Proceedings of the 17th World Congress
The International Federation of Automatic Control
Seoul, Korea, July 6-11, 2008

Sum-of-Squares Approximations to Robust Semidefinite Programs with Functional Variables: A Region-Dividing Approach ⋆

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Abstract: In this paper, we consider robust semidefinite programs with functional variables. In the proposed approach, an approximate semidefinite program is constructed based on approximation with the sum-of-squares technique. Unlike the conventional use of the sum-of-squares technique, the quality of approximation is improved by dividing the parameter region into several subregions. The idea is generalized from that of Jennawasin and Oishi (in Proceedings of the 46th IEEE Conference on Decision and Control, New Orleans, USA, December 2007), where robust semidefinite programs without functional variables are considered. The present approach is asymptotically exact in the sense that the optimal value of the approximate problem converges to that of the original problem as the resolution of the division becomes finer. Our approach also gives an *a priori* upper bound on the discrepancy between the optimal values of the two problems in terms of the resolution of the division.

1. INTRODUCTION

Optimization problems with polynomials have been known to cover a wide range of applications. Some of concrete examples in control are robust stability/performance analysis of linear systems affected by real parametric uncertainties [7], [5], [19], and analysis of nonlinear systems [15], [16]. Due to the development of the computational tool called "the sum-of-squares (SOS) approach" [17], a wide class of optimization problems involving polynomials can be efficiently solved. This is based on a connection between the SOS approach and a semidefinite program (SDP), which is computationally tractable and easy to handle using current solvers [21]. More discussions on the SOS approach can be found in [16], [11].

Here we consider an application of the SOS approach to robust semidefinite programs (robust SDPs) [1], [2], [6], [19], [9] which play an important role in robustness analysis/synthesis of linear systems. However, robust SDPs considered in the literature are limited to a specific class that does not cover some important applications in control.

In this paper, we consider a wider class of problems than those considered in [1], [2], [6], [19], [9]. The problems considered in this paper are described as follows:

$$\left.\begin{array}{ll}\text{minimize} & c^{\mathrm{T}}x \\ \text{subject to} & \mathcal{F}(x,\phi(\theta),\theta)\succeq 0, \quad \forall\theta\in\Theta,\end{array}\right\} \qquad (1)$$

where the optimization variables are a vector $x \in \mathbb{R}^n$ and a function ϕ. The function ϕ belongs to the space of piecewise continuous functions which map from $\Theta \in \mathbb{R}^p$ to $\mathbb{R}^{n_\phi}$. A parameter θ, which represents the uncertainties in the given system, can take any value in the set Θ. We assume throughout this paper that the set Θ is a given multidimensional interval in $\mathbb{R}^p$. The function $\mathcal{F}(x, a, \theta)$ is an $m \times m$ symmetric-matrix-valued function which is affine in $x \in \mathbb{R}^n$ and $a \in \mathbb{R}^{n_\phi}$, and depends polynomially on θ. This problem has an important application to robustness analysis/synthesis for a parameter-dependent linear systems with parameter-dependent Lyapunov functions. The problem (1) is also called a robust SDP because the constraint has to be satisfied for all possible values of $\theta \in \Theta$. Note here that the robust SDPs in the literature [1], [2], [6], [19], [9] can be considered as (1) without the functional variable ϕ. A robust SDP in the form (1) is difficult to solve due to its infinite-dimensional nature caused by the functional variable ϕ. One way to deal with this problem is to reduce the problem into a finite-dimensional one by choosing an appropriate functional basis; for example, polynomial basis, for ϕ. Bliman [4] showed that the resulting finite-dimensional robust SDP is asymptotically exact to the original one as the degree grows up. This convergence result is available when the system smoothly depends on parameters. Once the finite-dimensional robust SDP is obtained, one can apply the SOS approach to solve it in an asymptotically exact fashion (see [20], [9] for details). Besides the SOS approach, the Kalman-Yakubovich-Popov (KYP) Lemma [3], the Pólya's lemma [18] or the matrix-dilation approach [12], [13] can be applied with guaranteed asymptotic exactness. However, little is known about the tradeoff between the computational complexity and the amount of conservatism. This might be due to the difficulty to investigate such relationship in general.

⋆ Part of this work is supported by a Grant-in-Aid of the Ministry of Education, Culture, Sports, Science and Technology of Japan and the Pache Research Fund 2007-I-A-2 of Nanzan University, Japan.

