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Control under Quantization, Saturation and Delay: An LMI Approach

Emilia Fridman * **Michel Dambrine** **

* *Department of Electrical Engineering-Systems, Tel-Aviv University, Tel-Aviv 69978, Israel (email: emilia@eng.tau.ac.il)*
** *LAMIH CNRS UMR 8530, Valenciennes University, 59313 Valenciennes cedex 9, France (email: michel.dambrine@univ-valenciennes.fr)*

Abstract: This paper studies quantized and delayed state-feedback control of linear systems. We consider two types of quantization: quantized feedback and quantized state. The quantizer may be either unconstrained or saturated with a given quantization error bound. The delay is supposed to be time-varying and bounded. The controller is designed with the following property: all the states of the closed-loop system (starting from a neighborhood of the origin in the saturated case) exponentially converge to some bounded region in R^n. The design procedure is given in terms of Linear Matrix Inequalities (LMIs), derived via Lyapunov-Krasovskii functional and the comparison principle.

1. INTRODUCTION

Quantization in control systems has recently become an active research topic. The need for quantization arises when digital networks are part of the feedback loop. In this paper we study linear control systems with either quantized state or quantized control input. See e.g. Brocket and Liberzon [2000], Ishii and Francis [2003], Liberzon [2003], Bullo and Liberzon [2006] and the references therein for control under different types of quantization (in both, linear and nonlinear cases).

Time-delay often appears in control systems and, in many cases, delay is a source of instability [Hale and Verduyn-Lunel, 1993]. During the last decade, a considerable amount of attention has been paid to control of systems with uncertain constant or time-varying delays (see e.g. Boyd et al. [1994], Kolmanovskii and Myshkis [1999], Niculescu [2001], Fridman [2001], Gu et al. [2003], Fridman et al. [2004], He et al. [2007]). Delays often appears in networked control systems.

Delayed quantized control was recently studied in Liberzon [2006] by applying Input-To-State Stability (ISS) analysis (see Sontag and Wang [1995]). Sufficient conditions for ISS systems with time-varying delays were derived via Razumikhin approach in Teel [1998]. For systems with constant delays, ISS sufficient conditions were recently derived in terms of Lyapunov-Krasovskii functionals in Pepe and Jiang [2006]. For systems with time-varying delays ISS sufficient stability conditions via Krasovskii method were obtained in Fridman et al. [2007] in terms of matrix inequalities.

In the existing literature it is usually assumed that the stabilization problem in the absence of quantization and delay has been solved, i.e. a state-feedback that globally asymptotically stabilizes the system is known. Then the same feedback is applied in the presence of quantization and delay, and some ultimate bounds on the solutions are obtained. In Fridman et al. [2007], for the first time, LMI conditions have been derived directly for design of quantized and delayed control of linear systems.

It is the objective of the present paper to derive LMI conditions for state-feedback design in the cases of quantized control input or quantized state, in the presence of *saturation* and time-varying delay. We represent saturated quantizer as a quantizer that acts on the saturated input or state. Thus the problem is reduced to ISS analysis and design of systems with saturated input or state. In the case of saturated control input we employ a linear system representation with polytopic type uncertainty [Tarbouriech and Gomes da Silva, 2000, Cao et al., 2002]. The presented delay-dependent LMI conditions for ISS are based on improved Lyapunov-Krasovskii technique and, generally, lead to less restrictive conditions than in Fridman et al. [2007]. We note that recently exponential convergence of linear state-delay systems with bounded control and disturbances was studied in Oucheriah [2006], where delay-independent conditions were derived.

Notation: Throughout the paper the superscript 'T' stands for matrix transposition, R^n denotes the n dimensional Euclidean space with norm $|\cdot|$, $R^{n\times m}$ is the set of all $n\times m$ real matrices, and the notation $P>0$, for $P\in R^{n\times n}$ means that P is symmetric and positive definite. In symmetric block matrices we use $*$ as an ellipsis for terms that are induced by symmetry. $\bar{\lambda}(G)$ and $\underline{\lambda}(G)$ denote the largest and the smallest eigenvalues of the matrix $G\geq 0$.

We also denote $x_t(\theta) = x(t+\theta)$ $(\theta \in [-h, 0])$. For measurable function $w : [t_0, t] \to R$ we denote by $|w_{[t_0,t]}|_\infty$ the essential supremum of $|w(s)|$ for $s\in[t_0,t]$.

Given $\bar{q} = [\bar{q}_1, ..., \bar{q}_k]^T$, $0 < \bar{q}_i$, $i = 1, ..., m$, for any $z = [z_1, ..., z_k]^T$ we denote by $sat(z, \bar{q})$ the vector with coordinates $\mathrm{sign}(z_i)\min(|z_i|,\ \bar{q}_i)$.

