is called $(A, \mathscr{B})$(or controlled)-invariant if there exists $F : \mathscr{X} \to \mathscr{U}$ such that $(A + BF)\mathscr{V} \subset \mathscr{V}$. F is called a friend of $\mathscr{V}$ and we denote $\mathscr{F}(\mathscr{V})$ the set of all such F.

A subspace $\mathscr{R} \subset \mathscr{X}$ is called an $(A, \mathscr{B})$ controllability subspace if there exist $F : \mathscr{X} \to \mathscr{U}$, and $G : \mathscr{Y} \to \mathscr{U}$, with $\mathscr{Y} \subset \mathscr{U}$, such that: $\mathscr{R} := \langle A + BF | Im(BG)\rangle$.

A subspace $\mathscr{V}_a \subset \mathscr{X}$ is called an almost $(A, \mathscr{B})$(or controlled)-invariant subspace if for any $x_0 \in \mathscr{V}_a$ and for any $\epsilon > 0$ there exists a state trajectory $x_\epsilon \in \Sigma_{(A,B)x}$ with the properties that $x_\epsilon(0) = x_0$ and $d(x_\epsilon(t), \mathscr{V}_a) \le \epsilon$, for any $t \ge 0$.

A subspace $\mathscr{R}_a \subset \mathscr{X}$ is called an almost $(A, \mathscr{B})$ controllability subspace if for any $x_0 \in \mathscr{R}_a$, and any $x_1 \in \mathscr{R}_a$ there exists $T > 0$ such that, for any $\epsilon > 0$ there exists a state trajectory of $x_\epsilon \in \Sigma_{(A,B)x}$ with the properties that $x_\epsilon(0) = x_0, x_\epsilon(T) = x_1$ and $d(x_\epsilon(t), \mathscr{R}_a) \le \epsilon, \forall t \ge 0$.

The supremal $(A, \mathscr{B})$(or controlled)-invariant subspace contained in $KerE$ is denoted by $\mathscr{V}^*$. It is the limit of the following non increasing algorithm, see G.Basile and G.Marro (1992) and W.M.Wonham (1985):

$$\begin{cases} \mathscr{V}^0 = \mathscr{X} \\ \mathscr{V}^{i+1} = KerE \cap A^{-1}\left(ImB + \mathscr{V}^i\right) \end{cases}$$

Similarly, $\mathscr{R}^*$, the supremal $(A, \mathscr{B})$ controllability subspace contained in $KerE$, is the limit of the following non decreasing algorithm, see W.M.Wonham (1985):

$$\begin{cases} \mathscr{R}^0 = 0 \\ \mathscr{R}^{i+1} = \mathscr{V}^* \cap \left(A\mathscr{R}^i + ImB\right) \end{cases}$$

$\mathscr{R}_a^*$, the supremal almost $(A, \mathscr{B})$ controllability subspace contained in $KerE$, is the limit of the following non decreasing algorithm, see J.C.Willems (1981):

$$\begin{cases} \mathscr{R}_a^0 = 0 \\ \mathscr{R}_a^{i+1} = kerE \cap \left(A\mathscr{R}_a^i + ImB\right) \end{cases}$$

Let us denote by $\mathscr{S}^*$ the limit of the following algorithm:

$$\begin{cases} \mathscr{S}^0 = 0 \\ \mathscr{S}^{i+1} = ImB + A\left(KerE \cap \mathscr{S}^i\right) \end{cases}$$

$\mathscr{S}^*$ is usually introduced in the context of $(\mathscr{K}, A)$ invariance (dual to $(A, \mathscr{B})$ invariance). In our present context, we prefer to handle it through its almost controllability properties, as established by J.C.Willems (1981), and namely: $\mathscr{S}^* = A\mathscr{R}_a^* + ImB$.

Note that all these notions of exact/almost controlled invariance or controllability properties, can easily be defined, similarly, for the "composite" system, say $\Sigma_c := (A, [B|D], E)$, i.e. with $\mathscr{U} \oplus \mathscr{Q}$ in place of $\mathscr{U}$. They will be noted, respectively, $\mathscr{V}_c^*, \mathscr{R}_c^*, \mathscr{R}_{ca}^*, \mathscr{S}_c^*$. They are, respectively, the limits of the following algorithms:

$$\mathscr{V}_c^* : \begin{cases} \mathscr{V}_c^0 = \mathscr{X} \\ \mathscr{V}_c^{i+1} = KerE \cap A^{-1}\left(Im[B|D] + \mathscr{V}_c^i\right) \end{cases}$$

$$\mathscr{R}_c^* : \begin{cases} \mathscr{R}_c^0 = 0 \\ \mathscr{R}_c^{i+1} = \mathscr{V}_c^* \cap \left(A\mathscr{R}_c^i + Im[B|D]\right) \end{cases}$$

$$\mathscr{R}_{ca}^* : \begin{cases} \mathscr{R}_{ca}^0 = 0 \\ \mathscr{R}_{ca}^{i+1} = KerE \cap \left(A\mathscr{R}_{ca}^i + Im[B|D]\right) \end{cases}$$

$$\mathscr{S}_c^* : \begin{cases} \mathscr{S}_c^0 = 0 \\ \mathscr{S}_c^{i+1} = Im[B|D] + A\left(KerE \cap \mathscr{S}_c^i\right) \end{cases}$$

Some particular structures of Σ play a key role in the solution of control problems. These are mainly the invariant zeros. The finite invariant zeros of (A, B, E), i.e. from u to z, are equal to the dynamics [1] of the system in the part of $\mathscr{V}^*$ which is "outside" $\mathscr{R}^*$ (more rigorously in the quotient space $\mathscr{V}^*/\mathscr{R}^*$):

$$Zeros(A, B, E) := \sigma(A + BF|(\mathscr{V}^*/\mathscr{R}^*)),$$

for any $F \in \mathscr{F}(\mathscr{V}^*)$.

