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Stability of Hybrid Impulsive Systems With Time Delays and Stochastic Effects ⋆

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Abstract: In this paper, a class of hybrid impulsive systems with time delays and stochastic effects are considered. We obtain some criteria on the global exponential stability in mean square for the impulsive stochastic delayed systems. To do this, differential inequalities and $\mathcal{L}$-operator inequalities are developed. An example is given to illustrate the effectiveness of our results.

1. INTRODUCTION

In many evolutionary systems, there exist two common phenomena, that is, time delays and stochastic effects. For example, delay effects are inevitable in the implementation of electronic networks due to the finite switching speed of the hardware (see Baldi et al. [1994] for example). Moreover, stochastic effects can also be found widely in many dynamical systems in various fields such as medicine and biology, economics, electrical engineering, telecommunications. They can be observed from the phenomena including stochastic failures and repairs of the components, changes in the interconnections of subsystems, and sudden environment changes. These complexities, sometimes, can be described by time delay systems and/or stochastic systems. With such background, the theoretical studies and the applications of delayed stochastic systems have attracted much attention (see, e.g., Richard [2003], Mohammed [2006]).

On the other hand, some state variables of systems are often subject to instantaneous perturbations and experience critical changes at certain instants, which may be caused by switching mechanism, abrupt frequency change or other sudden disturbance, and exhibit impulsive effects (see Lakshmikantham et al. [1989], Bainov et al. [1993]). From the mathematical view, impulsive systems belongs to a new category of dynamical systems, which is neither continuous-time nor discrete-time ones in the traditional sense. Instead, it displays a combination of the characteristics of both the continuous-time and discrete-time systems, and therefore, it is regarded as a class of hybrid dynamical systems (see Michel [1999]). Hybrid impulsive systems with time delays and stochastic effects are certainly with more accuracy in the modeling of the evolutionary process in practical systems, but they are harder to be studied than the traditional dynamics. In some cases, these complex factors cannot be neglected since they may directly affect the dynamical behaviors of the system by leading to oscillatory and instability. Thus, it may become necessary to investigate such hybrid impulsive systems with delays and stochastic factors for some practical problems.

In recent years, the stability of stochastic systems with delays has been an interesting problem, and some stability conditions have been reported for such systems (referring to Richard [2003], Mohammed [2006], Nair et al. [2004], Kolmanovskii [1986], Mao [1995]). Moreover, the stability and attractivity of impulsive differential equations have been deeply investigated in the works including Lakshmikantham et al. [1989], Bainov et al. [1993], Samoilenko et al. [1995], and Obolenskii et al. [1988], and a full discussion of this subject for impulsive delay differential equations has been further carried out (see, e.g., Yan et al. [1999], Liu et al. [2001], Yu [2001], Guan et al. [2002])). However, to the best of our knowledge, there are few results about the stability for hybrid impulsive systems with delays and stochastic effects (Hespanha [2005], Boukas [2006]). Therefore, techniques and methods to study the stability and dynamic behaviors of impulsive stochastic differential equations with delays should be developed and explored.

In this paper, we consider a class of hybrid impulsive systems with delays and stochastic terms. The paper is organized as follows. In Section 2, we present the problem formulation and related preliminary knowledge. Then in Section 3, by developing differential inequalities and L-operator inequalities, we obtain the stability in mean square of the origin for the hybrid systems. A simple example is shown to illustrate the feasibility and effectiveness of our results. Finally, concluding remarks are given in Section 4.

2. PRELIMINARIES

In this section, we will give our model description and preliminary knowledge for the following analysis.

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As usual, let $N = \{1, 2, 3, \ldots\}$, $R_+ = [0, \infty)$, and I be an identity matrix. R^n is the space of n-dimensional real column vectors and $R^{m\times n}$ is the set of $m\times n$ real matrices. If A is a vector or a matrix, A^T stands for the transpose of A, trace(A) is the sum of the diagonal elements of A. If A is a real symmetric matrix $\lambda_M(A)$ and $\lambda_m(A)$ denotes the largest eigenvalue and the smallest of A. Additionally, E represents the expectation of a stochastic process, and L is a Kolmogorov's backup differential operator generated by the corresponding stochastic systems. Also, $\delta(t)$ is the Dirac impulsive function.

