Proceedings of the 17th World Congress
The International Federation of Automatic Control
Seoul, Korea, July 6-11, 2008

On the Accessibility of Distributed Parameter Systems

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Abstract: This contribution is devoted to the accessibility analysis of distributed parameter systems. A formal system theoretical approach is proposed by means of differential geometry, which allows an intrinsic representation for the class of infinite dimensional systems. Beginning with the introduction of a convenient representation form, in particular, the accessibility along a trajectory is discussed generally. In addition, the derivation of (local) (non-)accessibility criteria via utilizing transformation groups is shown. In order to illustrate the developed theory the proposed method is applied to an example.

Keywords: Distributed parameter systems; nonlinear systems; differential geometry; accessibility; system analysis.

1. INTRODUCTION

In this contribution systems are considered whose evolution (in time) is allowed to be governed by (nonlinear) ordinary differential equation (ODEs) and/or partial differential equations (PDEs). From the control point of view the analysis of infinite dimensional systems is not straightforward, see, e.g., Barbu [1993]. A corresponding approximative system stated by ODEs could be calculated. Hence, some essential system properties might be lost by utilizing only the approximative system, see, e.g., Curtain and Zwart [1995]. Therefore, we propose a different approach, where the equations themselves are analyzed without any approximation by ODEs. A basis for our analysis is provided by a geometric description of a dynamic system such that it is independent of the chosen representation. Especially, it is the objective of this paper to emphasize that differential geometric methods are appropriate for the study of distributed parameter systems. In particular, for the system theoretical analysis we focus our attention on the investigation of the accessibility property. The crucial observation is that the formal Lie group approach for the analysis of lumped parameter systems, as used in Schlacher et al. [2002], is applicable in the distributed parameter case as well. It is shown how accessibility conditions based on the infinitesimal invariance principle can be provided.

The paper is organized as follows: In Section 2 some mathematical preliminaries are summarized, which are used in the sequel. Further, we introduce a convenient representation for distributed parameter systems with respect to the subsequent theoretical system analysis. Then, in Section 3 the property accessibility of dynamic systems is discussed, in general, and we outline how (local) conditions on (non)accessibility can be derived by means of transformation groups. In addition, the developed formal approach is applied to an example in order to show its practicability. The contribution finishes with some conclusions.

2. MATHEMATICAL PRELIMINARIES

This contribution applies the concept of smooth manifolds and bundles, see, e.g., Boothby [1986] and Saunders [1989] for an introduction and much more details. A bundle resp. a fibred manfold is a triple $(\mathcal{E}, \pi, \mathcal{M})$ with the total manifold $\mathcal{E}$, the base manifold $\mathcal{M}$ and the projection $\pi : \mathcal{E} \to \mathcal{M}$, where $\pi^{-1}(p)$ for any $p \in \mathcal{M}$ denotes the fiber over p. The manifold $\mathcal{E}$ possesses the coordinates (X^i, x^α) with the independent coordinates X^i, $i = 1 \ldots n_X$ and the dependent coordinates x^α, $\alpha = 1 \ldots n_x$. A section γ of the bundle $\mathcal{E} \to \mathcal{M}$ is a map $\gamma : \mathcal{M} \to \mathcal{E}$ such that $\pi \circ \gamma = \mathrm{id}_{\mathcal{M}}$ with the identity map $\mathrm{id}_{\mathcal{M}}$ on $\mathcal{M}$. The tangent and cotangent bundle of a smooth n-dimensional manifold $\mathcal{N}$ are denoted by $\mathcal{T}(\mathcal{N}) \to \mathcal{N}$ and $\mathcal{T}^*(\mathcal{N}) \to \mathcal{N}$, which are equipped with the coordinates $(X^i, \dot{X}^i)$ and $(X^i, \dot{X}_i)$ with respect to the holonomic bases $\{\partial_i\}$ and $\{\mathrm{dX}^i\}$. For brevity, the Einstein summation convention is used throughout the paper. The exterior algebra over an n-dimensional manifold $\mathcal{N}$ is denoted by $\wedge(\mathcal{T}^*(\mathcal{N}))$ with the exterior derivative $\mathrm{d} : \wedge_k(\mathcal{T}^*(\mathcal{N})) \to \wedge_{k+1}(\mathcal{T}^*(\mathcal{N}))$, the interior product $\rfloor : \wedge_{k+1}(\mathcal{T}^*(\mathcal{N})) \to \wedge_k(\mathcal{T}^*(\mathcal{N}))$ written as $v \rfloor \omega$, $v : \mathcal{N} \to \mathcal{T}(\mathcal{N})$ and $\omega : \mathcal{N} \to \wedge_{k+1}(\mathcal{T}^*(\mathcal{N}))$, and the exterior product $\wedge$. Further, $\wedge_k(\mathcal{T}^*(\mathcal{N})) \to \mathcal{N}$ is the exterior k-form bundle on $\mathcal{N}$. The canonical product equals the map $\langle \cdot, \cdot \rangle : \mathcal{T}^*(\mathcal{N}) \times \mathcal{T}(\mathcal{N}) \to C^\infty(\mathcal{N})$ and the Lie derivative of $\omega : \mathcal{N} \to \wedge(\mathcal{T}^*(\mathcal{N}))$ along a vector field $f : \mathcal{N} \to \mathcal{T}(\mathcal{N})$ is identified by $f(\omega)$.

For a section $\gamma : \mathcal{M} \to \mathcal{E}$ the k^{th}-order partial derivatives are given by

$$\gamma_J^\alpha = \partial_J \gamma^\alpha = \frac{\partial^k}{\partial_1^{j_1} \ldots \partial_{n_X}^{j_{n_X}}} \gamma^\alpha, \ \partial_i = \frac{\partial}{\partial X^i}$$

with the ordered multi-index $J = j_1, \ldots, j_{n_X}$, and $k = \#J = \sum_{i=1}^{n_X} j_i$. For brevity $j_i = \delta_{ij}$, $j \in \{1, \ldots, n_X\}$ will be denoted as 1_j and $j_i + \delta_{ij}$ as $J + 1_j$ with the Kronecker symbol δ. Further, we will use $[1, \mathbf{0}]$ and $[0, \mathbf{1}]$

for $[1,0,\ldots,0]$ and $[0,1,\ldots,1]$. The section γ can be extended to a map $j^n\gamma = (X^i,\gamma^\alpha(X),\partial_J\gamma^\alpha(X))$ with $1 \leqslant \#J \leqslant n$, the n^{th} jet of γ. The set of n^{th} jets (or n^{th}-order prolongations) of sections $\mathcal{M}\to\mathcal{E}$ is the manifold $J^n(\mathcal{E})$ with the coordinates (X^i,x_J^α), $0\leqslant \#J\leqslant n$ where $x^\alpha = x_J^\alpha$ for $\#J=0$. By means of $J^n(\mathcal{E})$ the bundles $\pi^n : J^n(\mathcal{E})\to\mathcal{M};\ (X^i,x_J^\alpha)\mapsto (X^i)$ and $\pi_0^n : J^n(\mathcal{E})\to\mathcal{E};\ (X^i,x_J^\alpha)\mapsto (X^i,x^\alpha)$ among others can be constructed.