For more information about those structures, including also zeros at infinity, the reader can easily refer to M.Malabre (2006) and the main references that are cited therein (e.g. F.R.Gantmacher (1959), H.H.Rosenbrock (1970), T.Kailath (1980),H.Aling and J.M.Schumacher (1984), C.Commault and J.M.Dion (1982) and J.F.Lafay, C.Commault and M.Malabre (1991)).

We just recall here an important result concerning pole placement when using feedbacks of a controlled-invariant subspace $\mathscr{V}$. This result comes from J.M.Schumacher (1980), which itself is a particular case of a more general result given in A.S.Morse (1973).

Proposition 1. Let $\mathscr{V}$ be a controlled-invariant subspace, and let $\mathscr{R}^*(\mathscr{V})$ be the supremal controllability subspace included in $\mathscr{V}$. For any given spectra of ad-hoc lengths, say Λ_1 and Λ_2, there always exist in $\mathscr{F}(\mathscr{V})$ a feedback, say F_0, such that:
- the spectrum of $(A + BF_0)$ in $\mathscr{R}^*(\mathscr{V})$ equals Λ_1(free)
- the spectrum of $(A + BF_0)$ in $\mathscr{V} + \langle A|\mathscr{B}\rangle/\mathscr{V}$ equals Λ_2 (free)
- the spectra of $(A + BF_0)$ in $\mathscr{V}/\mathscr{R}^*(\mathscr{V})$ and in $\mathscr{X}/(\mathscr{V} + \langle A|\mathscr{B}\rangle)$ are fixed (same values for any $F_0 \in \mathscr{F}(\mathscr{V})$).

2.2 Self-Bounded Controlled-Invariant subspaces

In G.Basile and G.Marro (1992), another important refinement has been introduced which is the concept of self-boundedness. The following definition is an obvious variant of that initially proposed by Basile and Marro.

[1] Which indeed are fixed, after having applied any state feedback, say F, i.e. replacing A by $A + BF$. They are also invariant after any change of basis in $\mathscr{X}$, $\mathscr{U}$ and $\mathscr{Z}$

Definition 2. A controlled-invariant subspace included in $KerE$, say $\mathscr{V}$, is called self-bounded if and only if $\mathscr{R}^*$, the supremal $(A, \mathscr{B})$ controllability subspace contained in $KerE$, is included in $\mathscr{V}$. We denote the set of all the self-bounded controlled-invariant subspaces included in $KerE$ by $SBCI(A, B, \mathscr{K})$.

Self-bounded controlled-invariant subspaces in $KerE$ are thus output nulling controlled-invariant subspaces with "maximal" internal (i.e. inside $\mathscr{R}^*(\mathscr{V})$) pole placement abilities.

2.3 Exact and Almost Disturbance Decoupling

Definition 3. The exact version of the Disturbance Decoupling Problem (**DDP**), see W.M.Wonham (1985), is concerned with constructing a state feedback matrix F, such that, in the closed-loop system, the impulse response matrix, say W(t), from the disturbance q to the output z is zero.

Among the first geometric contributions to the solution of this problem let us cite S.Bhattacharyya (1974), which considers state feedback laws of the type $u(t) = Fx(t) + Gd(t)$.

Theorem 4. (**DDP**) is solvable with state feedback and disturbance feed-forward (i.e. $\exists F$ and $\exists G$ such that $E[sI - (A + BF)]^{-1}(BG + D) = 0$) if and only if:

$$ImD \subset \mathscr{V}^* + ImB$$

This condition is obviously equivalent to the existence of at least one $(A, \mathscr{B})$ controlled-invariant subspace $\mathscr{V}$ in $KerE$ such that $ImD \subset \mathscr{V} + ImB$, such a $\mathscr{V}$ is called a solution of (**DDP**).

When (**DDP**) is solvable, it has been shown in G.Basile and G.Marro (1992), that $\mathscr{R}_c^*$ is a self-bounded controlled-invariant subspace solution of (**DDP**), i.e. satisfies $ImD \subset \mathscr{R}_c^* + ImB$ and obviously contains $\mathscr{R}^*$. Moreover, $\mathscr{R}_c^*$ is the smallest self-bounded controlled-invariant subspace solution of (**DDP**). This means that it is the best solution of (**DDP**) in terms of pole placement since its controllable part is as large as possible ($\mathscr{R}^*$), and because it is the smallest of that kind, the poles can be placed maximally outside of it (see Proposition 1).

When (**DDP**) is not solvable, i.e. when no such "classical" solution exists (typically insuring both properness and good pole location), one must look for solutions of a generalised type. One way is to consider the (**ADDP**).

Definition 5. $(\mathbf{ADDP})'$ is solvable if the following holds: $\forall \epsilon > 0$, there exists a sequence of state feedback matrices $\{F_\epsilon; \epsilon > 0\}$, such that, in the closed loop system $\|z(t)\|_{L_q} \leq \epsilon \|q(t)\|_{L_p}$, for all L_p measurable disturbance input $q(t)$ and for all $1 \leq p \leq q \leq \infty$. If we let $p = q$ and $1 \leq p = q < \infty$, this problem is called $(\mathbf{ADDP})_p$.

It is well known J.C.Willems (1981) that $(\mathbf{ADDP})'$ is solvable if and only if:

$$ImD \subset \mathscr{V}_a^* \qquad (1)$$

and $(\mathbf{ADDP})_p$ is solvable if and only if:

$$ImD \subset \mathscr{V}^* + \mathscr{S}^* \qquad (2)$$

Note that (2) is very similar to the condition of Theorem 4, with $\hat{B}$ in place of B, where $\hat{B}$ denotes some extended control input matrix such that $Im\hat{B} := \mathscr{S}^*$.