Our approach aims to address the above issue for the SOS approach, which is the most widely used among the mention approaches due to its efficacy in implementation with the software SOSTOOLS [17]. Our strategy is based on extending the idea of [9] to the robust SDP (1) in a straightforward manner. We first construct a corresponding finite-dimensional problem by dividing the parameter set, and consider a piecewise polynomial consistent with the division. Then we construct a finite number of LMI constraints from the semi-infinite constraint using the SOS approach. Therefore, a standard SDP which approximates the original robust SDP is constructed. In contrast with the conventional approach, however, the quality of approximation is improved by making subdivision on subregions, while the degree of polynomial remains unchanged. This reveals the main feature of our approach. The convergence result of our approach to (1) is similarly obtained as that of [9]. In particular, we show that the optimal value of the approximate problem converges to that of the original problem, as the resolution of the division becomes higher. Moreover, an upper bound on the discrepancy between the two optimal values can be *a priori* obtained in terms of the resolution of the division. Therefore, the tradeoff between the computational complexity and the amount of conservatism can be understood via this bound. Moreover, this bound can be used to construct an efficient division, which attains good approximation with moderate computational cost. We emphasize here that the procedure to obtain the main result of this paper can be done in a similar fashion to [9], though major difference between a robust SDP with a functional variable and that without a functional-variable. Finally, we provide an example to demonstrate a benefit of our approach in the sense of computational complexity.

A closely related result was proposed in [14]. The approach in [14] is also based on the division on the parameter set, and a piecewise polynomial solution candidate is considered in this setting. Construction of an approximate problem in [14], however, relies on the matrix-dilation approach, which is difficult to implement than the SOS approach due to the lack of user-friendly software packages.

The paper is structured as follows. We first introduce the concept of sum-of-squares matrices in Section 2. Our approach is presented in Section 3, where an upper bound on the approximation error is derived here. Section 4 presents a numerical example. Section 5 concludes the paper.

2. SUM-OF-SQUARES POLYNOMIAL MATRICES

Let $\mathbb{R}[\theta]^{m\times n}$ denote the set of $m \times n$ polynomial matrices in $\theta \in \mathbb{R}^p$ and $\mathbb{S}[\theta]^n$ denote the set of $n \times n$ symmetric polynomial matrices. We define the notion of *sum-of-squares (SOS)* polynomial matrices as follows.

Definition 1. [10], [20] A polynomial matrix $S \in \mathbb{S}[\theta]^m$ is said to be a sum of squares (SOS) if there exists a polynomial matrix $T \in \mathbb{R}[\theta]^{q\times m}$ such that

$$S(\theta) = T(\theta)^{\mathrm{T}}T(\theta).$$

This is a generalization of the SOS representation for scalars [11], [16]. We use $\Sigma[\theta]^m$ to represent the set of $m\times m$ SOS polynomial matrices. It is clear that any polynomial matrix $S \in \Sigma[\theta]^m$ is globally positive semidefinite, but the converse is not true in general.

A computational procedure for verifying whether $S(\theta)$ is an SOS proceeds as follows. Choose pairwise different monomials $u_1(\theta), \ldots, u_{n_u}(\theta)$ and search for the coefficient matrix Y in the representation

$$T(\theta) = Y(u(\theta) \otimes I_m)$$

with $Y = (Y_1, \ldots, Y_{n_u})$ and $u(\theta) = (u_1(\theta), \ldots, u_{n_u}(\theta))^{\mathrm{T}}$. In [20], the matrix $S(\theta)$ is said to be an SOS with respect to $u(\theta)$ if there exists some Y satisfying $S(\theta) = (u(\theta) \otimes I_m)^{\mathrm{T}}(Y^{\mathrm{T}}Y)(u(\theta)\otimes I_m)$. Substituting $Z = Y^{\mathrm{T}}Y$ yields the following result.

Proposition 1. [10], [20] A polynomial matrix $S \in \mathbb{S}[\theta]^m$ is an SOS with respect to the monomial basis $u(\theta)$ if and only if there exists a symmetric matrix $Z \succeq O$ with

$$S(\theta) = (u(\theta) \otimes I_m)^{\mathrm{T}}Z(u(\theta) \otimes I_m). \qquad (2)$$

The condition (2) can be interpreted as an affine constraint in Z. This implies that the problem to find $Z \succeq O$ with (2) can be formulated as an SDP. In other words, we can check whether $S \in \Sigma[\theta]^m$ with respect to some monomial basis by solving an SDP.