The vertical bundle $\mathcal{V}(\mathcal{E})\to\mathcal{E}$ as a subbundle of $\mathcal{T}(\mathcal{E})\to\mathcal{E}$ is generated by all $v:\mathcal{E}\to\mathcal{T}(\mathcal{E})$ fulfilling $\pi_*(v)=0$, where v is said to be π-vertical. If $\exists w:\mathcal{M}\to\mathcal{T}(\mathcal{M})\ \pi_*(v)=w\circ\pi$ for a $v:\mathcal{E}\to\mathcal{T}(\mathcal{E})$, the vector field v is referred as π-projectable. A π-projectable vector field generates locally a 1-parameter group with the parameter ε. This also induces the bundle automorphism $(\exp(\varepsilon\pi_*(v),\exp(\varepsilon v)))$, see, e.g., Saunders [1989]. In particular, for a $v:\mathcal{E}\to\mathcal{V}(\mathcal{E})$ the fiber preserving automorphism $(\mathrm{id}_\mathcal{M},\exp(\varepsilon v)))$ is obtained. Instead of prolonging the group-induced bundle automorphism to $J^n(\mathcal{E})$, its infinitesimal generator can be prolonged,

$$\begin{aligned} j^n v &= d_J(v^\alpha)\partial_\alpha^J,\ 1\leqslant\#J\leqslant n,\\ v &= v^\alpha\partial_\alpha,\ d_J=(d_1)^{j_1}\circ\cdots\circ(d_{n_X})^{j_{n_X}} \end{aligned}\tag{1}$$

where $d_i : J^\infty(\mathcal{E})\to\mathcal{T}(J^\infty(\mathcal{E}))$ and $d_J(v^\alpha)=v^\alpha$ for $\#J=0$. The operator d_i fulfills $(d_i f)\circ j^{n+1}\gamma = \partial_i f(j^n\gamma)$ for $\forall f\in C^\infty(J^n(\mathcal{E}))$ and $\forall\gamma:\mathcal{M}\to\mathcal{E}$, and is called the total derivative with respect to the independent variables X^i. In coordinates (X^i,x_J^α) it is defined by $d_i=\partial_i + x^\alpha_{J+1_i}\partial^J_\alpha,\ \partial^J_\alpha=\frac{\partial}{\partial x^\alpha_J}$. By means of the total derivatives d_i we can introduce the horizontal derivative d_h, given as $(j^{n+1}\gamma)^*(\mathrm{d}_h(\omega)) = \mathrm{d}((j^n\gamma)^*(\omega))$, $\omega : J^n(\mathcal{E})\to\wedge(\mathcal{T}^*(J^n(\mathcal{E})))$ or in local coordinates $\mathrm{d}_h=\mathrm{d}X^i\wedge d_i$, see, e.g., Giachetta et al. [1997]. In addition, we will use Stokes' Theorem, see, e.g., Boothby [1986],

$$\int_\mathcal{C} j^{n+1}\gamma^*(\mathrm{d}_h\omega) = \int_\mathcal{C}\mathrm{d}(j^n\gamma^*(\omega)) = \int_{\partial\mathcal{C}}\iota_\mathcal{C}^*(j^n\gamma^*(\omega))\tag{2}$$

for $\omega : J^n(\mathcal{E})\to\pi^{n,*}(\wedge_{n_X-1}\mathcal{T}^*(\mathcal{M}))$, an orientable, bounded and compact manifold $\mathcal{C}\subset\mathcal{M}$ with global volume form, a coherently oriented boundary manifold $\partial\mathcal{C}$ and the corresponding inclusion map $\iota_\mathcal{C}:\partial\mathcal{C}\to\mathcal{C}$.

In this paper the coordinate X^1 represents the time t and $X_\mathcal{D}^j = X^{1+j}$, $j=1\ldots n_X-1$ the spatial coordinates. A dynamic system is described by a set of ODEs and PDEs, which may refer to different spatial domains $\mathcal{D}(\alpha)$ with $\mathcal{D}=\bigcup_\alpha\mathcal{D}(\alpha)\subseteq\mathcal{M}$, $\alpha=1,\ldots,n_x$. Let the areas $\mathcal{D}(\alpha)$ be $n_{\mathcal{D}(\alpha)}$-dimensional, compact manifolds ($\forall\alpha\in[1,\ldots,n_x]\ n_{\mathcal{D}(\alpha)}\in\mathbb{N}^+$) with global volume form and the coherently orientable boundaries $\partial\mathcal{D}(\alpha)$, respectively. Further, the boundary domains are denoted by $\mathcal{B}(\eta)$, $\eta=1,\ldots,n_b$ with $\mathcal{B}(\eta)\subseteq\bigcup_\alpha\partial\mathcal{D}(\alpha)$. For brevity and legibility the range of the indices will not always be stated explicitly, if it is clear from the context.

Throughout the contribution we consider dynamic systems of the form

$$\begin{aligned} x^\alpha_{[1,\mathbf{0}]} &= F^\alpha_{\mathcal{D}(\alpha)}\left(X,x_{[0,J]},u_{[0,K]}\right) &&,\ \alpha=1,\ldots,n_x\\ 0 &= F^\eta_{\mathcal{B}(\eta)}\left(X,x_{[0,J_\partial]},u_{[0,K_\partial]}\right) &&,\ \mu=1,\ldots,n_b \end{aligned}\tag{3}$$

with $0\leq\#J\leq n$, $0\leq\#K\leq n$, $0\leq\#J_\partial\leq m$ and $0\leq\#K_\partial\leq m$. The subscripts refer to the validity area of the equations, respectively. We call (3) an k^{th}-order differential equation with $k=\max\{n_x,n\}$. For the geometric interpretation of such a dynamic system, the bundles $\mathcal{E}_\mathcal{X}\to\Omega$ and $\mathcal{E}_\mathcal{U}\to\Omega$ are introduced with the adapted coordinates (X^i,x^α) and (X^i,u^ς) on $\mathcal{E}_\mathcal{X}$ and $\mathcal{E}_\mathcal{U}$ at least locally, where the independent coordinates are X^i, $i=1\ldots n_X$ and the dependent coordinates are x^α, $\alpha=1\ldots n_x$ and u^ς, $\varsigma=1\ldots n_u$. The so-called time-space cylinder Ω, where the system (3) evolutes in, is assumed to be a compact n_X-dimensional submanifold of $\mathcal{M}$ with $\Omega\subset\mathcal{M}$. Further, the product bundle $\mathcal{E}=\mathcal{E}_\mathcal{X}\times\mathcal{E}_\mathcal{U}\to\Omega$ is introduced. From the geometric point of view the system domain equations (3) represent a fibred submanifold of $J^n(\mathcal{E})$, see, e.g., Giachetta et al. [1997]. According to the bundle construction a solution of the system (3) follows as a section $\gamma:\Omega\to\mathcal{E}_\mathcal{X}$ that satisfies for an input function $u=\mu(X)$, $\mu:\Omega\to\mathcal{E}_\mathcal{U}$, the equations

$$\begin{aligned} x^\alpha_{[1,\mathbf{0}]}\circ j^1\gamma &= \left(F^\alpha_{\mathcal{D}(\alpha)}\left(X,x_{[0,J]},u_{[0,K]}\right)\circ j^n\mu\right)\circ j^n\gamma\\ 0 &= \left(F^\eta_{\mathcal{B}(\eta)}\left(X,x_{[0,J_\partial]},u_{[0,K_\partial]}\right)\circ j^m\mu\right)\circ j^m\gamma. \end{aligned}$$