Furthermore, $C[X, Y]$ denotes the space of continuous mappings from the topological space X to the topological space Y. Especially, $C[\,[-\tau, 0], R^n]$ denotes the family of all continuous R^n-valued functions ψ on $[-\tau, 0]$ with norm $||\psi|| = \sup_{-\tau\le\theta\le 0} |\psi(\theta)|$. Moreover,

$PC[\hat{I}, R] := \{\psi : \hat{I} \to R^n \mid \psi(t^+) = \psi(t)$ for $t \in \hat{I}$, $\psi(t^-)$ exists for $t \in \hat{I}$, $\psi(t^-) = \psi(t)$ for all but points $t_k \in \hat{I}\}$, where the domain $\hat{I} \subseteq R$, $\psi(t^+)$ and $\psi(t^-)$ denote the left-hand and right-hand limits of scalar function $\psi(t)$, respectively.

Consider a class of hybrid impulsive stochastic systems with time delays:

$$\begin{aligned} dx_i(t) = & f_i(t, x_i(t))dt + g_i(t, x(t), x(t-\mu))dt \\ & + \sigma_i(t, x(t), x(t-\nu))dw_i(t) \\ & + \sum_{k=1}^{\infty} I_{ik}(t, x(t), x(t-\varsigma))\delta(t-t_k)dt, \end{aligned} \tag{1}$$

where $t \ge 0$, $x_i \in R^{n_i}, i = 1, \ldots, m$, $\sum_{i=1}^m n_i = n$, $x = (x_1^T, \ldots, x_m^T)^T \in R^n$, x_i is the state of the ith subsystem, which is described by $\dot{x}_i = f_i(t, x_i)$, $g_i : R\times R^n\times R^n \to R^{n_i}$ is the deterministic interconnected function, and $\sigma_i : R \times R^n \times R^n \to R^{n_i\times m_i}$ and $I_i : R \times R^n \times R^n \to R^{n_i\times m_i}$ represent the strength of the stochastic interconnection and the impulsive one, respectively. The time delays μ, ν, ς may be unknown (constant or time-varying), but are bounded by a known constant, i.e., $0 \le \mu, \nu, \varsigma \le \tau$, and $w_i = (w_{i1}, \ldots, w_{im_i})^T$ with $w(t) = (w_1^T(t), \ldots, w_m^T(t))$ as an $\sum m_i-$dimensional Brownian motion defined on a complete probability space $(\Omega, \{\mathcal{F}_t\}_{t\ge 0}, P)$ with a natural filtration $\{\mathcal{F}_t\}_{t\ge 0}$ (i.e. $\{F_t\} = \sigma\{w(s), 0 \le s \le t\}$). Moreover, t_k are the impulsive moments satisfying

$$t_0 = 0 < t_1 < \ldots < t_k < t_{k+1} < \ldots, \lim_{k\to\infty} t_k = \infty.$$

If $I_i(t, \cdot) = 0$, the hybrid system (1) becomes a stochastic delayed system

$$\begin{aligned} dx_i(t) = & f_i(t, x_i(t))dt + g_i(t, x(t), x(t-\mu))dt \\ & + \sigma_i(t, x(t), x(t-\nu))dw_i(t). \end{aligned} \tag{2}$$

In this paper, we assume that $f_i(t, 0) = 0, g_i(t, 0) = 0, \sigma(t, 0) = 0$, and there exists a solution for any given initial values.