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2. PROBLEM FORMULATION

We consider the linear system

$$\dot{x}(t) = Ax(t) + Bu(t - \tau(t)), \tag{1}$$

where $x(t) \in R^n$ is the state, $u(t) \in R^m$ is the control input and $\tau(t)$ is an unknown piecewise-continuous delay that satisfies $0 \le \tau(t) \le h$. For example sampled-data control law may be represented as control input with time-varying delay, where h is the maximum sampling period [Fridman et al., 2004].

A saturated quantizer is a piecewise constant function $q = [q_1, ..., q_k]^T$ with $q_i : R \to \mathcal{Q}_i,\ i = 1, ..., k$, where $\mathcal{Q}_i$ is a finite subset in R. We will consider $k = m$ in the case of quantized control input or $k = n$ in the case of quantized state measurements. Similar to Brocket and Liberzon [2000], we assume that there exist real numbers $\bar{q}_i > \Delta > 0$ such that the following two conditions hold:

$$\begin{array}{l} |z| \le \bar{q}_i \Rightarrow |q_i(z) - z| \le \Delta,\ i = 1, ..., k, \\ |z| > \bar{q}_i \Rightarrow |q_i(z)| > \bar{q}_i - \Delta. \end{array} \tag{2}$$

We will design either a quantized control law

$$u(t) = q(sat(Kx(t), \bar{q})),\ \bar{q} = [\bar{q}_1, ..., \bar{q}_m], \tag{3}$$

or a control law with quantized state

$$u(t) = Kq(sat(x(t), \bar{q})),\ \bar{q} = [\bar{q}_1, ..., \bar{q}_n]. \tag{4}$$

The problem of interest is to design a controller of the form (3) or (4) to achieve the following property: there exist a region $\mathcal{R}_0 \subset R^n$ of initial conditions $x(t_0)$ such that all the states starting from this region exponentially converge to another (attractive) region $\mathcal{R}_a \subset R^n$. We note that in the unsaturated case ($\bar{q}_i = \infty, i = 1, ..., k$) the region $\mathcal{R}_0$ coincides with R^n.

We represent the closed-loop systems (1)-(3) and (1)-(4) in the following forms

$$\begin{array}{l} \dot{x}(t) = Ax(t) + Bsat(Kx(t - \tau(t)), \bar{q})) + Bw(t), \\ w(t) = q(sat(Kx(t - \tau(t)), \bar{q})) - sat(Kx(t - \tau(t)), \bar{q}) \end{array} \tag{5}$$

and

$$\begin{array}{l} \dot{x}(t) = Ax(t) + BKsat(x(t - \tau(t)), \bar{q})) + BKw(t), \\ w(t) = q(sat(x(t - \tau(t)), \bar{q})) - sat(x(t - \tau(t)), \bar{q}) \end{array} \tag{6}$$

respectively. In both cases $|w_{[t_0,t]}|_\infty \le \sqrt{k}\Delta$. Since

$$u(t - \tau(t)) = 0,\ t - \tau(t) < t_0,$$

the initial condition for the closed-loop systems is given by

$$x(t_0) = x_0,\ x(s) = 0,\ s < t_0. \tag{7}$$

Following Brocket and Liberzon [2000], we characterize the desired property by an ISS property [Sontag and Wang, 1995] and derive LMI conditions by using Lyapunov-Krasovskii approach started in Fridman et al. [2007].

Moreover, we will derive an ellipsoidal lower bound $\mathcal{X}_0 \subset \mathcal{R}_0$ on this region of initial conditions (and we are interested to "enlarge" this ellipsoid). We will also derive an ellipsoidal (upper) bound $\mathcal{X}_\infty$ on the attractive region $\mathcal{R}_a$ (trying to "minimize" the latter ellipsoid). We note that, in the unsaturated case, we show that the latter ellipsoid is attractive from R^n. Given time $T > t_0$, we will find also a *reachable ellipsoid* $\mathcal{X}_T$, in which all solutions starting from $\mathcal{X}_0$ will enter in time $t = T$ and will not leave it.

3. BOUNDS ON THE SOLUTIONS OF SYSTEMS WITH TIME-VARYING DELAYS

3.1 Main result

We first consider an auxiliary linear system without saturation

$$\dot{x}(t) = Ax(t) + A_1 x(t - \tau(t)) + B_1 w(t), \tag{8}$$

with initial condition given by (7), where $x(t) \in R^n$, $w(t) \in R^k$ and $0 \le \tau(t) \le h$.

We will derive the conditions that guarantee the following: the solutions of (8) satisfy the following bound

$$|x(t)|^2 \le ce^{-a(t-t_0)}|x_0|^2 + \gamma^2 |w_{[t_0,t]}|^2_\infty \tag{9}$$

for some constants $c \ge 1,\ a > 0$ and γ (thus the resulting closed-loop system is ISS). More precisely, given $a > 0$, we will derive LMI conditions that guarantee the bound (9) for the solution of (8) and we will find c and γ.