A solution $\gamma:\Omega\to\mathcal{E}_\mathcal{X}$ of the system (3) is always related with some input $\mu:\Omega\to\mathcal{E}_\mathcal{U}$ and an initial condition $\gamma(X)|_{X^1=t_0}=\gamma_{t_0}$. Hence, it is still denoted by γ, where, concurrently, the latter information is disregarded in order to simplify matters. In addition, we will use $\gamma(\tau)$ and γ_τ to denote $\gamma(X)|_{X^1=\tau}$ as well as γ to denote a mapping $(\gamma,\mu):\Omega\to\mathcal{E}$, if it is clear from the context. In this geometrical framework an admissible change of coordinates of the explicit dynamic system (3) is a bijective map $\psi=(\psi_X^i,\psi_x^\alpha,\psi_u^\varsigma)$,

$$\begin{aligned} \bar{X}^{\bar{1}} &= \psi_X^{\bar{1}}\left(X^1\right),\\ \bar{X}_\mathcal{D}^{1+\bar{j}} &= \psi_X^{1+\bar{j}}\left(X_\mathcal{D}\right)\ ,\ \bar{j}=1,\ldots,n_X-1\\ \bar{x}^{\bar{\alpha}} &= \psi_x^{\bar{\alpha}}\left(X_\mathcal{D},x\right)\ ,\ \bar{\alpha}=1,\ldots,n_x\\ \bar{u}^{\bar{\varsigma}} &= \psi_u^{\bar{\varsigma}}\left(X_\mathcal{D},u\right)\ ,\ \bar{\varsigma}=1,\ldots,n_u. \end{aligned}\tag{4}$$

Then, the system in new coordinates follows to

$$\begin{aligned} \bar{x}^{\bar{\alpha}}_{[\bar{1},\mathbf{0}]} &= \partial_{\bar{1}}(\psi_X^{\bar{1}})^{-1}d_{[1,\mathbf{0}]}\left(\psi_x^{\bar{\alpha}}\left(X_\mathcal{D},x\right)\right)\circ(j^n\psi)^{-1}\\ &= \bar{F}^{\bar{\alpha}}_{\mathcal{D}(\bar{\alpha})}\left(\bar{X},\bar{x}_{[0,\bar{J}]},\bar{u}_{[0,\bar{K}]}\right)\\ 0 &= F^{\bar{\eta}}_{\mathcal{B}(\bar{\eta})}\left(X,x_{[0,J_\partial]},u_{[0,K_\partial]}\right)\circ(j^m\psi)^{-1}\\ &= \bar{F}^{\bar{\eta}}_{\mathcal{B}(\bar{\eta})}\left(\bar{X},\bar{x}_{[0,\bar{J}_\partial]},\bar{u}_{[0,\bar{K}_\partial]}\right). \end{aligned}$$

3. ACCESSIBILITY ANALYSIS VIA TRANSFORMATION GROUPS

Motivated by the geometric framework for distributed parameter systems it is the intention to investigate this class of systems from the control point of view. In this paper a system theoretical analysis based on a Lie group approach is considered, as it is utilized in, e.g., Schlacher et al. [2002] for the study of lumped parameter systems. Conditions for (local) (non-)accessiblity of dynamic systems are derived by means of an infinitesimal invariance criterion for distributed parameter systems. The idea behind the analysis is to relate the property accessiblity of dynamic systems to the non-existence of an invariant with respect to a set of transformation groups.

The requirements for the analysis by transformation groups are a mathematical model represented by (3) and suitable normed function spaces $\mathcal{H}_{\mathcal{U}} = \mathcal{H}_{\mathcal{U}}(t_0, t_0+T)$ and $\mathcal{H}_t([t_0, t_0+T], \mathcal{H}_{\mathcal{D}}) = \mathcal{H}_t$ for its inputs and solutions, i.e., ensuring the existence and uniqueness of the solution with respect to the initial condition, boundary conditions and the input. Here, $\mathcal{H}_{\mathcal{D}}$ denotes a function space based on the spatial domain $\mathcal{D}$ and $\mathcal{H}_t$ the function space for the evolution. For a specific coordinate system all considered function spaces are assumed to be Banach spaces, i.e., normed and complete vector spaces. Let $\gamma_\mu : \Omega \to \mathcal{E}_{\mathcal{X}}$ be the solutions of (3) on the time interval $\bar{T} = [t_0, t_0+T]$, $T > 0$, for some input $u = \mu(X)$, $\mu : \Omega \to \mathcal{E}_{\mathcal{U}}$. With respect to accessibility, in particular, we consider the mapping $f_{x_{t_0+T}} : \mathcal{H}_{\mathcal{U}} \to \mathcal{H}_{\mathcal{D}};\ u \longmapsto \gamma_{t_0+T}$, which represents a mapping between Banach spaces.

For lumped parameter systems it is well known that, in general, the accessibility property depends on the system trajectory, see, e.g., Schlacher et al. [2002] for similar definitions in the lumped parameter case, and so we define accessibility along a trajectory for distributed parameter systems. For an investigation of the accessibility property it is assumed that there exists an input function μ such that the system can be steered from an initial point γ_{t_0} to some $\bar{\gamma}_{t_0+T}$ in some time $T > 0$.

Definition 1. Let γ denote a solution of the dynamic system (3) on the time interval $\bar{T} = [t_0, t_0+T]$ for some $T > 0$ that meets the time boundary conditions $\gamma(t_0) = \gamma_{t_0}$ and $\gamma(t_0+T) = \gamma_{t_0+T}$ for the input $u = \mu(X)$, $\mu : \Omega \to \mathcal{E}_{\mathcal{U}}$. The system (3) will be called *accessible along the trajectory* γ, if locally there exists an input function $\bar{\mu} : \Omega \to \mathcal{E}_{\mathcal{U}}$ such that any point $\bar{\gamma}_{t_0+T}$ in an arbitrary small neighborhood $V_{\gamma_{t_0+T}} \subset \mathcal{H}_{\mathcal{D}}$ of γ_{t_0+T} is reachable. The system (3) will be called *approximately accessible along the trajectory* γ, if locally there exists an input function $\bar{\mu} : \Omega \to \mathcal{E}_{\mathcal{U}}$ to steer to any point $\bar{\gamma}_{t_0+T}$ in an arbitrary small neighborhood $V_{\gamma_{t_0+T}} \subset \mathcal{H}_{\mathcal{D}}$ of γ_{t_0+T} within a distance ε for any $\varepsilon > 0$, i.e. the corresponding endpoint $\tilde{\gamma}_{t_0+T}$ satisfies $\|\tilde{\gamma}_{t_0+T} - \bar{\gamma}_{t_0+T}\|_{\mathcal{H}_{\mathcal{D}}} \leq \varepsilon$ [1].

A weaker notion of exact accessibility is essential for distributed parameter systems since the image of the mapping $f_{x_{t_0+T}}$ might be not closed with respect to the topology of $\mathcal{H}_{\mathcal{D}}$ due to the infinite dimensional nature of $\mathcal{H}_{\mathcal{D}}$. In the finite dimensional case the closure of a (connected) subset is the subset itself. Thus, exact and approximative accessibility are equivalent concepts for lumped parameter systems. According to Definition 1 the system property accessibility is defined locally and serves as the basis for our subsequent investigations.

In order to illustrate the idea of applying the approach by transformation groups for this problem we consider the following situation.

Remark 1. Let γ be a solution of the dynamic system (3) on the finite time interval $\bar{T} = [t_0, t_0+T]$, $T > 0$ for the input $u = \mu(X)$, $\mu : \Omega \to \mathcal{E}_{\mathcal{U}}$ and consider vertical variations of the form

$$\tilde{\gamma}(X) = \Phi(X, \gamma(X), \mu(X))$$

for the given trajectory γ, where by arbitrarily varying the input and by fixing γ_{t_0} the distorted solution $\bar{\gamma}$ is another solution of the system (3). If at least one functional of the form $\mathcal{I}(\gamma) = \int_{\mathcal{D}} (I \circ \gamma)\, \mathrm{dX}_{\mathcal{D}}$ with $I \in C^\infty(\mathcal{E})$ can be found, which is left invariant despite all admissible distortions of γ, the system (3) will be called non-accessible along the trajectory γ.