3. THE PROPOSED APPROACH

3.1 Construction of an approximate problem by dividing the region

An approximate approach to the robust SDP (1) is described in this section. In our approach, we make the problem finite-dimensional by choosing ϕ as a polynomial with low degree. In order to improve the quality of approximation, we divide the parameter set Θ into several subregions and allow ϕ to be a piecewise polynomial consistent to the division. Then we have a finite-dimensional robust SDP with several semi-infinite constraints corresponding to the division. To deal with the semi-infinite constraints, we apply the concept of sum-of-squares matrices in Section 2. Finally, we obtain an approximate problem which is a standard SDP. The optimal value of the approximate problem converges to that of the original problem, as the resolution of the division becomes higher. This convergence result will be quantitatively investigated in the next subsection.

For the functional variable $\phi(\theta)$, we use a fixed-degree polynomial $\sum_{\alpha\in V} u_\alpha\theta^\alpha$ for some finite set $V \subset \mathbb{Z}_+^p$. Here the symbol θ^α stands for the product $\theta_1^{\alpha_1}\theta_2^{\alpha_2}\cdots\theta_p^{\alpha_p}$ with $\alpha = [\alpha_1\ \alpha_2\ \cdots\ \alpha_p] \in \mathbb{Z}_+^p$. We use the coefficients $u = (u_\alpha) \in \mathbb{R}^{n_u}$ to characterize the polynomial and write $\phi_u(\theta) = \sum_{\alpha\in V} u_\alpha\theta^\alpha$. Substitution of $\phi_u(\theta)$ into $\mathcal{F}$ makes this function dependent on finite-dimensional variables x and u. In particular, we define the notation

$$F(x, u, \theta) := \mathcal{F}(x, \phi_u(\theta), \theta).$$

Note that F is affine in x and u while polynomial in θ.

The definition of *a division* Δ of the parameter set Θ is given here for the succeeding discussion. We define a *division* $\Delta = \{\Theta^{[j]}\}_{j=1}^J$ of Θ as a set of closed convex polytopes such that $\Theta = \cup_{j=1}^J\Theta^{[j]}$ holds and $\Theta^{[j]} \cap \Theta^{[k]}$ has no interior point whenever $j \neq k$. Each element $\Theta^{[j]}$ of a division Δ is called a *subregion*. We assume that each $\Theta^{[j]}$ is a p-dimensional interval $\Pi_{i=1}^p[\underline{\theta}_i^{[j]}, \overline{\theta}_i^{[j]}]$. Here, the coefficients u is allowed to take a difference value $u^{[j]}$ depending on the subregion $\Theta^{[j]}$, for each $j = 1, 2, \ldots, J$. Hence, the function ϕ is a piecewise polynomial.

We then consider the following finite-dimensional problem:

$$P_0(\Delta): \text{minimize } c^{\mathrm{T}} x \quad \text{subject to } F(x, u^{[j]}, \theta) \succeq 0, \quad \forall \theta \in \Theta^{[j]}, \quad \forall j = 1, 2, \ldots, J,$$

where the optimization variables are $x \in \mathbb{R}^n$ and $u^{[1]}, u^{[2]}, \ldots, u^{[j]} \in \mathbb{R}^{n_u}$. Since only a specific class of function are considered for ϕ in this problem, we immediately have $\inf P_0(\Delta) \geq v_{\mathrm{opt}}$, where $\inf P_0(\Delta)$ and v_{opt} denote the optimal values of $P_0(\Delta)$ and (1) respectively.

The problem $P_0(\Delta)$ is still difficult to solve due to its semi-infinite constraint. Here, we apply the result of Proposition 1 to overcome this difficulty. In particular, an approximate problem for $P_0(\Delta)$ with the notion of SOS matrices:

$$P(\Delta): \text{minimize } c^{\mathrm{T}} x \quad \text{subject to } F(x, u^{[j]}, \theta) = S_0^{[j]}(\theta) + \sum_{i=1}^{p} (\theta_i - \underline{\theta}_i^{[j]})(\overline{\theta}_i^{[j]} - \theta_i) S_i^{[j]}(\theta), \quad \forall j = 1, 2, \ldots, J,$$

where $S_0^{[j]}, S_1^{[j]}, \ldots, S_p^{[j]} \in \Sigma[\theta]^m$, for all $j = 1, 2, \ldots, J$. In our setting, we use the same monomial basis, say $u_i(\theta)$, for the SOS matrices $S_i^{[j]}(\theta)$, for all $j = 1, 2, \ldots, J$. This leads to the parameterization $S_i^{[j]} = (u_i(\theta) \otimes I_m)^{\mathrm{T}} Z_i^{[j]} (u_i(\theta) \otimes I_m)$, $\forall j = 1, 2, \ldots, J$, for some positive semidefinite matrices $Z_i^{[j]}$'s. With this parameterization, it is not difficult to express the problem $P(\Delta)$ as an SDP in the decision variables x, $u^{[j]}$, and $Z_0^{[j]}, Z_1^{[j]}, \ldots, Z_p^{[j]}$, for $j = 1, 2, \ldots, J$, using the idea discussed at the end of Section 2.