3. GEOMETRIC SOLUTION AND POLE PLACEMENT OF ALMOST INVARIANT SUBSPACES

It is well known, since the seminal paper from J.C.Willems (1981), that any almost controlled-invariant subspace, say $\mathscr{V}_a$, can be written as the direct sum of a controllability subspace, say $\mathscr{R}$, plus a coasting subspace [2], say $\mathscr{C}$, plus a sliding subspace [3], say $\mathscr{S}$, which can be seen as the limit, say when ϵ tends to zero, of a family of controlled-invariant subspaces $\mathscr{S}_\epsilon$, on which the dynamics are infinitely fast as ϵ tends to zero.

Definition 6. If $\mathscr{V}_a$ is an almost invariant subspace, the class of all these static feedbacks $F_\epsilon : \mathscr{X} \to \mathscr{U}$ such that, for any $x_0 \in \mathscr{V}_a$ and for any $t \geq 0$, $d(e^{(A+BF_\epsilon)t}x_0, \mathscr{V}_a) \leq \epsilon, 1 \leq p \leq \infty$, is denoted by $\mathscr{F}_\epsilon(\mathscr{V}_a)$. We call $F_\epsilon \in \mathscr{F}_\epsilon(\mathscr{V}_a)$ an ϵ-distance friend of the almost invariant subspace $\mathscr{V}_a$.

As an altenative to (1), we can write:

Proposition 7. $(\mathbf{ADDP})'$ is solvable if and only if there exist an almost controlled-invariant subspace $\mathscr{V}_a$ included in $KerE$ such that: $ImD \subset \mathscr{V}_a$. Such a $\mathscr{V}_a$ is called a particular geometric solution for $(\mathbf{ADDP})'$.

In order to consider the spectral assignability properties associated to a given almost controlled invariant subspace $\mathscr{V}_a$, we need the following:

Lemma 8. Suppose $\mathscr{V}_a$ is an almost invariant subspace, $\mathscr{V}$ is an $(A, \mathscr{B})$-invariant subspace and $\mathscr{R}_a$ is an almost controlllability subspace such that $\mathscr{V}_a = \mathscr{V} + \mathscr{R}_a$, then $\langle A|\mathscr{V}_a + \mathscr{B}\rangle = \mathscr{V}_a + \langle A|\mathscr{B}\rangle = \mathscr{V} + \langle A|\mathscr{B}\rangle$.

Proof.

$$\begin{aligned}
\langle A|\mathscr{V}_a + \mathscr{B}\rangle &= \langle A|\mathscr{V} + \mathscr{R}_a + \mathscr{B}\rangle \\
&= \mathscr{V} + \mathscr{R}_a + \mathscr{B} + \\
&\quad A\mathscr{V} + A(\mathscr{R}_a) + A\mathscr{B} + \\
&\quad A^2\mathscr{V} + A^2(\mathscr{R}_a) + A^2\mathscr{B} + \\
&\quad \cdots \\
&= \mathscr{V} + \langle A|\mathscr{B}\rangle + \mathscr{R}_a + A(\mathscr{R}_a) + \\
&\quad A^2(\mathscr{R}_a) + \cdots \\
&= \mathscr{V}_a + \langle A|\mathscr{B}\rangle \\
&= \mathscr{V} + \langle A|\mathscr{B}\rangle
\end{aligned}$$

Here we have used the fact that any almost controllability subspace is contained in $\langle A|\mathscr{B}\rangle$, see J.C.Willems (1981).■

Now, we can see that $\mathscr{V}_a + \langle A|\mathscr{B}\rangle$ is A-invariant, and thus, of course it is also an $(A, \mathscr{B})$-invariant subspace, and is even an $(A + BF)$-invariant subspace for any F.

Lemma 9. Let $\mathscr{R}_a$ be an almost controllability subspace, and suppose Λ is a symmetric set of $dim(\langle A|\mathscr{B}\rangle) - dim(\mathscr{R}_a)$

[2] a controlled-invariant subspace $\mathscr{C}$ will be called a coasting subspace if and only if $\mathscr{R}^*(\mathscr{C}) = \{0\}$, i.e. the supremal controllability subspace in $\mathscr{C}$ is 0

[3] an almost controlled-invariant subspace $\mathscr{S}$ will be called a sliding subspace if and only if $\mathscr{V}^*(\mathscr{S}) = \{0\}$, i.e. the supremal controlled-invariant subspace in $\mathscr{S}$ is 0

complex numbers. There then exists an $(A, \mathscr{B})$-invariant subspace $\mathscr{V}$ and a map $F \in \mathscr{F}(\mathscr{V})$ such that

$$\mathscr{V} \oplus \mathscr{R}_a = \langle A|\mathscr{B}\rangle$$

and

$$\sigma(A + BF|\mathscr{V}) = \Lambda$$

Proof. The proof of this lemma can be found in Theorem 2.39 of H.L.Trentelman (1985). ■

Theorem 10. Let $\mathscr{V}_a$ be an almost invariant subspace, $\mathscr{V}$ is the maximal $(A, \mathscr{B})$-invariant subspace contained in $\mathscr{V}_a$, and suppose Λ is a symmetric set of $dim(\langle A|\mathscr{V}_a + \mathscr{B}\rangle) - dim(\mathscr{V}_a)$ complex numbers. There then exist a subspace $\mathscr{W}$ and, for each map $F_0 \in \mathscr{F}(\mathscr{V})$, a map $F_1 : \mathscr{X} \to \mathscr{U}$ such that:

$$F_1|\mathscr{V} = F_0|\mathscr{V}$$
$$\mathscr{V}_a \oplus \mathscr{W} = \mathscr{V}_a + \langle A|\mathscr{B}\rangle$$
$$(A + BF_1)(\mathscr{V} \oplus \mathscr{W}) \subset \mathscr{V} \oplus \mathscr{W}$$