With respect to Remark 1 such distortions of the solution γ is identified here by a symmetry group, i.e., which generates a flow on $\mathcal{E}$ and, thus, can be used to derive solutions from given ones. This is the reason why the theory of symmetry groups (Lie groups) is applied in this approach. In order to formulate the above considerations and to derive criterions for (non-)accessibility along a trajectory of a system (3) let us consider a set of one-parameter (Lie) groups $\Phi : \mathcal{G} \times \mathcal{E} \to \mathcal{E}$ of the form

$$\Phi : \mathcal{G} \times \mathcal{E} \to \mathcal{E}; (\varepsilon,\ X,\ x,\ u) \mapsto (X,\ \bar{x},\ \bar{u}) \tag{5}$$

and

$$\begin{aligned}&\left(X^i,\ \bar{x}^{\bar{\alpha}},\ \bar{u}^{\bar{\varsigma}}\right)\\ &\quad = \left(X^i,\ \Phi_\varepsilon^{\bar{\alpha}}(X,\ x,\ u),\ \Phi_\varepsilon^{\bar{\varsigma}}(X,\ x,\ u)\right)\end{aligned}$$

where ε is the group parameter and $\Phi_\varepsilon = \Phi(\varepsilon, \cdot)$ that operates on the state variables x^α and the input variables u^ς, respectively. In order to state accessibility conditions appropriately later on, the infinitesimal generator of (5) can be calculated, which is given by

$$\begin{aligned} v_{\mathcal{C}} &= \left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Phi_\varepsilon(X,\ x,\ u)\right|_{\varepsilon=0} \\ &= v_{\mathcal{X}}^\alpha(X,\ x,\ u)\partial_\alpha + v_{\mathcal{U}}^\varsigma(X,\ x,\ u)\partial_\varsigma \end{aligned} \tag{6}$$

with suitable functions $v_{\mathcal{X}}^\alpha, v_{\mathcal{U}}^\varsigma \in C^\infty(\mathcal{E})$. Henceforth, we consider the infinitesimal generator $v_{\mathcal{C}}$ of the transformation group Φ_ε (5) and its corresponding n^{th} prolongation $j^n(v_{\mathcal{C}}) : J^n(\mathcal{E}) \to \mathcal{V}(J^n(\mathcal{E})) \subset \mathcal{T}(J^n(\mathcal{E}))$, given by

$$j^n v_{\mathcal{C}} = d_J(v_{\mathcal{X}}^\alpha)\,\partial_\alpha^J + d_J(v_{\mathcal{U}}^\varsigma)\,\partial_\varsigma^J, \tag{7}$$

see (1), instead of Φ_ε itself, to investigate variations from a nominal solution γ.

Here, a valid variational vector field $v_{\mathcal{C}}$ is subject to restrictions. It has to be ensured that the considered transformation group is also a symmetry group of (3) since a distorted solution has to be a solution of the system (3) again. Thus, the prolonged vertical vector field (7) is applied to the system equations (3) and by the vanishing of

$$\begin{aligned} &j^n v_{\mathcal{C}}\left(x_{[1,\mathbf{0}]}^\alpha - F_{\mathcal{D}(\alpha)}^\alpha\left(X, x_{[0,J]}, u_{[0,K]}\right)\right) \circ j^n\gamma \\ &\quad = d_{[1,\mathbf{0}]}(v_{\mathcal{X}}^\alpha) - \\ &\qquad\quad d_{[0,L]}\left(v_{\mathcal{X}}^\beta\right)\partial_\beta^{[0,L]}F_{\mathcal{D}(\alpha)}^\alpha - d_{[0,L]}(v_{\mathcal{U}}^\varsigma)\,\partial_\varsigma^{[0,L]}F_{\mathcal{D}(\alpha)}^\alpha \\ &\quad = 0 \end{aligned} \tag{8}$$

on the domains $\mathcal{D}(\alpha)$ and

$$\begin{aligned} &j^m v_{\mathcal{C}}\left(F_{\mathcal{B}(\eta)}^\eta\left(X, x_{[0,J\partial]}, u_{[0,K\partial]}\right)\right) \circ j^m\gamma \\ &\quad = d_{[0,L\partial]}(v_{\mathcal{X}}^\alpha)\,\partial_\alpha^{[0,L\partial]}F_{\mathcal{B}(\eta)}^\eta + d_{[0,L\partial]}(v_{\mathcal{U}}^\varsigma)\,\partial_\varsigma^{[0,L\partial]}F_{\mathcal{B}(\eta)}^\eta \\ &\quad = 0. \end{aligned} \tag{9}$$

on the boundaries $\mathcal{B}(\eta)$, respectively, with $0 \leq \#L \leq n$ and $0 \leq \#L_\partial \leq m$, it is ensured that $v_{\mathcal{C}}$ is a symmetry group of the system, see, e.g., Olver [1993]. Obviously, the coefficients of the vertical vector field $v_{\mathcal{X}}^\alpha$ and $v_{\mathcal{U}}^\varsigma$ obey the equations (8) and (9) on the domain and boundary with respect to their validity area.

[1] $\|\cdot\|_{\mathcal{H}_{\mathcal{D}}}$ denotes the norm on $\mathcal{H}_{\mathcal{D}}$.

So far we have guaranteed that a valid vertical variation of a section implicate a new solution of the system (3). For the further analysis we are interested in the existence of local invariants of the form

$$\mathcal{I}(\gamma) = \int_{\mathcal{D}} (I(X,\ x,\ u) \circ \gamma)\, \mathrm{dX}_{\mathcal{D}} \tag{10}$$

where $\mathrm{dX}_{\mathcal{D}} = \mathrm{d}X^2 \wedge \cdots \wedge \mathrm{d}X^{n+1}$, $I \in C^{\infty}(\mathcal{E})$ and $\gamma, \tilde{\gamma} : \Omega \to \mathcal{E}$ satisfying

$$\mathcal{I}(\underbrace{\Phi \circ \gamma}_{\tilde{\gamma}}) = \int_{\mathcal{D}} (I(X,\ x,\ u) \circ (\Phi \circ \gamma))\, \mathrm{dX}_{\mathcal{D}} = \mathcal{I}(\gamma).$$

Note that for systems, whose subsystems may be valid on different spatial domains, the functional is normally of the form $\mathcal{I}(\gamma) = \int_{\mathcal{D}(\alpha)} \left(I_{\mathcal{D}(\alpha)}(X,\ x^{\alpha},\ u) \circ \gamma\right) \mathrm{dX}_{\mathcal{D}(\alpha)}$, where $\mathrm{dX}_{\mathcal{D}(\alpha)}$ is the volume on $\mathcal{D}(\alpha)$ and equations, or equivalently x^{α}, which refer to the same domain, are arranged, respectively. Henceforth, for brevity and legibility we consider one spatial domain to develop our theory, which can be extended to multi spatial domain problems (coupled systems) in a straight forward manner, like it is shown in the presented example.

Based on the geometric picture, the system (3) is accessible along a trajectory if there exists a subset of the considered transformation groups Φ_{ε} such that the initial condition is unchanged and locally the corresponding infinitesimal generators span the tangent space of $\mathcal{E}_{\mathcal{X}}$ at $(X,\ \gamma(X))|_{X^1 = t_0 + T}$ $\forall X_{\mathcal{D}} \in \mathcal{D}$. Equivalently, if there exists a non-trivial invariant function $I \in C^{\infty}(\mathcal{E})$ on the considered subset of the transformation groups Φ_{ε}, then the system (3) is not accessible since any neighborhood $V_{\gamma_{t_0+T}}$ of γ_{t_0+T}, which contains all points reached by an input function $\bar{\mu} : \Omega \to \mathcal{E}_{\mathcal{U}}$, is not open.