Consider a Lyapunov-Krasovskii functional (applied in Fridman and Orlov [2008] to exponential stability analysis):

$$V(x_t,\dot{x}_t) = x^T(t)Px(t) + \int_{t-h}^{t} e^{a(s-t)} x^T(s)Sx(s)ds + h\int_{-h}^{0}\int_{t+\theta}^{t} e^{a(s-t)}\dot{x}^T(s)R\dot{x}(s)dsd\theta \tag{10}$$

where $P > 0$ and $R, S \ge 0$. Similar to Fridman et al. [2007] we obtain the following result.

Proposition 1. If there exist $a > 0, b > 0$ and $n \times n$-matrices $P > 0$, $S > 0$ and $R > 0$ such that the Lyapunov-Krasovskii functional (10) satisfies the condition

$$W \triangleq \dot{V} + aV - b|w|^2 < 0, \tag{11}$$

then the solution of (8), (7) satisfies the following inequality

$$x^T(t)Px(t) < e^{-a(t-t_0)}x_0^T P x_0 + \frac{b}{a}|w_{[t_0,t]}|^2_\infty \tag{12}$$

for $t \ge t_0$ and $|x_0|^2 + |w_{[t_0,t]}|^2_\infty > 0$, i.e. the ellipsoid

$$x^T(t)Px(t) < \frac{b}{a}|w_{[t_0,t]}|^2_\infty$$

is (exponentially) attractive for all $x_0 \in R^n$.

Proof. By applying the comparison principle [Lakshmikantham and Leela, 1969], we have

$$x^T(t)Px(t) \le V(x_t) < e^{-a(t-t_0)}V(x_{t_0}) + \int_{t_0}^{t} e^{-a(t-s)}b|w(s)|^2 ds,$$

that implies (12). □

Since

$$V(x_t, \dot{x}_t) \ge x^T(t)Px(t) \ge \underline{\lambda}(P)|x(t)|^2,$$

the inequality (12) yields the following value of γ^2 in (9):

$$\gamma^2 = \frac{b}{a\underline{\lambda}(P)}. \tag{13}$$

We will derive now LMI conditions that guarantee that $W < 0$. Differentiating V, and applying standard arguments (see e.g. Fridman and Orlov [2008] and the references therein), we obtain that

$$W \le \eta^T(t)\Phi\eta(t) \le 0,\\ \eta(t) = col\{x(t), \dot{x}(t), x(t-h), x(t-\tau(t)), w(t)\}, \tag{14}$$

if the LMI

$$\Phi = \begin{bmatrix} \Phi_{11} & \Phi_{12} & 0 & P_2^T A_1 + Re^{-ah} & P_2^T B_1 \\ * & \Phi_{22} & 0 & P_3^T A_1 & P_3^T B_1 \\ * & * & -(S+R)e^{-ah} & Re^{-ah} & 0 \\ * & * & * & -2Re^{-ah} & 0 \\ * & * & * & * & -bI \end{bmatrix} < 0 \tag{15}$$

is feasible, where

$$\Phi_{11} = A^T P_2 + P_2^T A + aP + S - Re^{-ah},\\ \Phi_{12} = P - P_2^T + A^T P_3,\ \Phi_{22} = -P_3 - P_3^T + h^2 R. \tag{16}$$

Thus, the following result is obtained.

Lemma 2. Given $a > 0$, let there exist $n \times n$ matrices $P > 0$, P_2, P_3, $R \ge 0$, $S \ge 0$ and a scalar $b > 0$ such that the LMI (15) with notations given in (16) holds. Then the solution of (8) satisfies (12) for all piecewise-continuous delays $0 \le \tau \le h$. The ellipsoid $\mathcal{X}_\infty$ defined by

$$x^T P x \le \frac{b}{a} k\Delta^2 \tag{17}$$

is attractive from R^n for all $|w(t)|^2 \le k\Delta^2$.

Remark 1. Since LMI (15) is affine in the system matrices the criterion of Lemma 2 can be applied to the case where these matrices are uncertain. In this case we denote

$$\Omega = [\, A\ \ A_1\ \ B_1 \,]$$

and assume that $\Omega \in \mathcal{C}o\{\Omega_j,\ j = 1, ...N\}$, namely,

$$\Omega = \sum_{j=1}^{N} f_j \Omega_j \quad \text{for some} \quad 0 \le f_j \le 1,\ \sum_{j=1}^{N} f_j = 1 \tag{18}$$

where the N vertices of the polytope are described by

$$\Omega_j = \left[\, A^{(j)}\ \ A_1^{(j)}\ \ B_1^{(j)} \,\right].$$

In the case of time-varying uncertainty with $f_j = f_j(t)$, one has to solve the LMI (15) simultaneously for all the N vertices, applying the same matrices P, P_2, P_3, S and R for all vertices. In the case of time-invariant uncertainty, one has to solve the LMI (15) simultaneously for all the N vertices, applying the same matrices P_2, P_3 and different matrices $P^{(j)}, R^{(j)}, S^{(j)}$.