and

$$\sigma(A + BF_1|(\mathscr{V} \oplus \mathscr{W})/\mathscr{V}) = \Lambda$$

Proof. From Lemma 8, $\mathscr{V}_a + \langle A|\mathscr{B}\rangle$ is an A-invariant subspace, and here $\mathscr{V}$ is the maximal $(A, \mathscr{B})$-invariant subspace contained in $\mathscr{V}_a$, so we can let $P : (\mathscr{V}_a + \langle A|\mathscr{B}\rangle) \to (\mathscr{V}_a + \langle A|\mathscr{B}\rangle)/\mathscr{V}$ denote the canonical projection and let $F_0 \in \mathscr{F}(\mathscr{V})$. Let $\overline{B} := PB$ and let $\overline{A}$ denote the quotient map induced by $A + BF_0$ in $(\mathscr{V}_a + \langle A|\mathscr{B}\rangle)/\mathscr{V}$. Also from Lemma 8, we have $P(\mathscr{V}_a + \langle A|\mathscr{B}\rangle) = P(\mathscr{V} + \langle A|\mathscr{B}\rangle) = P\langle A|\mathscr{B}\rangle = P\langle A + BF_0|\mathscr{B}\rangle = \langle\overline{A}|\overline{\mathscr{B}}\rangle$.

Let $\mathscr{R}_a$ be an almost controllability subspace such that $\mathscr{V} \oplus \mathscr{R}_a = \mathscr{V}_a$, then $P\mathscr{V}_a = P\mathscr{R}_a$. It is well known that, see J.C.Willems (1981), because of the independence between $\mathscr{V}$ and $\mathscr{R}_a$, we can always let $\mathscr{R}_a = \mathscr{B}_1 + (A + BF_0)\mathscr{B}_2 + (A + BF_0)^2\mathscr{B}_3 + \cdots$, where F_0 is defined as above, and with $\{\mathscr{B}_i\}$ a chain in $\mathscr{B}$, i.e. $\mathscr{B} \supset \mathscr{B}_1 \supset \mathscr{B}_2 \cdots \supset \mathscr{B}_i \supset \mathscr{B}_{i+1} \cdots$. Now, if we define $\overline{\mathscr{B}}_i := P\mathscr{B}_i$, we get immediately that $P\mathscr{R}_a = P\mathscr{B}_1 + P(A + BF_0)\mathscr{B}_2 + P(A + BF_0)^2\mathscr{B}_3 + \cdots = \overline{\mathscr{B}}_1 + \overline{A}\overline{\mathscr{B}}_2 + \overline{A}^2\overline{\mathscr{B}}_3 + \cdots$, with $\{\overline{\mathscr{B}}_i\}$ a chain in $\overline{\mathscr{B}}$, also from J.C.Willems (1981), we can conclude that $P\mathscr{R}_a$ is an almost controllability subspace in the quotient system $(\overline{A}, \overline{B})$.

Let Λ be defined as above, obviously, Λ contains $dim(\langle\overline{A}|\overline{\mathscr{B}}\rangle) - dim(P\mathscr{R}_a)$ complex numbers and thus we can apply Lemma 9 and find an $(\overline{A}, \overline{\mathscr{B}})$-invariant subspace $\overline{\mathscr{W}} \subset (\mathscr{V}_a + \langle A|\mathscr{B}\rangle)/\mathscr{V}$ and a map $\overline{F}$ such that $P\mathscr{R}_a \oplus \overline{\mathscr{W}} = \langle\overline{A}|\overline{\mathscr{B}}\rangle$, $(\overline{A} + \overline{BF})\overline{\mathscr{W}} \subset \overline{\mathscr{W}}$ and $\sigma(\overline{A} + \overline{BF}|\overline{\mathscr{W}}) = \Lambda$. Now, let $\mathscr{W} \subset (\mathscr{V}_a + \langle A|\mathscr{B}\rangle)$ be any subspace such that $P\mathscr{W} = \overline{\mathscr{W}}$ and $\mathscr{W} \cap \mathscr{V} = 0$. Define a map $F_1 : \mathscr{X} \to \mathscr{U}$ such that $F_1|(\mathscr{V}_a + \langle A|\mathscr{B}\rangle) := F_0|(\mathscr{V}_a + \langle A|\mathscr{B}\rangle) + \overline{F}P$, it can easily be verified that we obtain the results. ■

As a summary of this part, we now give our first main result, described as the following Proposition:

Proposition 11. Let $\mathscr{V}_a$ be an almost controlled-invariant subspace, and let $\mathscr{V}^*(\mathscr{V}_a)$ and $\mathscr{R}^*(\mathscr{V}_a)$ denote, respectively, the supremal controlled-invariant (resp. controllability subspace) included in $\mathscr{V}_a$. For any given spectra of ad-hoc lengths, say Λ_1 and Λ_2, for any ϵ there always exist a feedback, say F_ϵ, such that:
- the spectrum of $(A + BF_\epsilon)$ in $\mathscr{R}^*(\mathscr{V}_a)$ equals Λ_1 (free)
- the spectrum of $(A + BF_\epsilon)$ in $\mathscr{V}_a + \langle A|\mathscr{B}\rangle/\mathscr{V}^*(\mathscr{V}_a)$ equals Λ_2 (free)
- the spectra of $(A + BF_\epsilon)$ in $\mathscr{V}^*(\mathscr{V}_a)/\mathscr{R}^*(\mathscr{V}_a)$ and in $\mathscr{X}/(\mathscr{V}_a + \langle A|\mathscr{B}\rangle)$ are fixed and finite (the same for any F_ϵ)
- the spectrum of $(A+BF_\epsilon)$ in $\mathscr{V}_a/\mathscr{V}^*(\mathscr{V}_a)$ is "infinite" but stable, in the sense that $\mathscr{V}_a/\mathscr{V}^*(\mathscr{V}_a)$ can be identified with a sliding subspace for which on any approximation $\mathscr{S}_\epsilon$, all the dynamics tend to "minus infinity" as ϵ tends to zero.

The (three) fixed parts of this spectrum splitting form the so-called "fixed poles of the almost invariant subspace $\mathscr{V}_a$". This is summarized in Figure 1.