For each j, the existence of the SOS matrices $S_0^{[j]}, S_1^{[j]}, \ldots, S_p^{[j]}$ implies that $F(x, u^{[j]}, \theta) \succeq 0, \forall \theta \in \Theta^{[j]}$. This is immediately obtained from the definition of SOS matrices and the assumption on $\Theta^{[j]}$. Hence the feasible region of $P(\Delta)$ projected into the space of x and $u^{[1]}, u^{[2]}, \ldots, u^{[J]}$ is included in the feasible region of $P_0(\Delta)$. In particular, $\inf P(\Delta) \geq \inf P_0(\Delta) \geq v_{\mathrm{opt}}$.

We now have an approximate problem, which is a standard SDP, for (1). In order to improve the approximation, we make subdivision on Δ and solve again the new approximate problem $P(\Delta)$. This procedure is repeatedly performed until the obtained optimal value $\inf P(\Delta)$ is satisfactory. In the next subsection, we show that the approximate optimal value $\inf P(\Delta)$ converges to v_{opt} when the resolution of division is fine enough. The convergence result is shown in a quantitative manner, that is, an *a priori* upper bound on the approximation error $|\inf P(\Delta) - v_{\mathrm{opt}}|$ is available with some mild assumptions on the original problem.

3.2 An upper bound on the approximation error

In this subsection, we provide an *a priori* upper bound on the approximation error $|\inf P(\Delta) - v_{\mathrm{opt}}|$ and discuss its implications. This is a generalization of the upper bound in [9], which is for a robust SDP without a functional variable.

We need the following assumption in order to obtain the result. The implications of these assumption can be found in [14].

Assumption 1.

(a) There exist $x \in \mathbb{R}^n$ and ϕ such that $\mathcal{F}(x, \phi(\theta), \theta) \succ 0, \ \forall \theta \in \Theta$.
(b) There exist three positive numbers $\bar{\epsilon}$, $\bar{x}$, and $\bar{\phi}$ such that, for any $0 \leq \epsilon \leq \bar{\epsilon}$ and any $v \in \mathbb{R}^n$, the set

$$\{(x, \phi) \mid c^{\mathrm{T}} x \leq v, \ \ \mathcal{F}(x, \phi(\theta), \theta) \succeq \epsilon I, \ \ \forall \theta \in \Theta\}$$

is either empty or having an element (x, ϕ) with $||x|| < \bar{x}$ and $||\phi(\theta)|| < \bar{\phi}$ for any $\theta \in \Theta$.
(c) The set V in the polynomial $\phi_u(\theta) = \sum_{\alpha \in V} u_\alpha \theta^\alpha$ contains the origin. □

We next need a measure of the resolution of the division. For a division $\Delta = \{\Theta^{[j]}\}_{j=1}^{J}$ of Θ, The *radius* of the subregion $\Theta^{[j]}$ is defined as $\mathrm{rad}\ \Theta^{[j]} := \max_i \frac{\overline{\theta}_i^{[j]} - \underline{\theta}_i^{[j]}}{2}$. The *maximum radius* of a division Δ is defined as $\overline{\mathrm{rad}}\ \Delta := \max_j \mathrm{rad}\ \Theta^{[j]}$. We use the maximum radius to measure the resolution of Δ.

Our main result, which provides the desired upper bound, is given in the following theorem.

Theorem 2. Suppose that Assumption 1 holds. Then, with the monomial bases

$$u_0(\theta) = \begin{bmatrix} 1 & \theta_1 & \theta_2 & \cdots & \theta_1^{d_1+1} \theta_2^{d_2+1} \cdots \theta_p^{d_p+1} \end{bmatrix}^{\mathrm{T}},$$
$$u_i(\theta) = \begin{bmatrix} 1 & \theta_1 & \theta_2 & \cdots & \theta_1^{d_1} \theta_2^{d_2} \cdots \theta_p^{d_p} \end{bmatrix}^{\mathrm{T}}, \ i = 1, \ldots, p,$$

the approximate problem $P(\Delta)$ satisfies

$$|\inf P(\Delta) - v_{\mathrm{opt}}| \leq C \overline{\mathrm{rad}}\ \Delta \tag{3}$$

for any division Δ with $\overline{\mathrm{rad}}\ \Delta \leq C_1$, where C_1 and C are positive numbers independent of Δ.