Since the transformation group acts only on the dependent variables, (6) qualifies for a vertical vector field $v_{\mathcal{C}} : \mathcal{E} \to \mathcal{V}(\mathcal{E}) \subset \mathcal{T}(\mathcal{E})$ on the manifold $\mathcal{E}$, as shown before. It follows that by admitting only fiber preserving variations that

$$\begin{aligned} \left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon} \mathcal{I}(\exp(\varepsilon v_{\mathcal{C}})(\gamma))\right|_{\varepsilon=0} &= \int_{\mathcal{D}} v_{\mathcal{C}}(\gamma)^{*} (I \mathrm{dX}_{\mathcal{D}}) \\ &= \int_{\mathcal{D}} \gamma^{*} (v_{\mathcal{C}} (I \mathrm{dX}_{\mathcal{D}})) \\ &= 0 \end{aligned} \tag{11}$$

if $v_{\mathcal{C}}$ is an admissible vector field. By using the relation $v_{\mathcal{C}}(I\mathrm{dX}_{\mathcal{D}}) = \mathrm{d}(v_{\mathcal{C}} \rfloor (I\mathrm{dX}_{\mathcal{D}})) + v_{\mathcal{C}} \rfloor \mathrm{d}(I\mathrm{dX}_{\mathcal{D}})$ and Stokes' Theorem (2) one can write the functional (11) in the form

$$\underbrace{\int_{\mathcal{D}} \gamma^{*}(v_{\mathcal{C}} \rfloor \mathrm{d}I \wedge \mathrm{dX}_{\mathcal{D}})}_{=\mathcal{I}_1(\gamma)=0} + \underbrace{\int_{\partial\mathcal{D}} (\gamma \circ \iota_{\mathcal{D}})^{*} (v_{\mathcal{C}} \rfloor (I\mathrm{dX}_{\mathcal{D}}))}_{=\mathcal{I}_2(\gamma)=0} = 0 \tag{12}$$

with the inclusion $\iota_{\mathcal{D}} : \partial\mathcal{D} \to \mathcal{D}$. The functional $\mathcal{I}_2(\gamma)$ vanishes since $v_{\mathcal{C}} \rfloor (I\mathrm{dX}_{\mathcal{D}}) = 0$ due to the use of a vertical vector field $v_{\mathcal{C}}$. The system analysis is based on proofing the (non-)existence of an invariant functional $\mathcal{I}$ for the considered transformation groups, i.e., since $\mathcal{I}_1(\gamma) = 0$

$$\int_{\mathcal{D}} \gamma^{*}(v_{\mathcal{C}}(I) \wedge \mathrm{dX}_{\mathcal{D}}) = \int_{\mathcal{D}} \gamma^{*}(\langle \mathrm{d}I, v_{\mathcal{C}} \rangle \wedge \mathrm{dX}_{\mathcal{D}}) = 0 \tag{13}$$

with

$$\mathrm{d}I = \partial_i I \mathrm{d}X^i + \underbrace{\partial_{\alpha} I \mathrm{d}x^{\alpha}}_{=\omega^{\mathcal{X}}} + \underbrace{\partial_{\varsigma} I \mathrm{d}u^{\varsigma}}_{=\omega^{\mathcal{U}}}$$

where $\mathrm{d}I : \mathcal{E} \to \mathcal{T}^{*}(\mathcal{E})$ is the differential of I. Hence, due to the special structure of (6) the equations (13) become

$$\begin{aligned} &\int_{\mathcal{D}} \gamma^{*}\left((\langle \omega_{\mathcal{X}}, v_{\mathcal{X}} \rangle + \langle \omega_{\mathcal{U}}, v_{\mathcal{U}} \rangle) \wedge \mathrm{dX}_{\mathcal{D}}\right) \\ &= \int_{\mathcal{D}} \gamma^{*}((v^{\alpha}_{\mathcal{X}} \underbrace{\partial_{\alpha} I}_{=\omega^{\mathcal{X}}_{\alpha}} + v^{\varsigma}_{\mathcal{U}} \underbrace{\partial_{\varsigma} I}_{\omega^{\mathcal{U}}_{\varsigma}}) \wedge \mathrm{dX}_{\mathcal{D}}) \\ &= 0. \end{aligned} \tag{14}$$

In order to incorporate the restrictions of the vertical vector field $v_{\mathcal{C}}$ we prolongate the functional (14) with respect to time, i.e. we obtain the terms

$$\begin{aligned} &\int_{\mathcal{D}} j^1\gamma^{*}\left(d_{[1,\mathbf{0}]}\left(v^{\alpha}_{\mathcal{X}} \omega^{\mathcal{X}}_{\alpha} + v^{\varsigma}_{\mathcal{U}} \omega^{\mathcal{U}}_{\varsigma}\right) \mathrm{dX}_{\mathcal{D}}\right) \\ &\quad = \int_{\mathcal{D}} j^1\gamma^{*}\left(d_{[1,\mathbf{0}]}(v^{\alpha}_{\mathcal{X}})\, \omega^{\mathcal{X}}_{\alpha} + v^{\alpha}_{\mathcal{X}} d_{[1,\mathbf{0}]}\left(\omega^{\mathcal{X}}_{\alpha}\right) \mathrm{dX}_{\mathcal{D}}\right) \\ &\qquad + \int_{\mathcal{D}} j^1\gamma^{*}\left(d_{[1,\mathbf{0}]}(v^{\varsigma}_{\mathcal{U}})\, \omega^{\mathcal{U}}_{\varsigma} + v^{\varsigma}_{\mathcal{U}} d_{[1,\mathbf{0}]}\left(\omega^{\mathcal{U}}_{\varsigma}\right) \mathrm{dX}_{\mathcal{D}}\right) \\ &\quad = 0 \end{aligned}$$

on the domain. Due to the explicit structure of the domain restrictions we can directly plug in the equations (8) yielding the relation

$$\begin{aligned} &\int_{\mathcal{D}} j^1\gamma^{*}\left(d_{[1,\mathbf{0}]}(v^{\alpha}_{\mathcal{X}})\, \omega^{\mathcal{X}}_{\alpha} \mathrm{dX}_{\mathcal{D}}\right) \\ &\quad = \int_{\mathcal{D}} j^n\gamma^{*}\left(\left(d_{[0,L]}\left(v^{\beta}_{\mathcal{X}}\right) \partial^{[0,L]}_{\beta} F^{\alpha}_{\mathcal{D}(\alpha)}\right) \omega^{\mathcal{X}}_{\alpha} \mathrm{dX}_{\mathcal{D}}\right) \\ &\qquad + \int_{\mathcal{D}} j^n\gamma^{*}\left(\left(d_{[0,L]}(v^{\varsigma}_{\mathcal{U}})\, \partial^{[0,L]}_{\varsigma} F^{\alpha}_{\mathcal{D}(\alpha)}\right) \omega^{\mathcal{X}}_{\alpha} \mathrm{dX}_{\mathcal{D}}\right). \end{aligned}$$