$\mathscr{X}$
| fixed finite
$\mathscr{V}_a + \langle A|\mathscr{B}\rangle$
| free finite
$\mathscr{V}_a$
| infinite (but stable)
$\mathscr{V}^*(\mathscr{V}_a)$
| fixed finite
$\mathscr{R}^*(\mathscr{V}_a)$
| free finite
0

Figure 1: Pole placement freedom related to $\mathscr{V}_a$.

4. NEW GEOMETRIC RESULTS ABOUT (**ADDP**)

Let us now generalize the notion of self-boundedness.

Definition 12. An almost controlled-invariant subspace included in $KerE$, say $\mathscr{V}_a$, is called almost self-bounded if and only if $\mathscr{R}_a^*$, the supremal almost $(A, \mathscr{B})$ controllability subspace contained in $KerE$, is included in $\mathscr{V}_a$. We denote the set of all the almost self-bounded controlled-invariant subspaces included in $KerE$ by $ASBCI(A, B, \mathscr{K})$.

It is obvious, that, the set of all the $ASBCI$ subspaces included in $KerE$ is closed with respect to the intersection and to the sum, it is a lattice(non distributive) with respect to $\subseteq, +, \cap$. The supremum of $ASBCI(A, B, \mathscr{K})$ is $\mathscr{V}_a^*$, and the infimum is $\mathscr{R}_a^*$.

Proposition 13. Any subspace $\mathscr{L} \in KerE$ is an $ASBCI$ subspace included in $KerE$ if and only if $\mathscr{L} = \mathscr{V}^*(\mathscr{L}) + \mathscr{R}_a^*$, i.e.

$$\mathscr{L} \in ASBCI(A, B, \mathscr{K}) \Leftrightarrow \begin{cases} \mathscr{L} \subset \mathscr{K} \\ \mathscr{L} = \mathscr{V}^*(\mathscr{L}) + \mathscr{R}_a^* \end{cases}$$

Proof. i. $\Rightarrow$. Let $\mathscr{L} \subset KerE$ be any almost controlled-invariant subspace, it is well known that $\mathscr{L} = \mathscr{R}_a^*(\mathscr{L}) + \mathscr{V}^*(\mathscr{L})$, see J.C.Willems (1981), where $\mathscr{R}_a^*(\mathscr{L})$ is the supremal almost controllability subspace in $\mathscr{L}$ and $\mathscr{V}^*(\mathscr{L})$ is the supremal controlled-invariant subspace in $\mathscr{L}$, now because of self-boundedness, we also have $\mathscr{R}_a^* \subset \mathscr{L}$, and because $\mathscr{R}_a^*$ is the supremal almost controllability subspace in $KerE$, of course $\mathscr{R}_a^*(\mathscr{L}) \subset \mathscr{R}_a^*$, since $\mathscr{L} \subset KerE$, so we get $\mathscr{R}_a^*(\mathscr{L}) = \mathscr{R}_a^*$, i.e.$\mathscr{L} = \mathscr{R}_a^* + \mathscr{V}^*(\mathscr{L})$. ii. $\Leftarrow$ is obvious from the definition of an $ASBCI$ subspace. ■

Proposition 14. When $(\mathbf{ADDP})_p$ is solvable, the set of all the almost self-bounded $(A, \mathscr{B})$ controlled-invariant subspaces included in $KerE$ is exactly identical to the set of all the "exact" self-bounded controlled-invariant subspaces included in $KerE$ with $\hat{B}$ in place of B, where $Im\hat{B} := \mathscr{S}^*$.

Proof. Under the assumption that $(\mathbf{ADDP})_p$ is solvable, we have proved that $\hat{\mathscr{R}}^* = \mathscr{R}_a^* = \mathscr{K} \cap \mathscr{S}^*$, see M.Malabre and R.M.Zou (2007).

i: $\forall \mathscr{L} \in ASBCI(A, B, \mathscr{K}) \Rightarrow \mathscr{L} \in SBCI(A, \hat{B}, \mathscr{K})$
Since $\mathscr{L} \in ASBCI(A, B, \mathscr{K})$, we have $A\mathscr{L} = A(\mathscr{V}^*(\mathscr{L}) + \mathscr{R}_a^*) = A\mathscr{V}^*(\mathscr{L}) + A(\mathscr{K} \cap \mathscr{S}^*) \subset \mathscr{V}^*(\mathscr{L}) + \mathscr{S}^* = \mathscr{V}^*(\mathscr{L}) + \mathscr{K} \cap \mathscr{S}^* + \mathscr{S}^* = \mathscr{L} + \hat{\mathscr{B}}$, so $\mathscr{L}$ is an $(A, \hat{\mathscr{B}})$ controlled-invariant subspace in $KerE$. On the other hand, $(\mathbf{ADDP})_p$ solvable means $\hat{\mathscr{R}}^* = \mathscr{K} \cap \mathscr{S}^*$, and since $\mathscr{K} \cap \mathscr{S}^* \subset \mathscr{L}$, we have $\hat{\mathscr{R}}^* \subset \mathscr{L}$. This means that $\mathscr{L} \in SBCI(A, \hat{B}, \mathscr{K})$.

ii: $\forall \mathscr{L} \in SBCI(A, \hat{B}, \mathscr{K}) \Rightarrow \mathscr{L} \in ASBCI(A, B, \mathscr{K})$
For the proof, we first show by induction that $\hat{\mathscr{V}}^i(\mathscr{L}) = \mathscr{K} \cap \mathscr{S}^* + \mathscr{V}^i(\mathscr{L}), \forall i = 0, 1, \cdots$, with:

$$\begin{cases} \hat{\mathscr{V}}^0(\mathscr{L}) = \mathscr{X} \\ \hat{\mathscr{V}}^{i+1}(\mathscr{L}) = \mathscr{L} \cap A^{-1}\left(Im\hat{B} + \hat{\mathscr{V}}^i(\mathscr{L})\right) \end{cases}$$

$$\begin{cases} \mathscr{V}^0(\mathscr{L}) = \mathscr{X} \\ \mathscr{V}^{i+1}(\mathscr{L}) = \mathscr{L} \cap A^{-1}\left(ImB + \mathscr{V}^i(\mathscr{L})\right) \end{cases}$$