4. QUANTIZED CONTROL INPUT

4.1 Unconstrained State-Feedback

We first consider unsaturated closed-loop system (5)

$$\dot{x}(t) = Ax(t) + BKx(t-\tau(t)) + Bw(t), \tag{19}$$

We apply conditions of Lemma 2, where $A_1 = BK$ and $B_1 = B$. To find the unknown gain K we choose $P_3 = \epsilon P_2$, where ϵ is a tuning scalar parameter (which may be restrictive). Then P_2 is non-singular due to the fact that the only matrix which can be negative definite in Φ_{22} of (15) is $-\epsilon(P_2 + P_2^T)$. Moreover, $\epsilon > 0$. Defining:

$$Q = P_2^{-1},\ \bar{P} = Q^T P Q,\ \bar{R} = Q^T R Q,\\ \bar{S} = Q^T S Q, \quad Y = KQ, \tag{20}$$

we multiply (15) by $diag\{P_2^{-1}, P_2^{-1}, P_2^{-1}, P_2^{-1}, I\}$ and its transpose, from the right and the left, respectively. We obtain:

Theorem 3. Given $a > 0$ and $\epsilon > 0$ let there exist $n \times n$ matrices $\bar{P} > 0$, Q, $\bar{R} \ge 0$, $\bar{S} \ge 0$, a $m \times n$-matrix Y and a scalar $b > 0$ such that the following LMI holds:

$$\begin{bmatrix} \Psi_{11} & \Psi_{12} & 0 & BY + \bar{R}e^{-ah} & B \\ * & \Psi_{22} & 0 & BY & \epsilon B \\ * & * & -(\bar{S}+\bar{R})e^{-ah} & \bar{R}e^{-ah} & 0 \\ * & * & * & -2\bar{R}e^{-ah} & 0 \\ * & * & * & * & -bI \end{bmatrix} < 0 \tag{21}$$

where

$$\Psi_{11} = Q^T A^T + AQ + a\bar{P} + \bar{S} - \bar{R}e^{-ah},\\ \Psi_{12} = \bar{P} - Q + \epsilon Q^T A^T,\ \Psi_{22} = -\epsilon Q - \epsilon Q^T + h^2 \bar{R}. \tag{22}$$

Then, for all piecewise-continuous delays $0 \le \tau \le h$, the solution of (19) with $K = YQ^{-1}$ satisfies (12), where $P = Q^{-T}\bar{P}Q^{-1}$, and the ellipsoid (17) with $k = m$ is attractive from R^n for all $|w(t)|^2 \le m\Delta^2$.

4.2 Saturated State-Feedback

Consider now the saturated closed-loop system (5)

$$\dot{x}(t) = Ax(t) + Bsat(Kx(t-\tau(t)),\ \bar{q}) + Bw(t). \tag{23}$$

We seek conditions for the existence of a gain matrix K which leads to local ISS, i.e. the bound (9) holds for x_0 from some domain including origin. Having met these conditions, a simple procedure for finding the gain K should be presented. Moreover, we derive an estimate $\mathcal{X}_0$ on this domain of initial conditions.

We solve the problem by using *a linear system representation with polytopic type uncertainty* introduced in Cao et al. [2002].

Denoting the i-th row of K by k_i, we define the polyhedron

$$\mathcal{L}(K, \bar{q}) = \{x \in R^n\ : |k_i x| \le \bar{q}_i,\ i = 1, ..., m\}.$$

If the control and the disturbance are such that $x \in \mathcal{L}(K, \bar{q})$ then the system (23) admits the linear representation. Following Cao et al. [2002], we denote the set of all diagonal matrices in $R^{m\times m}$ with diagonal elements that are either 1 or 0 by Υ, then there are 2^m elements D_i in Υ, and for every $i = 1, ..., 2^m$ $D_i^- \triangleq I_m - D_i$ is also an element in Υ.

Lemma 4. Cao et al. [2002] Given K and H in $R^{m\times n}$. Then

$$sat(Kx(t),\ \bar{q}) \in \mathcal{C}o\{D_i Kx + D_i^- Hx,\ i = 1, ..., 2^m\}$$

for all $x \in \mathcal{L}(H, \bar{q})$.

The following is obtained from Lemma 4.