Hence, it may happen that the latter functional is not appropriate in this form for this task since the prolongated functional $\mathcal{I}_1(\gamma)$ can depend on the derivatives of components of the vector field $v_{\mathcal{C}}$, and, thus, can contribute to the so far vanishing integral $\mathcal{I}_2(\gamma)$ over the boundary. This is a well known problem in the calculus variation, which could be solved alternatively by using Lepagian forms, see, e.g., Giachetta et al. [1997]. Considering $d_{[0,J]} = d_i \circ d_j \circ \cdots \circ d_l$ $i, j, \ldots, l \in \{2, \ldots, n_X\} \neq 1$ the identity

$$\begin{aligned} &\int_{\mathcal{D}} j^n\gamma^{*}\left(\left(d_i \circ d_j \circ \cdots \circ d_l\left(v^l\right)\right) \omega_l \mathrm{dX}_{\mathcal{D}}\right) = \\ &\quad = -\int_{\mathcal{D}} j^n\gamma^{*}\left(\left(d_j \circ \cdots \circ d_l\left(v^l\right)\right) d_i(\omega_l)\, \mathrm{dX}_{\mathcal{D}}\right) \\ &\qquad + \int_{\mathcal{D}} j^n\gamma^{*}(\underbrace{d_i\left(\left(d_j \circ \cdots \circ d_l\left(v^l\right)\right) \omega_l \mathrm{dX}_{\mathcal{D}}\right)}_{\tilde{\mathrm{d}}_h((d_j \circ \cdots \circ d_l(v)) \rfloor \omega \wedge \partial_i \rfloor \mathrm{dX}_{\mathcal{D}})}) \\ &\quad = -\int_{\mathcal{D}} j^n\gamma^{*}\left((d_j \circ \cdots \circ d_l(v_{\alpha}))\, d_i(\omega^{\alpha})\, \mathrm{dX}_{\mathcal{D}}\right) \\ &\qquad + \underbrace{\int_{\mathcal{D}} \tilde{\mathrm{d}}\left(j^{n-1}\gamma^{*}(d_j \circ \cdots \circ d_l(v)) \rfloor \omega \wedge \partial_i \rfloor \mathrm{dX}_{\mathcal{D}}\right)}_{=\int_{\mathcal{D}} \iota^{*}_{\mathcal{D}}(j^{n-1}\gamma^{*}(d_j \circ \cdots \circ d_l(v)) \rfloor \omega \wedge \partial_i \rfloor \mathrm{dX}_{\mathcal{D}})}, \end{aligned} \tag{15}$$

for $v = v^l \partial_l : \mathcal{E} \to \mathcal{T}(\mathcal{E})$ and $\omega_l \mathrm{d}x^l : \mathcal{E} \to \mathcal{T}^*(\mathcal{E})$, can be derived, where Stokes' Theorem (2) is applied and $\tilde{\mathrm{d}}_h = \mathrm{d}X^i \wedge d_i$ for $i \in \{2, \ldots, n_X\}$ is induced by the original horizontal derivative d_h as well as $\tilde{\mathrm{d}}$ from d, correspondingly. In order to obtain a domain form depending only on the components of the vector field $v_\mathcal{C}$, but not on their derivatives, we are successively using (15). This delivers the following terms on the domain

$$\begin{aligned}&\int_\mathcal{D} j^n\gamma^* d_{[1,\mathbf{0}]}\left(\omega_\alpha^\mathcal{X}\right) v_\mathcal{X}^\alpha \mathrm{dX}_\mathcal{D} \\ &\quad + \int_\mathcal{D} j^n\gamma^* \left((-1)^{\#L} d_{[0,L]}\left(\omega_\alpha^\mathcal{X} \partial_\beta^{[0,L]} F_{\mathcal{D}(\alpha)}^\alpha\right)\right) v_\mathcal{X}^\beta \mathrm{dX}_\mathcal{D} \\ &\quad + \int_\mathcal{D} j^n\gamma^* \left((-1)^{\#L} d_{[0,L]}\left(\omega_\alpha^\mathcal{X} \partial_\varsigma^{[0,L]} F_{\mathcal{D}(\alpha)}^\alpha\right) v_\mathcal{U}^\varsigma \mathrm{dX}_\mathcal{D}\right) \\ &\quad + \int_\mathcal{D} j^1\gamma^* \left(d_{[1,\mathbf{0}]}\left(v_\mathcal{U}^\varsigma\right) \omega_\varsigma^\mathcal{U} \mathrm{dX}_\mathcal{D}\right) = 0\end{aligned} \tag{16}$$

for $\forall L \; 0 \leqslant \#L \leqslant n$ and on the boundary

$$\begin{aligned}&\int_{\partial\mathcal{D}} \iota_\mathcal{D}^* \Big(j^{n-1}\gamma^* \Big((-1)^{\#L} d_{[0,L]}\left(\partial_\beta^{[0,L]} F_{\mathcal{D}(\alpha)}^\alpha \omega_\alpha^\mathcal{X}\right) \cdot \\ &\qquad\qquad d_{[0,L-M-\mathbf{1}]}\left(v_\mathcal{X}^\beta\right) \mathrm{dX}_\mathcal{D}\Big)\Big) \\ &+ \int_{\partial\mathcal{D}} \iota_\mathcal{D}^* \Big(j^{n-1}\gamma^* \Big((-1)^{\#L} d_{[0,L]}\left(\partial_\varsigma^{[0,L]} F_{\mathcal{D}(\alpha)}^\alpha \omega_\alpha^\mathcal{X}\right) \cdot \\ &\qquad\qquad d_{[0,L-M-\mathbf{1}]}\left(v_\mathcal{U}^\varsigma\right) \mathrm{dX}_\mathcal{D}\Big)\Big) = 0\end{aligned} \tag{17}$$

for $\forall L \; 1 \leqslant \#L \leqslant n$ and $0 \leqslant \#M \leqslant n-1$. The equations on the domain (16) must now hold along a given trajectory γ and independently from a special choice of the input u, i.e., independent from the choice of $v_\mathcal{X}$, $v_\mathcal{U}$ and $d_{[1,\mathbf{0}]}(v_\mathcal{U})$, since there are no further restrictions on the domain. Thus, for equations (16) to hold we require that the corresponding braced terms vanish along a trajectory γ and so the following geometric domain conditions can be extracted as

$$\begin{aligned} d_{[1,\mathbf{0}]}\left(\omega_\alpha^\mathcal{X}\right) + (-1)^{\#L} d_{[0,L]}\left(\omega_\beta^\mathcal{X} \partial_\alpha^{[0,L]} F_{\mathcal{D}(\beta)}^\beta\right) &= 0 \\ d_{[1,\mathbf{0}]}\left(\omega_\varsigma^\mathcal{U}\right) + (-1)^{\#L} d_{[0,L]}\left(\omega_\beta^\mathcal{X} \partial_\varsigma^{[0,L]} F_{\mathcal{D}(\beta)}^\beta\right) &= 0 \\ \omega_\varsigma^\mathcal{U} &= 0.\end{aligned} \tag{18}$$

The latter equations (18) show that the function part I of the invariant functionals $\mathcal{I}$ (10) of interest has in fact the form $I = I(X, x)$ and does not depend on the input u. Next, we can substitute the solution $\omega_\varsigma^\mathcal{U} = 0$ and gather a new system of equations for the unknowns $\omega_\alpha^\mathcal{X}$

$$\begin{aligned} d_{[1,\mathbf{0}]}\left(\omega_\alpha^\mathcal{X}\right) + (-1)^{\#L} d_{[0,\bar{J}]}\left(\omega_\beta^\mathcal{X} \partial_\alpha^{[0,L]} F_{\mathcal{D}(\beta)}^\beta\right) &= 0 \\ (-1)^{\#L} d_{[0,\bar{J}]}\left(\omega_\beta^\mathcal{X} \partial_\varsigma^{[0,L]} F_{\mathcal{D}(\beta)}^\beta\right) &= 0.\end{aligned} \tag{19}$$

In order to obtain the boundary conditions for the invariant, we pay attention to the vector field restrictions (9) on the boundary. According to (9) we consider all non-trivial terms

$$d_{[0,L_\partial]}\left(v_\mathcal{X}^\alpha\right) \partial_\alpha^{[0,L_\partial]} F_{\mathcal{B}(\eta)}^\eta = -d_{[0,L_\partial]}\left(v_\mathcal{U}^\varsigma\right) \partial_\varsigma^{[0,L_\partial]} F_{\mathcal{B}(\eta)}^\eta, \tag{20}$$

with which we have to successively substitute all these terms in (17) to incorporate all present boundary conditions of the variational vector field $v_\mathcal{C}$.