Equality is clearly true for $i = 0$ and $i = 1$, since $\hat{\mathscr{V}}^1(\mathscr{L}) = \mathscr{L} = \mathscr{V}^1(\mathscr{L})$ and $\mathscr{K} \cap \mathscr{S}^* + \mathscr{L} = \mathscr{L}$. Now suppose it is true for $i \geq 0$, then

$$\begin{aligned} \hat{\mathscr{V}}^{i+1}(\mathscr{L}) &= \mathscr{L} \cap A^{-1}\left(Im\hat{B} + \hat{\mathscr{V}}^i(\mathscr{L})\right) \\ &= \mathscr{L} \cap A^{-1}\left(\mathscr{S}^* + \mathscr{K} \cap \mathscr{S}^* + \mathscr{V}^i(\mathscr{L})\right) \\ &= \mathscr{L} \cap A^{-1}\left[(ImB + \mathscr{V}^i(\mathscr{L})) + A(\mathscr{K} \cap \mathscr{S}^*)\right] \\ &= \mathscr{L} \cap \left[A^{-1}(ImB + \mathscr{V}^i(\mathscr{L})) + \mathscr{K} \cap \mathscr{S}^*\right] \\ &= \mathscr{K} \cap \mathscr{S}^* + \mathscr{V}^{i+1}(\mathscr{L}) \end{aligned}$$

Here, we used the fact that $A(\mathscr{K} \cap \mathscr{S}^*) \subset ImA$ and the fact that if $(\mathscr{U} + \mathscr{W}) \cap ImA = \mathscr{U} \cap ImA + \mathscr{W} \cap ImA$ then $A^{-1}(\mathscr{U} + \mathscr{W}) = A^{-1}\mathscr{U} + A^{-1}\mathscr{W}$, and also used the fact that $\mathscr{K} \cap \mathscr{S}^* \subset \mathscr{L}$.
Since $\hat{\mathscr{V}}^i(\mathscr{L}) = \mathscr{K} \cap \mathscr{S}^* + \mathscr{V}^i(\mathscr{L}), \forall i = 0, 1, \cdots$, of course we have $\hat{\mathscr{V}}^*(\mathscr{L}) = \mathscr{K} \cap \mathscr{S}^* + \mathscr{V}^*(\mathscr{L})$, and due to $\mathscr{L} \in SBCI(A, \hat{B}, \mathscr{K})$ we have $\hat{\mathscr{V}}^*(\mathscr{L}) = \mathscr{L}$, so we get $\mathscr{L} = \mathscr{K} \cap \mathscr{S}^* + \mathscr{V}^*(\mathscr{L}) = \mathscr{R}_a^* + \mathscr{V}^*(\mathscr{L})$, from Proposition 13 we get the desired result $\mathscr{L} \in ASBCI(A, B, \mathscr{K})$. ■

Proposition 15. When $(\mathbf{ADDP})'$ is solvable, the following properties always hold:

i. $\mathscr{V}_{ca}^* = \mathscr{V}_a^*$
ii. $\mathscr{R}_{ca}^* = \mathscr{R}_a^* + \mathscr{S}_c^* \cap \mathscr{V}^*$
iii. $\forall \mathscr{L} \in ASBCI(A, Bc, \mathscr{K}), \quad ImD \subset \mathscr{L}$
iv. $ImD \subset \mathscr{R}_{ca}^*$

where, ImD is the image of disturbance input matrix, $Bc := [B|D]$.

Proof. Under the assumption that $(\mathbf{ADDP})_p$ is solvable, we have $\mathscr{V}_c^* + \mathscr{S}_c^* = \mathscr{V}^* + \mathscr{S}^*$, see M.Malabre and R.M.Zou (2007). This obviously holds when $(\mathbf{ADDP})'$ is solvable.
i: From this, we have $\mathscr{K} \cap (\mathscr{V}_c^* + \mathscr{S}_c^*) = \mathscr{K} \cap (\mathscr{V}^* + \mathscr{S}^*)$, and thus $\mathscr{V}_{ca}^* = \mathscr{V}_a^*$.
ii: $\mathscr{R}_{ca}^* = \mathscr{K} \cap \mathscr{S}_c^* = \mathscr{K} \cap \mathscr{S}_c^* \cap (\mathscr{V}_c^* + \mathscr{S}_c^*) = \mathscr{K} \cap \mathscr{S}_c^* \cap (\mathscr{V}^* + \mathscr{S}^*) = \mathscr{K} \cap (\mathscr{S}^* + \mathscr{S}_c^* \cap \mathscr{V}^*) = \mathscr{R}_a^* + \mathscr{S}_c^* \cap \mathscr{V}^*$.
iii: Since $(\mathbf{ADDP})'$ is solvable, we have $ImD \subset \mathscr{V}_a^* \subset \mathscr{V}_{ca}^*$, then $\mathscr{L} \in ASBCI(A, Bc, \mathscr{K}) \Rightarrow \mathscr{L} \supset \mathscr{R}_{ca}^* \supset Im[B|D] \cap \mathscr{R}_{ca}^* = Im[B|D] \cap (\mathscr{S}_c^* \cap \mathscr{V}_{ca}^*) = \mathscr{V}_{ca}^* \cap (\mathscr{B} + ImD) = \mathscr{V}_{ca}^* \cap \mathscr{B} + ImD \supset ImD$.
iv: It follows directly from the fact that $\mathscr{R}_{ca}^*$ is the supremal almost $(A, \mathscr{B}_c)$ controllability subspace contained in $KerE$, it is also the infimum of $ASBCI(A, Bc, \mathscr{K})$ and the result of (*iii*). ■

We can now formulate our second main (geometric) result.