Lemma 5. Let $\mathcal{X}_\beta$ be the ellipsoid $x^T P x \le \beta^{-1}$ for a given $\beta > 0$ and a $n \times n$ matrix $P > 0$. Assume that there exists H in $R^{m\times n}$ such that $\mathcal{X}_\beta \subset \mathcal{L}(H, \bar{q})$. Then, for $x(t) \in \mathcal{X}_\beta$, the system (23) admits the following representation

$$\dot{x}(t) = Ax(t) + \sum_{j=1}^{2^m} \lambda_j(t) A_j x(t-\tau(t)) + Bw(t) \tag{24}$$

where

$$A_j = B(D_j K + D_j^- H)\ j = 1, ..., 2^m,\\ \sum_{j=1}^{2^m} \lambda_j(t) = 1,\ 0 \le \lambda_j(t),\ \forall\, t > 0. \tag{25}$$

We denote

$$\Omega = \sum_{j=1}^{2^m} \lambda_j \Omega_j \quad \text{for all} \quad 0 \le \lambda_j \le 1, \ \sum_{j=1}^{2^m} \lambda_j = 1 \qquad (26)$$

where the vertices of the polytope are described by $\Omega_j = [\, A_j \,], \quad j = 1, ..., 2^m$. Since LMI (15) is affine in A_1 (that will be substituted by $\sum_{j=1}^{2^m} \lambda_j(t) A_j$), the problem becomes one of finding $\mathcal{X}_\beta$ and a corresponding H such that $|h_i x| \le \bar{q}_i, \ i = 1, ... 2^m$ for all $x \in \mathcal{X}_\beta$ and that the state of the system

$$\dot{x}(t) = Ax(t) + A_j x(t - \tau(t)) + Bw(t) \qquad (27)$$

remains in $\mathcal{X}_\beta$.

By modifying the derivations of Theorem 3, where BK should be substituted by $B(D_j K + D_j^- H)$, and denoting $G = HQ$, $\bar{b} = b^{-1}$, we obtain:

Theorem 6. Consider the linear system (1) with the quantized constrained delayed control law (3). Given $a > 0$ and $\epsilon \in R$, let there exist $n \times n$ matrices $\bar{P} > 0$, Q, $\bar{R} \ge 0$, $\bar{S} \ge 0$, $m \times n$-matrices Y, G and scalars $\bar{b} > 0, \beta > 0$ such that the following LMIs hold:

$$\begin{bmatrix} \Psi_{11} & \Psi_{12} & 0 & BZ_j + \bar{R}e^{-ah} & B\bar{b} \\ * & \Psi_{22} & 0 & BZ_j & \epsilon B\bar{b} \\ * & * & -(\bar{S} + \bar{R})e^{-ah} & \bar{R}e^{-ah} & 0 \\ * & * & * & -2\bar{R}e^{-ah} & 0 \\ * & * & * & * & -\bar{b}I \end{bmatrix} < 0, \qquad (28)$$

$j = 1, ..., 2^m$

$$a\bar{b} - \beta m \Delta^2 > 0, \qquad (29)$$

and

$$\begin{bmatrix} \beta & g_i \\ * & \bar{q}_i^2 \bar{P} \end{bmatrix} \ge 0, \ i = 1, ..., m, \qquad (30)$$

where $\Psi_{11}, \Psi_{12}, \Psi_{22}$ are given by (22) and $Z_j = D_j Y + D_j^- G$, for $j = 1, ..., 2^m$. Then, for all piecewise continuous delays $\tau(t) \in [0, h]$, and for all initial conditions from the ellipsoid $\mathcal{X}_0$ given by

$$x_0^T P x_0 \le \beta^{-1} - \frac{m\Delta^2}{a\bar{b}} \triangleq \delta, \qquad (31)$$

the solutions of the closed-loop system (5) satisfy the inequality (12), where $K = YQ^{-1}$ and $P = Q^{-T}\bar{P}Q^{-1}$. Moreover, for $T > t_0$, the solutions of (5) starting from $\mathcal{X}_0$ enter the reachable ellipsoid $x(t) \in \mathcal{X}_T, t \ge T$ given by

$$x^T P x \le \delta e^{-a(T - t_0)} + \frac{m\Delta^2}{\bar{b}a}, \qquad (32)$$

and the ellipsoid (17) with $b = \bar{b}^{-1}, k = m$ is attractive from $\mathcal{X}_0$.

Proof. For V given by (10), conditions are sought to ensure that $W = \dot{V} + aV - b\Delta^2 < 0$ for any $x(t) \in \mathcal{X}_\beta$.