A direct consequence of this theorem is the asymptotic exactness of our approach. As we can see from (3), the approximation error $|\inf P(\Delta) - v_{\mathrm{opt}}|$ converges to zero as the maximum radius of the division goes to zero. Evaluation of C_1 and C is available in [13] for the case of robust SDPs without functional variables, though the resulting bound is often conservative. However, it is difficult to compute the constants C_1 and C for the problem in this paper because of difficulty in evaluation of $\bar{x}$ and $\bar{\phi}$. Recall that the existing approach with the degree increase on polynomials does not provide a corresponding quantitative result.

The upper bound (3) also gives a relationship between the approximation error and the size of the approximate problem. Namely, in order to reduce the approximation error, we need to decrease the maximum radius. This increases the number of subregions and, then, the number of variables and constraints in the approximate problem $P(\Delta)$. Especially when the parameter dimension is high, this increase is rapid and makes the approximate problem more difficult to solve. In order to reduce the computational complexity, however, it is possible to apply the concept of adaptive division of the parameter region in a similar fashion to [12]. The details will be discussed as follows.

For a division Δ, consider a minimizer $(x, u^{[j]}, \{Z_i^{[j]}\}_{i=0}^{p})$ of the problem $P(\Delta)$. With this minimizer, a subregion $\Theta^{[j]}$ is said to be *active* if at least one of the matrices

$Z_0^{[j]}, Z_1^{[j]}, \ldots, Z_p^{[j]}$ has a zero eigenvalue. Since an active subregion is an important one, we now define a new index for the resolution of the division:

$$\overline{\text{a-rad}}\ \Delta := \max_j \text{rad}\ \Theta^{[j]},$$

where the maximum is taken over all j such that the subregion $\Theta^{[j]}$ is active.

We improve the quality of approximation by repeatedly dividing an active subregion until a good approximate optimal value is obtained. A heuristic algorithm for this purpose can be constructed in a similar fashion to [12], [14] as below.

Algorithm:
0. Consider a coarse division Δ.
1. Solve $P(\Delta)$ for the current division Δ.
2. Stop if the obtained optimal solution is satisfactory.
3. Find an active subregion for the obtained optimal solution.
4. Divide the found subregion into two subregions.
5. Go back to Step 1 with the updated division Δ.

This algorithm is justified by the next theorem, which guarantees that we do not need to decrease the maximum radius but the maximum active radius for the reduction of the approximation error.

Remark: A way to determine the stopping criterion in Step 2 is to compute the *a priori* error bound in (3). If that error bound is small enough, then we can conclude that a good approximate optimal value is attained in the division Δ. Another way is to compute a *lower bound* on the optimal value by randomly sampling in Θ, and solve an SDP corresponding to the sampled points. If the lower bound and the upper bound $\inf P(\Delta)$ are close to each other, then a good approximate optimal value can be obtained from $\inf P(\Delta)$.

Theorem 3. With the same assumptions and symbols as in Theorem 2, we have

$$|\inf P(\Delta) - v_{\text{opt}}| \le C\,\overline{\text{a-rad}}\ \Delta,$$

for any division Δ with $\overline{\text{a-rad}}\ \Delta \le C_1$.

Proof This theorem can be proved using the result of Theorem 2. The procedure for the proof follows in the same line as [12]. □

3.3 A proof on the main theorem

This subsection is devoted to prove Theorem 2. In order to prove the statement, some results of the matrix-dilation approach [12], [13], [14] are necessary. In particular, a relationship, which is explored by [9], between the SOS-based approach and the matrix-dilation approach plays a key role in the proof. Here, the procedure to compute the upper bound can be divided in two steps. In the first step, we construct an auxiliary approximate problem, say $P_1(\Delta)$, by utilizing the matrix-dilation approach suggested by [13]. Then we combine the results of [13] and [14] to prove the existence of an upper bound on the approximation error $|\inf P_1(\Delta) - v_{\text{opt}}|$. In the second step, we show some connections between $P_1(\Delta)$ and $P(\Delta)$. This gives a way to compute an upper bound on $|\inf P(\Delta) - v_{\text{opt}}|$ from that on $|\inf P_1(\Delta) - v_{\text{opt}}|$.