Theorem 16. If $(\mathbf{ADDP})'$ is solvable, then:
i. $\mathscr{R}_{ca}^*$ is a particular geometric solution for $(\mathbf{ADDP})'$, namely: $\mathscr{R}_{ca}^*$ is almost $(A, \mathscr{B})$ controlled-invariant included in $KerE$, and contains ImD .
ii. $\mathscr{R}_{ca}^*$ is an almost self-bounded $(A, \mathscr{B})$ controlled-invariant subspace solution of $(\mathbf{ADDP})'$
iii. $\mathscr{R}_{ca}^*$ is the smallest almost self-bounded $(A, \mathscr{B})$ controlled-invariant subspace solution of $(\mathbf{ADDP})'$

Proof. (sketch) i) and ii) follow directly from the obvious fact that $(\mathbf{ADDP})'$ solvable leads to $(\mathbf{ADDP})_p$ solvable, and our results in M.Malabre and R.M.Zou (2007) that $\mathscr{S}_c^* \cap \mathscr{V}^*$ is a controlled-invariant subspaces included in $KerE$, and Definition 12, Proposition 13, 15 and 7.

iii: To show this, le us consider the fictitious system $\hat{\Sigma}(A, \hat{B}_c, E)$, where $\hat{B}_c := [\hat{B}|D]$ and $Im\hat{B} := \mathscr{S}^*$. We will first show that $\hat{\mathscr{S}}_c^* = \mathscr{S}_c^*$ and $\hat{\mathscr{R}}_c^* = \mathscr{R}_{ca}^*$, then use these two equations to prove $ImD \subset \mathscr{R}_{ca}^*$. Here, $\hat{\mathscr{R}}_c^*$ stands for the supremal controllability subspace of the pair $(A, Im\hat{B}_c)$ included in $\mathscr{K}$, $\hat{\mathscr{S}}_c^*$ is the infimal $(\mathscr{E}, A)$-invariant subspace containning $Im\hat{B}_c$, and $\hat{\mathscr{V}}_c^*$ the supremal $(A, Im[\hat{B}|D])$(or controlled)-invariant subspace contained in $KerE$.

$\hat{\mathscr{S}}_c^* = \mathscr{S}_c^*$: $\hat{\mathscr{S}}_c^*$ is the infimal $(\mathscr{E}, A)$-invariant subspace containning $\mathscr{S}^* + ImD$, it obviously contains $\mathscr{S}_c^*$, the infimal $(\mathscr{E}, A)$-invariant subspace containning $ImB + ImD$, i.e. $\mathscr{S}_c^* \subset \hat{\mathscr{S}}_c^*$; for the reverse inclusion, just note that because $\mathscr{S}_c^* \supset \mathscr{S}^*$ and $\mathscr{S}_c^* \supset ImD$, $\mathscr{S}_c^*$ is a $(\mathscr{E}, A)$-invariant subspace containning $\mathscr{S}^* + ImD$, and thus contains the infimal one, $\hat{\mathscr{S}}_c^*$, i.e. $\mathscr{S}_c^* \supset \hat{\mathscr{S}}_c^*$, so we get $\hat{\mathscr{S}}_c^* = \mathscr{S}_c^*$.

$\hat{\mathscr{R}}_c^* = \mathscr{R}_{ca}^*$: Also under the assumption that $(\mathbf{ADDP})_p$ is solvable, we have $\hat{\mathscr{V}}_c^* = \mathscr{V}_c^* + \mathscr{R}_{ca}^*$, see M.Malabre and R.M.Zou (2007). Then, $\hat{\mathscr{R}}_c^* := \hat{\mathscr{V}}_c^* \cap \hat{\mathscr{S}}_c^* = (\mathscr{V}_c^* + \mathscr{R}_{ca}^*) \cap \mathscr{S}_c^* = \mathscr{V}_c^* \cap \mathscr{S}_c^* + \mathscr{R}_{ca}^* = \mathscr{R}_c^* + \mathscr{R}_{ca}^* = \mathscr{R}_{ca}^*$.

We have proved, see M.Malabre and R.M.Zou (2007), that when $(\mathbf{ADDP})_p$ is solvable, $\hat{\mathscr{V}}^* = \mathscr{V}_a^*$. This also holds when $(\mathbf{ADDP})'$ is solvable, thus $ImD \subset \hat{\mathscr{V}}^*$, where $\hat{\mathscr{V}}^*$ stands for the supremal $(A, \hat{\mathscr{B}})$(or controlled)-invariant subspace contained in $KerE$. Thanks to the result of M.Malabre et al. (1997), we can conclude, that $\hat{\mathscr{R}}_c^*$ is the infimal $(A, \hat{\mathscr{B}})$ self-bounded controlled-invariant subspace containing ImD and contained in $\mathscr{K}$, also because $\hat{\mathscr{R}}_c^* = \mathscr{R}_{ca}^*$ and Proposition 14, $\mathscr{R}_{ca}^*$ is the smallest almost self-bounded $(A, \mathscr{B})$ controlled-invariant subspace containing ImD and contained in $\mathscr{K}$.■

In order to characterize the set of fixed poles of $(\mathbf{ADDP})'$, we now extend a previous result from G.Basile and G.Marro (1992) to almost self-bounded controlled-invariants. This very important result expresses a key property of almost self-boundedness: among the set of all possible geometric solutions for $(\mathbf{ADDP})'$, the solutions which are almost self-bounded are always more efficient (at least as efficient) in terms of finite pole placement. The proof is skipped because of space limitation.

Proposition 17. Assume that $(\mathbf{ADDP})'$ is solvable and let $\mathscr{V}_a$ included in $KerE$ be any given geometric solution (in the sense of Proposition 7). Let us denote $\overline{\mathscr{V}_a} := \mathscr{V}_a + \mathscr{R}_a^*$. Then, $\overline{\mathscr{V}_a}$ is another geometric solution, $\overline{\mathscr{V}_a}$ is almost self-bounded controlled-invariant and its finite fixed spectra, as described in Proposition 11, always contain the finite fixed spectra of $\mathscr{V}_a$.