The inequalities (30) guarantee that the ellipsoid $\mathcal{X}_\beta$ is contained in the polyhedron $\mathcal{L}(H, \bar{q})$, where $g_i \triangleq h_i Q, \ i = 1, ..., m$ and $Q = P_2^{-1}$. This result follows from the fact that when $x \in \mathcal{X}_\beta$ the following inequalities

$$2\bar{q}_i \ge \bar{q}_i(1 + \beta x^T P x) \ge 2|h_i x|, \ i = 1, ..., m$$

imply that $|h_i x| \le \bar{q}_i$. The latter inequality, which can be written as

$$[1 \pm x^T] \begin{bmatrix} \bar{q}_i & h_i \\ * & \beta \bar{q}_i P \end{bmatrix} \begin{bmatrix} 1 \\ \pm x \end{bmatrix} \ge 0$$

is satisfied by (30)., where $g_i = h_i Q = h_i P_2^{-1}$ and $\bar{P} = P_2^{-T} P P_2^{-1}$ and the polytopic system representation of (24)is thus valid.

Moreover, (28) guarantees that $W < 0$ along the linear systems (27) and, thus, along (24) provided $x(t) \in \mathcal{X}_\beta$. From $W < 0$ it follows that (12) holds and therefore for the initial conditions of the form (39) the following inequalities hold:

$$x^T(t) P x(t) \le x_0^T P x_0 + \frac{m\Delta^2}{a\bar{b}} \le \beta^{-1}. \qquad (33)$$

We note that $\delta > 0$ due to LMI (29). Then for all initial values $x(0)$ from the ellipsoid (31), the trajectories of $x(t)$ remain within $\mathcal{X}_\beta$, and the polytopic system representation (24) is valid. Therefore, solutions of (23) starting from $x(0) \in \mathcal{X}_0$ satisfy the linear equation (24) and thus satisfy the bound (12). □

5. CONTROL UNDER QUANTIZED STATE

5.1 Unconstrained State-Feedback

As in the previous section, we first consider unsaturated closed-loop system (6)

$$\dot{x}(t) = Ax(t) + BKx(t - \tau(t)) + BKw(t), \qquad (34)$$

We apply conditions of Lemma 2, where $A_1 = BK$ and $B_1 = BK$. To find the unknown gain K we choose now $P_2 = \epsilon_2 I$ and $P_3 = \epsilon_3 I$, where ϵ_2 and ϵ_3 are tuning scalar parameters (which may be more restrictive than in the previous section). We obtain:

Theorem 7. Given $a > 0$, $\epsilon_2 \in R$ and $\epsilon_3 > 0$ let there exist $n \times n$ matrices $P > 0$, $R \ge 0$, $S \ge 0$, an $m \times n$-matrix K and a scalar $b > 0$ such that the following LMI holds:

$$\begin{bmatrix} \Xi_{11} & \Xi_{12} & 0 & \epsilon_2 BK + Re^{-ah} & \epsilon_2 BK \\ * & \Xi_{22} & 0 & \epsilon_3 BK & \epsilon_3 BK \\ * & * & -(S + R)e^{-ah} & Re^{-ah} & 0 \\ * & * & * & -2Re^{-ah} & 0 \\ * & * & * & * & -bI \end{bmatrix} < 0 \qquad (35)$$

where

$$\begin{aligned} \Xi_{11} &= \epsilon_2(A^T + A) + aP + S - Re^{-ah}, \\ \Xi_{12} &= P - \epsilon_2 I + \epsilon_3 A^T, \ \Xi_{22} = -2\epsilon_3 I + h^2 R. \end{aligned} \qquad (36)$$

Then the solution of (34) satisfies (12) for all piecewise-continuous delays $0 \le \tau \le h$. Moreover, the ellipsoid (17) with $k = n$ is attractive from R^n for all $|w(t)|^2 \le n\Delta^2$.

5.2 Saturated State

Consider now the saturated closed-loop system (5)

$$\dot{x}(t) = Ax(t) + BKsat(x(t - \tau(t)), \ \bar{q}) + BKw(t). \qquad (37)$$

As in the previous section, we seek conditions for the existence of a gain matrix K which leads to local ISS, i.e. the bound (9) holds for x_0 from some domain including origin. Having met these conditions, a simple procedure for finding the gain K should be presented. Moreover, we derive an estimate $\mathcal{X}_0$ on this domain of initial conditions.

For $x \in \mathcal{X}_\beta$, we want to guarantee that the following inequality holds

$$\bar{q}_i^2 \ge \bar{q}_i^2 \beta x^T P x \ge x_i^2, \ i = 1, ..., n.$$

The latter inequality can be written as $x^T(\bar{q}_i^2\beta P - E_i)x \geq 0$, where $E_i \in R^{n\times n}$ is a matrix with the only non-zero term (i,i), which is equal to 1. Hence, the following LMIs

$$\bar{q}_i^2\beta P - E_i \geq 0,\ i = 1, ..., n \tag{38}$$

guarantee that $x_i^2 \leq \bar{q}_i^2$ if $x \in \mathcal{X}_\beta$.