We now give the overview of the matrix-dilation approach, in order to construct the auxiliary approximate problem $P_1(\Delta)$. First, we expand $F(x,u,\theta)$ as a polynomial in θ:

$$F(x,u,\theta) = F_{00\cdots0}(x,u) + F_{10\cdots0}(x,u)\theta_1 + \cdots + F_{d_1 d_2 \cdots d_p}(x,u)\theta_1^{d_1}\theta_2^{d_2}\cdots\theta_p^{d_p}.$$

Based on it, we consider a decomposition $2F(x,u,\theta) = M(\theta)^{\mathrm{T}} G(x,u) M(\theta)$. The matrix $G(x,u)$ contains matrix coefficients of $F(x,u,\theta)$, while

$$M(\theta) = \left[I_m\ \theta_1 I_m\ \theta_2 I_m\ \cdots\ \theta_1^{d_1}\theta_2^{d_2}\cdots\theta_p^{d_p} I_m\right]^{\mathrm{T}},$$

i.e., the matrix $M(\theta)$ contains all of monomials whose each element θ_i has degree less than or equal to d_i. Moreover, we consider a matrix $H(\theta)$ such that the matrix $[M(\theta)\ \ H(\theta)]$ is nonsingular and the relation $M(\theta)^{\mathrm{T}} H(\theta) = O$ holds for all $\theta \in \mathbb{R}^p$. Such $H(\theta)$ is called an *orthogonal complement* of $M(\theta)$. An important fact is that the orthogonal complement $H(\theta)$ can be chosen to be affine in θ.

For a given division Δ, pick up one subregion $\Theta^{[j]}$, which is a multi-dimensional interval $\Pi_{i=1}^p[\underline{\theta}_i^{[j]}, \overline{\theta}_i^{[j]}]$ by assumption. We define θ^{c} as the center of $\Theta^{[j]}$, that is $\theta^{\mathrm{c}} := \left[\frac{\underline{\theta}_1^{[j]} + \overline{\theta}_1^{[j]}}{2}\ \frac{\underline{\theta}_2^{[j]} + \overline{\theta}_2^{[j]}}{2}\ \cdots\ \frac{\underline{\theta}_p^{[j]} + \overline{\theta}_p^{[j]}}{2}\right]$. Since $H(\theta)$ is affine in θ, it can be expanded around θ^{c} as

$$H(\theta) = H(\theta^{\mathrm{c}}) + (\theta_1 - \theta_1^{\mathrm{c}})H_1 + (\theta_2 - \theta_2^{\mathrm{c}})H_2 + \cdots + (\theta_p - \theta_p^{\mathrm{c}})H_p,$$

where $H_1, \ldots, H_p$ are constant matrices. We now consider the constraints

$$G(x,u^{[j]}) + H(\theta^{\mathrm{c}})(W^{[j]})^{\mathrm{T}} + W^{[j]}H(\theta^{\mathrm{c}})^{\mathrm{T}} - \sum_{i=1}^{p} V_i^{[j]} \succeq O, \quad (4)$$

$$V_i^{[j]} + (\overline{\theta}_i^{[j]} - \theta_i^{\mathrm{c}})(H_i(W^{[j]})^{\mathrm{T}} + W^{[j]}H_i^{\mathrm{T}}) \succeq O, \quad (5)$$

$$V_i^{[j]} - (\overline{\theta}_i^{[j]} - \theta_i^{\mathrm{c}})(H_i(W^{[j]})^{\mathrm{T}} + W^{[j]}H_i^{\mathrm{T}}) \succeq O. \quad (6)$$

Here the subscript i runs from 1 to p. By following the idea of Ben-Tal and Nemirovski [2], it is easy to see that if there exist $V_1^{[j]}, V_2^{[j]}, \ldots, V_p^{[j]}$ satisfying the inequalities (4)–(6), then $(x, W^{[j]})$ satisfies the constraint

$$G(x,u^{[j]}) + H(\theta)(W^{[j]})^{\mathrm{T}} + W^{[j]}H(\theta)^{\mathrm{T}} \succeq O \quad (7)$$

for any vertex of $\Theta^{[j]}$. For example, adding the p inequalities of (5) to (4), we can see that the inequality of (7) holds at the vertex $\overline{\theta}^{[j]} := \left[\overline{\theta}_1^{[j]}\ \overline{\theta}_2^{[j]}\ \cdots\ \overline{\theta}_p^{[j]}\right]$. Since the inequality (7) is affine in θ, $(x, W^{[j]})$ satisfies (7) for any point in $\Theta^{[j]}$. Premultiplication of $M(\theta)^{\mathrm{T}}$ and postmultiplication of $M(\theta)$ to this inequality give $F(x, u^{[j]}, \theta) \succeq O$, $\forall\theta \in \Theta^{[j]}$.