Then, from the geometrical point of view it follows that the system (3) is obviously (locally) not accessible along a trajectory γ, if at least one non trivial invariant for the set of all admissible symmetry groups Φ_ε, having $\left(x_{[1,\mathbf{0}]} - F_\mathcal{D}\right)$ and $F_\mathcal{B}$ as invariants, exists.

Theorem 1. On a time intervall $\bar{T} = [t_0, t_0 + T]$, $T > 0$ let $\gamma : \Omega \to \mathcal{E}_\mathcal{X}$ be a solution of the dynamic system (3) for some input $\bar{\mu} : \Omega \to \mathcal{E}_\mathcal{U}$. A dynamic system (3), which is (locally) accessible along the trajectory γ, must fulfill in an arbitrarily small neighborhood of γ that the equations (19) as well as the extracted conditions from (17), which already incorporate (20), imply the trivial invariant as the only admissible solution, i.e., a functional of the form $\mathcal{I}(\gamma) = \int_\mathcal{D} (I(X) \circ \gamma) \mathrm{dX}_\mathcal{D}$, which is independent of the system state x and the input u or, equivalently, $\omega_\alpha^\mathcal{X} = 0$ and $\omega_\varsigma^\mathcal{U} = 0$.

Proof. Assuming a non-trivial invariant functional $\mathcal{I}(\gamma)$ depending on x, then, we are able to choose a point $\tilde{\gamma}_{t_0+T}$ in the arbitrarily small open neighborhood $V_{\gamma_{t_0+T}}$ of γ_{t_0+T}, which fulfills $\mathcal{I}(\tilde{\gamma}_{t_0+T}) \neq \mathcal{I}(\gamma_{t_0+T})$. Hence, the mapping $f_{x_{t_0+T}}$ is obviously not surjective locally since we cannot find an input $\mu : \Omega \to \mathcal{E}_\mathcal{U}$ according to (5) steering the system to the point $\tilde{\gamma}_{t_0+T}$. I.e., if there exists such a $\mathcal{I}(\gamma)$, any neighborhood $V_{\gamma_{t_0+T}}$ of γ_{t_0+T} will not be open and so the system is not accessible.

Remark 2. It is worth mentioning that the accessibility conditions represent linear differential equations with homogeneous boundary conditions and for linear systems the domain and boundary conditions from Theorem 1 do not depend on the fixed trajectory.

As can easily be seen, by the accomplishment of the pull back in (16) resp. (17) along a given solution γ, the problem can be stated by (19) and (17), incorporating (20), in the unknowns $\omega_\alpha^\mathcal{X}$, which depend only on X, and on the normally unkown solution γ. In addition, all total derivatives d_J degenerate to partial derivatives ∂_J.

4. EXAMPLE

In order to get a better understanding for the developed theory let us consider an illustrative example, namely the gantry crane with two heavy chains, see Figure 1.

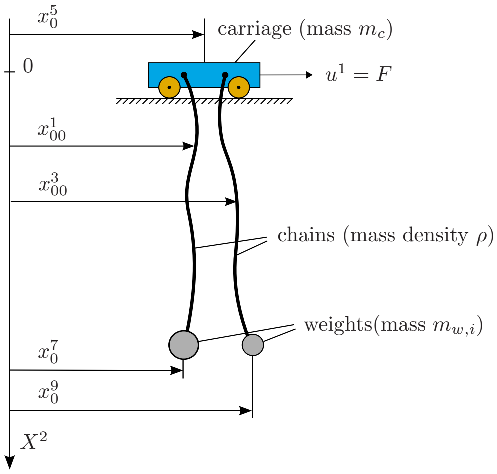

Fig. 1. Schematic diagram of the gantry crane

Similar to Thull et al. [2005] the system is given by

$$\begin{aligned}
x_{10}^1 &= F_{\mathcal{D}(1)}^1 = x_{00}^2 \\
x_{10}^2 &= F_{\mathcal{D}(2)}^2 = \rho^{-1} d_{01}\left(P_1\left(X^2\right) x_{01}^1\right) \\
x_{10}^3 &= F_{\mathcal{D}(3)}^3 = x_{00}^4 \\
x_{10}^4 &= F_{\mathcal{D}(4)}^4 = \rho^{-1} d_{01}\left(P_2\left(X^2\right) x_{01}^3\right) \\
x_1^5 &= F_{\mathcal{D}(5)}^5 = x_0^6 \\
x_1^6 &= F_{\mathcal{D}(6)}^6 = 2m_c^{-1}\left(P_1(0)\, x_{01}^1 + P_2(0)\, x_{01}^3\right) + u^1 \\
x_1^7 &= F_{\mathcal{D}(7)}^7 = x_0^8 \\
x_1^8 &= F_{\mathcal{D}(8)}^8 = -2m_{w,1}^{-1} P_1(L)\, x_{01}^1 \\
x_1^9 &= F_{\mathcal{D}(9)}^9 = x_0^{10} \\
x_1^{10} &= F_{\mathcal{D}(10)}^{10} = -2m_{w,2}^{-1} P_2(L)\, x_{01}^3
\end{aligned} \tag{21}$$

and

$$\begin{aligned}
0 &= F_{\mathcal{B}(1)}^1 = x_{00}^1 - x_0^5 & 0 &= F_{\mathcal{B}(5)}^5 = x_{00}^1 - x_0^7 \\
0 &= F_{\mathcal{B}(2)}^2 = x_{00}^2 - x_0^6 & 0 &= F_{\mathcal{B}(6)}^6 = x_{00}^2 - x_0^8 \\
0 &= F_{\mathcal{B}(3)}^3 = x_{00}^3 - x_0^5 & 0 &= F_{\mathcal{B}(7)}^7 = x_{00}^3 - x_0^9 \\
0 &= F_{\mathcal{B}(4)}^4 = x_{00}^4 - x_0^6 & 0 &= F_{\mathcal{B}(8)}^8 = x_{00}^4 - x_0^{10}
\end{aligned}$$

with $\mathcal{D}(1,\ldots,4) = [0,L]$, $\mathcal{D}(5,6) = \mathcal{B}(1,\ldots,4) = [0]$, $\mathcal{D}(7,\ldots,10) = \mathcal{B}(5,\ldots,8) = [L]$, $P_i(X^2) = g\left[\rho(L-X^2) + \frac{m_{w,i}}{2}\right]$ for $0 \leq X^2 \leq L$ and $i = 1,2$, and $m_c, m_{w,1}, m_{w,2}, g, \rho, L \in \mathbb{R}^+$. The system represents an interacting system, whose evolution is governed by ODEs and PDEs, respectively. The Hilbert space $\mathcal{H}_{\mathcal{D}} = \{z = (x_0^1,\ldots,x_0^7) \in L^2(0,L) \times H^1(0,L) \times L^2(0,L) \times H^1(0,L) \times \mathbb{R} \times \cdots \times \mathbb{R}\}$ equipped with a suitable inner product $\langle\cdot,\cdot\rangle_{\mathcal{H}_{\mathcal{D}}}$ can be introduced as a function space for the system state. For the corresponding evolution equation $\dot{z} = Az + Bu$ it can be shown that the linear operator $A : D(A) \subset \mathcal{H}_{\mathcal{D}} \to \mathcal{H}_{\mathcal{D}}$ is the infinitesimal generator of a C_0-operator-semigroup ($u = 0$), see Thull et al. [2005] for similar results.