Denoting $\bar{\beta} = \beta^{-1}$, and

$$\delta \triangleq \beta^{-1} - \frac{b}{a}n\Delta^2 > 0 \tag{39}$$

we derive from (38) and (39) the following inequalities:

$$\begin{aligned} &\bar{q}_i^2 P - E_i\bar{\beta} \geq 0,\ i = 1, ..., n,\\ &\bar{\beta} - \frac{b}{a}n\Delta^2 > 0. \end{aligned} \tag{40}$$

We obtain

Theorem 8. Consider the linear system (1) with the quantized constrained delayed control law (3). Given $a > 0$, $\Delta > 0$ and $\epsilon_2, \epsilon_3 \in R$, let there exist $n \times n$ matrices $P > 0$, $R \geq 0$, $S \geq 0$, an $m \times n$-matrix K, and scalars $b > 0$, $\bar{\beta} > 0$ such that the LMIs (40) and (35) with notations given in (36) are feasible.
Then for all piecewise continuous delays $\tau(t) \in [0, h]$ and for all initial conditions x_0 from the ellipsoid $\mathcal{X}_0$ given by

$$x_0^T P x_0 \leq \bar{\beta} - \frac{b}{a}n\Delta^2$$

solutions of the closed-loop system (6) satisfy the inequality (12). Moreover, for $T > t_0$ the solutions of (5) starting from $\mathcal{X}_0$ enter the reachable ellipsoid $x(t) \in \mathcal{X}_T, t \geq T$ given by

$$x^T P x \leq \delta e^{-a(T-t_0)} + \frac{n\Delta^2 b}{a} \tag{41}$$

and the ellipsoid (17) with $k = n$ is attractive from $\mathcal{X}_0$.

5.3 Example [Bullo and Liberzon, 2006]

We consider (1) with

$$A = \begin{bmatrix} 0 & 1 \\ 0.5 & 0.5 \end{bmatrix},\ B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

By applying Theorem 3, we find that the system is input-to-state stabilizable for the maximum value of $h = 0.9999$.

a) We consider first the case of *quantized state*. In Bullo and Liberzon [2006] the controller under quantized and saturated state was designed with $\Delta = 1$ and $|x_i| \leq 5$ for system without delay and the following attractive ball was found $|x| \leq 2\sqrt{5} \approx 4.47$. By applying Theorem 4 with $h = 0$ and $\epsilon_2 = 2.25$, $\epsilon_3 = 0.004, a = 0.98$ we find a smaller attractive ball $|x| \leq 2.5$, where the resulting $K = [-1.2821\ \ -1.7791]$.

In order to enlarge the ellipse of initial conditions, we denote by r_m the semi-minor axis of $\mathcal{X}_0$, that is

$$r_m = \frac{\bar{\beta} - bn\Delta^2/a}{\bar{\sigma}(P)}$$

Imposing the additional LMI constraint $P < \alpha I$ (so $\bar{\sigma}(P) < \alpha$), we obtain the inequality

$$-r_m\alpha < -\bar{\beta} + bn\Delta^2/a$$

Finding the minimum value of $-r_m$ satisfying the last inequality and the previous LMIs is a generalized eigenvalue minimization problem (see Boyd et al. [1994]). In order to obtain a reasonable attractive set by limiting the size of $\mathcal{X}_\infty$, we add also the LMI constraint $bn\Delta^2 < a\,r_{\max}\alpha$ with $r_{\max} = 20$. We find for $h = 0$ and $\epsilon_2 = 2.26$, $\epsilon_3 = 0.69$, $a = 0.74$ and $K = [-1.0343\ \ -1.5345]$. We depicted in Fig. 1 the resulting ellipses of initial conditions $\mathcal{X}_0$ (the outer ellipse), the attractive ellipse $\mathcal{X}_\infty$ (the inner ellipse), the ellipse reachable from $\mathcal{X}_0$ in $T = 2$ (the dashed one) and some solutions for $t \in [0, 2]$ (which are simulated in the case of an uniform quantizer). We see that in fact solutions reach essentially smaller region than the predicted by Theorem 8, that illustrates the conservativeness of the method.

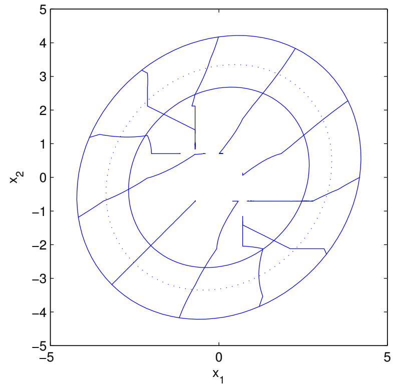

Fig. 1. Ellipsoids $\mathcal{X}_0$, $\mathcal{X}_\infty$ (solid) and $\mathcal{X}_T$ (dotted) in the case of quantized state and $h = 0$.