Define $S_1(\Theta^{[j]})$ as the set of all $(x, u^{[j]}, W^{[j]}, \{V_i^{[j]}\}_{i=1}^p)$ such that (4)–(6) hold. We now obtain the following new approximate problem:

$$\left.\begin{array}{ll} P_1(\Delta): \text{minimize} & c^{\mathrm{T}}x \\ \quad\quad\quad \text{subject to} & (x, u^{[j]}, W^{[j]}, \{V_i^{[j]}\}_{i=1}^p) \in S_1(\Theta^{[j]}), \\ & \forall j = 1, 2, \ldots, J. \end{array}\right\}$$

By construction, the feasible region of $P_1(\Delta)$ is included in that of (1), which implies $\inf P_1(\Delta) \ge v_{\text{opt}}$.

The result on the approximation error $|\inf P_1(\Delta) - v_{\text{opt}}|$ is given in the following proposition.

Proposition 4. Under Assumption 1, there exist constants C_1 and C such that, if a given division Δ satisfies $\overline{\mathrm{rad}}\ \Delta \leq C_1$, then

$$|\inf P_1(\Delta) - v_{\mathrm{opt}}| \leq C\overline{\mathrm{rad}}\ \Delta.$$

Proof This proposition can be proved by combining the results of Theorem 4 in [14] and Theorem 7 in [13]. □

We next prepare the following lemma, which explains the relationship between the problems $P_1(\Delta)$ and $P(\Delta)$. The subscript $[j]$ is omitted in the following lemma for convenience.

Lemma 5. If there exist x, u, W and $\{V_i\}_{i=1}^{p}$ such that

$$\begin{aligned} G(x,u) + H(\theta^{\mathrm{c}})W^{\mathrm{T}} + WH(\theta^{\mathrm{c}})^{\mathrm{T}} - \sum_{i=1}^{p} V_i \succeq O, \\ V_i + (\overline{\theta}_i - \theta_i^{\mathrm{c}})(H_i W^{\mathrm{T}} + WH_i^{\mathrm{T}}) \succeq O, \quad \forall i = 1,\ldots,p, \\ V_i - (\overline{\theta}_i - \theta_i^{\mathrm{c}})(H_i W^{\mathrm{T}} + WH_i^{\mathrm{T}}) \succeq O, \quad \forall i = 1,\ldots,p, \end{aligned} \tag{8}$$

then there exist SOS matrices $S_0(\theta), S_1(\theta), \ldots, S_p(\theta)$ satisfying

$$F(x,u,\theta) = S_0(\theta) + \sum_{i=1}^{p} (\theta_i - \underline{\theta}_i)(\overline{\theta}_i - \theta_i) S_i(\theta). \tag{9}$$

Moreover, the monomial bases $u_0(\theta), u_1(\theta), \ldots, u_p(\theta)$ are expressed as follows:

$$\begin{aligned} u_0(\theta) &= \begin{bmatrix} 1 & \theta_1 & \theta_2 & \cdots & \theta_1^{d_1+1}\theta_2^{d_2+1}\cdots\theta_p^{d_p+1} \end{bmatrix}^{\mathrm{T}}, \\ u_i(\theta) &= \begin{bmatrix} 1 & \theta_1 & \theta_2 & \cdots & \theta_1^{d_1}\theta_2^{d_2}\cdots\theta_p^{d_p} \end{bmatrix}^{\mathrm{T}}, \ i = 1,\ldots,p. \end{aligned} \tag{10}$$

Proof This is an immediate result from Lemmas 1 and 2 in [9], which were derived by the present authors. □

Now we give the proof of our main theorem.
Proof of Theorem 2. Suppose in the problem $P(\Delta)$ that the monomial bases $u_0(\theta), u_1(\theta), \ldots, u_p(\theta)$ of the SOS matrices $S_0(\theta), S_1(\theta), \ldots, S_p(\theta)$ are chosen as (10). Lemma 5 implies that, for each subregion $\Theta^{[j]}$, the constraint of the problem $P_1(\Delta)$ is just a sufficient condition of the constraint of $P(\Delta)$. Therefore, the feasible region of $P_1(\Delta)$ is included in that of $P(\Delta)$, and thus $v_{\mathrm{opt}} \leq \inf P(\Delta) \leq \inf P_1(\Delta)$. If a given division Δ satisfies $\overline{\mathrm{rad}}\ \Delta \leq C_1$, then we obtain from Proposition 4 that

$$|\inf P(\Delta) - v_{\mathrm{opt}}| \leq |\inf P_1(\Delta) - v_{\mathrm{opt}}| \leq C\overline{\mathrm{rad}}\ \Delta,$$

which completes the proof. □

4. NUMERICAL EXAMPLE

In this section, we demonstrate the effectiveness of our approach by an example on robust H_∞ analysis of a linear uncertain system. All the computations are executed using Matlab 6.1, SOSTOOLS [17] with Matrix Patch [8]. The computer is equipped with Celeron 897 MHz and 256 MByte memory.