We are now in a position to write down the third most important result of our contribution, without detailed proof due to space limitation. The proof uses the previous results on pole placement facilities in addition with the fact that, when $(\mathbf{ADDP})'$ is solvable, $\mathscr{R}_{ca}^*$ is the smallest almost self-bounded solution (see Theorem 16). To simplify the exposition, we assume here that $(A, \mathscr{B})$ is controllable.

Theorem 18. Assume that $(\mathbf{ADDP})'$ is solvable, and $(A, \mathscr{B})$ is controllable.

- Any feedback solution of $(\mathbf{ADDP})'$ contains a set of finite fixed poles, and an infinite but always stable part (in the sense of Proposition 11).
- It is possible to find particular feedback solutions for which all the other finite poles, other than the fixed and infinite poles of $(\mathbf{ADDP})'$, can be placed freely.
- The Fixed Poles of $(\mathbf{ADDP})'$ are characterized as $\sigma_{fixed}^{finite} = \sigma(A + B\Phi|(\mathscr{S}_c^* \cap \mathscr{V}^*)/\mathscr{R}^*)$ where Φ is any map which makes $\mathscr{S}_c^* \cap \mathscr{V}^*$ $(A + B\Phi)$ invariant.
- When using ϵ-distance friend of $\mathscr{R}_{ca}^*$, infinite but stable poles occur as $\sigma_{stable}^{\infty} = \lim_{\epsilon \to 0}(\sigma(A + BF_\epsilon|\mathscr{S}_\epsilon))$, where F_ϵ is any map which makes $(A + BF_\epsilon)$ invariant $\mathscr{S}_\epsilon$, and where $\mathscr{S}_\epsilon$ is the controlled-invariant approximation of the sliding part $\mathscr{R}_{ca}^*/(\mathscr{S}_c^* \cap \mathscr{V}^*)$.

The following result expresses the Fixed Poles of $(\mathbf{ADDP})'$ in terms of zero structures of the systems. Let us denote respectively by $Zeros(A, B, E)$ and $Zeros(A, [B|D], E)$, the corresponding set of finite invariant zeros of the considered systems. We have the following theorem (the simple proof is also skipped).

Theorem 19. Assume that $(\mathbf{ADDP})'$ is solvable, and $(A, \mathscr{B})$ is controllable. The Fixed Poles of $(\mathbf{ADDP})'$, say σ_{fixed}^{finite}, are characterized as follows:

$$Zeros(A, B, E) = \sigma_{fixed}^{finite} \uplus Zeros(A, [B|D], E)$$

where $\uplus$ denotes union of sets with common elements repeated.

5. CONCLUSION

We have extended the key notion of self-boundedness, as initially introduced by Basile and Marro, to the case of Almost Self-Bounded Controlled-Invariant subspaces. We have shown that, when $(\mathbf{ADDP})'$ is solvable, the "best" almost controlled-invariant subspace to choose in order to achieve $(\mathbf{ADDP})'$ and simultaneously place the "largest set" of finite poles for the closed loop solution, is $\mathscr{R}_{ca}^*$, the supremal output nulling almost controllability subspace obtained when the disturbance is (artificially) considered as an extra control input. We have characterized the finite fixed poles of $(\mathbf{ADDP})'$ and expressed them in terms of some zeros.

REFERENCES

H.Aling and J.M.Schumacher. A nine-fold canonical decomposition for linear systems. *International Journal of Control*, Vol.39, pages 779-805, 1984.

G.Basile and G.Marro. Controlled Invariants and Conditioned Invariants in Linear System Theory. *Prentice-Hall*, Englewood Cliffs, New Jersey, 1992.

S.Bhattacharyya. Disturbance rejection in linear systems. *International Journal of Systems Science*, Vol.5, pages 931-943,1974.

C.Commault and J.M.Dion. Structure at infinity of linear multivariable systems: a geometric approach. *IEEE Transactions on Automatic Control*, AC-27, pages 693-696,1982.

F.R.Gantmacher. Theory of Matrices Vol.1 and 2. *Chelsea*, New York, 1959

T.Kailath. Linear Systems. *Prentice-Hall*, Englewood Cliff, N.J., 1970.

J.F.Lafay, C.Commault and M.Malabre. Structure of linear systems: geometric and transfer matrix approaches. *Kybernetika*, Vol.27, pages 170-185,1991.

M.Malabre. Some issues of structure and control for linear time-invariant systems. *Annual Reviews in Control*, Vol.30, pages 131-141,2006.

M.Malabre, J.C.Martinez-Garcia, and B.Del-Muro-Cuellar. On the fixed poles for disturbance rejection. *Automatica*, Vol.33, No.6, pages 1209-1211,1997.

M.Malabre, and R.M.Zou. New geometric insight into the Almost Disturbance Decoupling Problem by state feedback. *Third IFAC Symposium on Systems Structure and Control*, Iguassu Falls, Brazil, October 2007.

A.S.Morse. Structure and design of linear model following systems. *IEEE Transactions on Automatic Control*, AC-18, pages 346-354,1973.

H.H.Rosenbrock. State Space and Multi-variable Theory. *John Wiley and Sons*, New York, 1970.

J.M.Schumacher. A complement on pole placement. *IEEE Transactions on Automatic Control*, AC-25, pages 281–282, 1980.

H.L.Trentelman. Almost Invariant Subspaces and High Gain Feedback. *PhD Dissertation*, Rijksuniversiteit Groningen, May 1985.

J.C.Willems. Almost invariant subspaces: an approach to high gain feedback design. Part I:almost controlled invariant subspaces. *IEEE Transactions on Automatic Control*, AC-26, pages 235-252,1981.

W.M.Wonham. Linear multivariable control: a geometric approach(3^{rd} ed. 1985). *Springer-Verlag*, New York, 1974.