Proposition 1. The dynamic system (21) is not accessible along any trajectory for $m_{w,1} = m_{w,2}$.

Proof. Considering a solution $\gamma : \Omega \to \mathcal{E}_{\mathcal{X}}$ of the system (21) we are interested in the (non-)existence of an invariant functional $\mathcal{I}$ of the form

$$\mathcal{I}(\gamma) = \int_{\mathcal{D}(1)} \left(I\left(X, x^1, x^2, x^3, x^4\right) \circ \gamma\right) \mathrm{d}X^2 \\
+ \left[I\left(X, x^5, x^6\right) \circ \gamma\right]_{\mathcal{D}(5)} + \left[I\left(X, x^7, \ldots, x^{10}\right) \circ \gamma\right]_{\mathcal{D}(7)},$$

which consists of three parts since the system (21) represents a coupled system. According to Theorem 1 the (local) accessibility conditions follow as

$$\begin{aligned}
v_{\mathcal{X}}^1 &: d_{10}\left(\omega_1^{\mathcal{X}}\right) + \rho^{-1} d_{01}\left(P_1\left(X^2\right) d_{01}\left(\omega_2^{\mathcal{X}}\right)\right) = 0 \\
v_{\mathcal{X}}^2 &: d_{10}\left(\omega_2^{\mathcal{X}}\right) + \omega_1^{\mathcal{X}} = 0 \\
v_{\mathcal{X}}^3 &: d_{10}\left(\omega_3^{\mathcal{X}}\right) + \rho^{-1} d_{01}\left(P_2\left(X^2\right) d_{01}\left(\omega_4^{\mathcal{X}}\right)\right) = 0 \\
v_{\mathcal{X}}^4 &: d_{10}\left(\omega_4^{\mathcal{X}}\right) + \omega_3^{\mathcal{X}} = 0
\end{aligned} \tag{22}$$

on the domain $\mathcal{D}(1,\ldots,4)$,

$$\begin{aligned}
v_{\mathcal{X}}^5 &: d_{10}\left(\omega_5^{\mathcal{X}}\right) - \rho^{-1} P_1(0)\, d_{01}\left(\omega_2^{\mathcal{X}}\right) \\
&\qquad - \rho^{-1} P_2(0)\, d_{01}\left(\omega_4^{\mathcal{X}}\right) = 0 \\
v_{\mathcal{X}}^6 &: d_{10}\left(\omega_6^{\mathcal{X}}\right) + \omega_5^{\mathcal{X}} = 0 \\
d_{01}\left(v_{\mathcal{X}}^1\right) &: \rho^{-1} P_1(0)\, \omega_2^{\mathcal{X}} + 2m_c^{-1} P_1(0)\, \omega_6^{\mathcal{X}} = 0 \\
d_{01}\left(v_{\mathcal{X}}^3\right) &: \rho^{-1} P_2(0)\, \omega_4^{\mathcal{X}} + 2m_c^{-1} P_2(0)\, \omega_6^{\mathcal{X}} = 0 \\
v_{\mathcal{U}}^1 &: \omega_6^{\mathcal{X}} = 0
\end{aligned} \tag{23}$$

on $\mathcal{D}(5,6)$ and

$$\begin{aligned}
v_{\mathcal{X}}^7 &: d_{10}\left(\omega_7^{\mathcal{X}}\right) - \rho^{-1} P_1(L)\, d_{01}\left(\omega_2^{\mathcal{X}}\right) = 0 \\
v_{\mathcal{X}}^8 &: d_{10}\left(\omega_8^{\mathcal{X}}\right) + \omega_7^{\mathcal{X}} = 0 \\
d_{01}\left(v_{\mathcal{X}}^1\right) &: \rho^{-1} P_1(L)\, \omega_2^{\mathcal{X}} - 2m_{w,1}^{-1} P_1(L)\, \omega_8^{\mathcal{X}} = 0 \\
v_{\mathcal{X}}^9 &: d_{10}\left(\omega_9^{\mathcal{X}}\right) - \rho^{-1} P_2(L)\, d_{01}\left(\omega_4^{\mathcal{X}}\right) = 0 \\
v_{\mathcal{X}}^{10} &: d_{10}\left(\omega_{10}^{\mathcal{X}}\right) + \omega_9^{\mathcal{X}} = 0 \\
d_{01}\left(v_{\mathcal{X}}^3\right) &: \rho^{-1} P_2(L)\, \omega_4^{\mathcal{X}} - 2m_{w,2}^{-1} P_2(L)\, \omega_{10}^{\mathcal{X}} = 0
\end{aligned} \tag{24}$$

on $\mathcal{D}(7,\ldots,10)$. Since $P_1\left(X^2\right) = P_2\left(X^2\right)$ for $m_{w,1} = m_{w,2}$ one obtains by means of the following coordinate transformation

$$\begin{aligned}
\bar{\omega}^{\mathcal{X}} = (&\omega_1^{\mathcal{X}} - \omega_3^{\mathcal{X}},\ \omega_2^{\mathcal{X}} - \omega_4^{\mathcal{X}},\ \omega_1^{\mathcal{X}} + \omega_3^{\mathcal{X}}, \\
&\omega_2^{\mathcal{X}} + \omega_4^{\mathcal{X}},\ \omega_5^{\mathcal{X}},\ \omega_6^{\mathcal{X}},\ \omega_7^{\mathcal{X}} - \omega_9^{\mathcal{X}}, \\
&\omega_8^{\mathcal{X}} - \omega_{10}^{\mathcal{X}},\ \omega_7^{\mathcal{X}} + \omega_9^{\mathcal{X}},\ \omega_8^{\mathcal{X}} + \omega_{10}^{\mathcal{X}})
\end{aligned}$$

and some basic modifications also the decoupled equation system

$$\begin{aligned}
\bar{F}_{\mathcal{D}(1)}^1 &= d_{10}\left(\bar{\omega}_1^{\mathcal{X}}\right) \\
&\quad + \rho^{-1} d_{01}\left(P_1\left(X^2\right) d_{01}\left(\bar{\omega}_2^{\mathcal{X}}\right)\right) = 0 \\
\bar{F}_{\mathcal{D}(1)}^2 &= d_{10}\left(\bar{\omega}_2^{\mathcal{X}}\right) + \bar{\omega}_1^{\mathcal{X}} = 0 \\
\bar{F}_{\mathcal{B}(1)}^3 &= \bar{\omega}_2^{\mathcal{X}} = 0 \\
\bar{F}_{\mathcal{B}(7)}^4 &= d_{10}\left(\bar{\omega}_7^{\mathcal{X}}\right) - \rho^{-1} P_1(L)\, d_{01}\left(\bar{\omega}_2^{\mathcal{X}}\right) = 0 \\
\bar{F}_{\mathcal{B}(7)}^5 &= d_{10}\left(\bar{\omega}_8^{\mathcal{X}}\right) + \bar{\omega}_7^{\mathcal{X}} = 0 \\
\bar{F}_{\mathcal{B}(7)}^6 &= \rho^{-1} P_1(L)\, \bar{\omega}_2^{\mathcal{X}} - 2m_{w,1}^{-1} P_1(L)\, \bar{\omega}_8^{\mathcal{X}} = 0.
\end{aligned} \tag{25}$$

The equations (25) are equal to the equations of a gantry crane with a single heavy chain and fixed carriage and admit non-trivial solutions implying non-trival invariants.

5. CONCLUSIONS

In this contribution the class of distributed parameter systems was investigated from the differential geometrical point of view which allows a covariant system representation. Based on this geometric picture a system analysis concerning the accessiblity of distributed parameter systems was motivated by a transformation group approach (Lie groups). By means of this framework conditions on the (local) (non-)accessibility can be provided.

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