Consider the following uncertain system borrowed from [5].

$$\left.\begin{aligned} \dot{x} &= A(\theta)x + Bw \\ y &= Cx + Dw, \end{aligned}\right\}$$

where $A(\theta) = \theta_1 A_1 + \theta_2 A_2 + (1 - \theta_1 - \theta_2)A_3$ with

$$A_1 = \begin{bmatrix} -0.42 & -1.68 & -2.24 & 2.92 \\ -0.74 & -1.74 & -4.58 & 1.44 \\ -2.92 & 3.84 & -6.98 & 2 \\ -4.92 & -2.68 & -8.66 & -0.78 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} -0.78 & 5.52 & 1.36 & 5.8 \\ -5.42 & -4.62 & -0.26 & -1.08 \\ 2.48 & 6 & -7.7 & -7.72 \\ -1.32 & 3.8 & 2.14 & 2.1 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} -4.2 & -3.12 & -2.96 & 1.84 \\ 4.48 & -1.02 & -2.78 & -7.38 \\ 1.22 & -0.12 & -2.66 & -0.34 \\ 2.1 & 4.52 & -1.28 & -1.5 \end{bmatrix},$$

$$B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, C = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}^{\mathrm{T}}, D = 0.$$

The parameter set is given by

$$\Theta = \{\theta \in \mathbb{R}^2 \mid 0 \leq \theta_1 \leq 0.5,\ 0 \leq \theta_2 \leq 0.5\}.$$

The robust H_∞ performance of the above system can be computed by solving the following optimization problem:

$$\left.\begin{aligned} &\text{minimize} \ \ \gamma \\ &\text{subject to} \ \ P(\theta) \succ 0, \quad \forall\theta \in \Theta, \\ &\qquad G(\theta, P(\theta)) + \begin{bmatrix} C^{\mathrm{T}} \\ D^{\mathrm{T}} \end{bmatrix} [C \ \ D] \prec 0, \quad \forall\theta \in \Theta, \end{aligned}\right\} \tag{11}$$

where

$$G(\theta, P(\theta)) = \begin{bmatrix} A(\theta)^{\mathrm{T}}P(\theta) + P(\theta)A(\theta) & P(\theta)B \\ B^{\mathrm{T}}P(\theta) & -\gamma^2 \end{bmatrix}.$$

The problem (11) is a robust SDP with a functional variable $P(\theta)$. Solving (11) with restricting $P(\theta)$ to be a polynomial yields an upper bound on the robust H_∞ performance.

With the conventional approach, which is based on the degree increase of $P(\theta)$, we obtain the following results.

Table 1. The results of the conventional approach

Degree of $P(\theta)$	Upper bound	Number of variables	Times (sec.)
1	1.2236	482	3.525
2	1.2152	1587	12.688

We now apply the proposed approach by dividing Θ into two subregions $\Theta^{[1]} = [0, 0.5] \times [0, 0.25]$ and $\Theta^{[2]} = [0, 0.5] \times [0.25, 0.5]$, and considering $P(\theta)$ being piecewise affine in θ. We obtain the upper bound 1.2152, which is the same as that obtained by $P(\theta)$ of degree two. The results are summarized in the following table.

Table 2. The results of the proposed approach

Number of subregions	Upper bound	Number of variables	Times (sec.)
1	1.2236	482	3.525
2	1.2152	963	5.758

It can be seen that improving the upper bound by the region-dividing approach requires less computational cost than that by the degree-increasing approach. This is due to the fact that the degree increase of $P(\theta)$ leads to increase of the required degrees of SOS matrices, which results in rapid growth of the number of decision variables in the approximate problem.

5. CONCLUSION

We have provided an approximate approach for robust SDPs with functional variables. Our approximation scheme is shown to be asymptotically exact. The convergence result is quantitative in the sense that approximation error can be explicitly obtained in terms of the resolution of the division. Extension of our approach to problems with derivative of the functional variables will be a possible research direction in the near future.

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