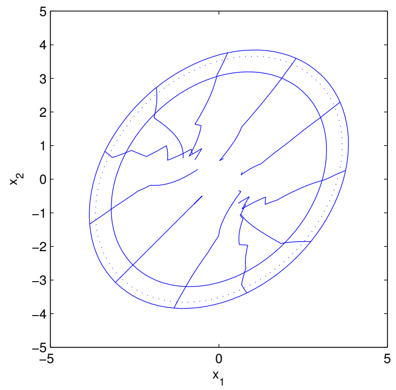

Fig. 2. Ellipsoids $\mathcal{X}_0$, $\mathcal{X}_\infty$ (solid) and $\mathcal{X}_T$ (dotted) in the case of quantized state and $h = 0.2$.

For $h > 0$, we find that conditions of Theorem 8 are feasible for the following maximum value of $h = 0.3923$, where $\epsilon_2 = 0.1033$, $\epsilon_3 = 0.1455$, $a = 0.5865$, $K = [-0.5540\ \ -1.0539]$. Hence, the delayed state-feedback guarantees the attractiveness of the ball from some neighborhood of the origin for all $0 \leq \tau(t) \leq 0.3923$. For $h = 0.2$ the resulting initial, attractive and reachable in $T = 2$ ellipses are depicted on Fig.2. The solutions are simulated in the case of an uniform quantizer and an uniform sampling $t_1, ..., t_k, ...$ with $t_{k+1} - t_k = h$.

b) Consider next the case of *quantized saturated feedback* with $\Delta = 1$ and $|Kx| \leq 5$. We find that conditions of Theorem 6 are feasible for the following maximum value of $h = 0.4745$.

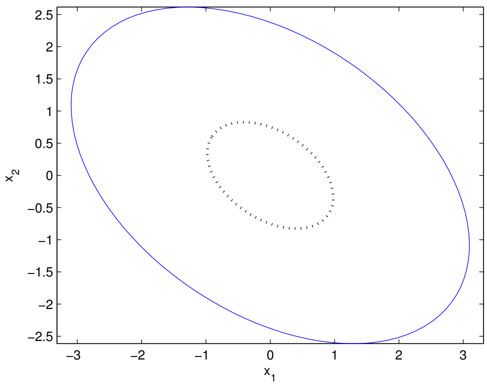

Fig. 3. Ellipsoids $\mathcal{X}_\infty$ (dotted) and $\mathcal{X}_0$ (solid) in the case of input quantization and $h = 0$

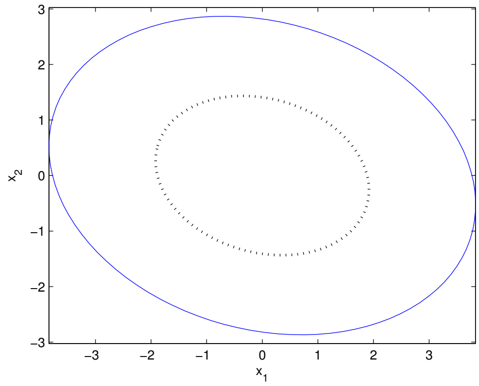

Fig. 4. Ellipsoids $\mathcal{X}_\infty$ (dotted) and $\mathcal{X}_0$ (solid) in the case of input quantization and $h = 0.2$

For $h = 0$, by applying Theorem 6 and taking $a = 0.9$ and $\epsilon = 0.6$, we obtain a gain $K = [-1.4331 \ \ -1.6752]$. The attractive ellipsoid $\mathcal{X}_\infty$ is given by $x^T P x < 1$, where

$$P = \begin{bmatrix} 1.2613 & 0.6200 \\ 0.6200 & 1.7668 \end{bmatrix}$$

The ellipsoid $\mathcal{X}_\infty$ is 10 times bigger (defined by $x^T P x < 10$), these results are illustrated in Fig. 3. For $h = 0.2$, with $a = \epsilon = 0.7$, we obtain the gain $K = [-0.7809 \ \ -1.3662]$. The ellipsoids $\mathcal{X}_\infty$ and $\mathcal{X}_0$ are defined respectively as $x^T P x < 1$ and $x^T P x < 4$ with

$$P = \begin{bmatrix} 0.2811 & 0.0689 \\ 0.0689 & 0.5032 \end{bmatrix}.$$

These results are illustrated Fig. 4.

6. CONCLUSIONS

In this paper, we have proposed a new methodology for design of quantized delayed controller with saturation. LMI solutions are derived via the comparison principle and Lyapunov-Krasovskii method. A numerical example illustrates the efficiency of the new method.

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