Let $C^{2,1}(J \times R^n; R_+)$ denote the family of all nonnegative functions $V(x, t)$, which are twice continuously differentiable in x and once in t, $t \in J \subset R, x = (x_1, x_2, \ldots, x_m) \in R^n$. For each $V(x, t) \in C^{2,1}(R^n \times R_+; R_+)$, define an operator LV_i along the trajectory of system (2) by

$$\begin{aligned} LV_i(t, x) = & \frac{\partial V_i(t, x)}{\partial t} + V_x(t, x)[f_i(t, x_i) + g_i(t, x, x(t-r)) \\ & + \frac{1}{2} trace[\sigma_i^T(t, x, x(t-s))V_{xx}\sigma_i(t, x, x(t-s))] \end{aligned}$$

where

$$V_x(x, t) = (\frac{\partial V(x, t)}{\partial x_1}, \cdots, \frac{\partial V(x, t)}{\partial x_n}),$$

$$V_t(x, t) = \frac{\partial V(x, t)}{\partial t},\ V_{xx}(x, t) = (\frac{\partial V^2(x, t)}{\partial x_i \partial x_j})_{n\times n}.$$

For a right-hand continuous function $v : R \to R$, define its upper Dini derivative as follows

$$D^+v(t) = \limsup_{s\to 0^+} \frac{v(t+s) - v(t)}{s}.$$

The following lemma is a modification of the continuous delay differential inequality (see Xu [1988]) and the proof is given in Appendix.

Lemma 1. Let $u_i(t) \in C[[\sigma, b), R_+]$ satisfy

$$\begin{cases} D^+u_i(t) \le \sum_{j=1}^n p_{ij}(t)u_j(t) + \sum_{j=1}^n q_{ij}(t)\widetilde{u_j}(t),\ t \in [\sigma, b), \\ u_i(\sigma + s) = \phi_i(s), \phi \in PC,\ s \in [-\tau, 0], \end{cases}$$

where $\tau \ge 0$, $\sigma < b \le +\infty$, $p_{ij}(t) \ge 0$ for $i \ne j$, $q_{ij}(t) \ge 0$, $\widetilde{u}_i(t) = \sup_{\theta\in(0,\tau]} u_i(t+\theta), i = 1, 2, \ldots, n$. Suppose that there exists an integrable function $r(t), t \in [\sigma - \tau, b)$ such that for $t \in [\sigma, b)$

$$\sum_{j=1}^n p_{ij}(t) + e^{\sup_{\theta\in[-\tau,0]}\{\int_{t+\theta}^t r(s)ds\}} \sum_{j=1}^n q_{ij}(t) < -r(t). \tag{3}$$

If the initial condition satisfies

$$u_i(t) \le \kappa e^{-\int_\sigma^t r(s)ds},\quad \kappa \ge 0,\ t \in [\sigma - \tau, \sigma], \tag{4}$$

then for $i = 1, \ldots, n$

$$u_i(t) \le \kappa e^{-\int_\sigma^t r(s)ds},\ t \in [\sigma, b). \tag{5}$$

3. STABILITY ANALYSIS

In this section, we will give our main results on stability conditions.

In order to obtain the exponential stability of (1), we first show the following L-operator inequalities.

Theorem 1. Let $V_i(t, x) \in C^{2,1}([t_{k-1}, t_k) \times R^n; R_+)$ satisfy

$$\begin{cases} LV_i(t, x) \le \sum_{j=1}^m [p_{ij}(t)V_j(t, x) + q_{ij}(t)\widetilde{V_j}(t, x)], t \ne t_k, \\ V_i(t_k, x_i + I_{ik}(t_k, \cdot)) \le \sum_{j=1}^m [b_{ij}^{(k)}V_j(t_k, x) + d_{ij}^{(k)}\widetilde{V}_j(t_k, x)], \end{cases}$$

where $\widetilde{V}_i = \sup_{\theta\in(0,\tau]} V_i(t, x(t+\theta)), i = 1, 2, \ldots, m, k \in N$. If there exist $\gamma \geq 1$ and $\alpha(t) \in PC[R, R^+]$ satisfying

$$\sum_{j=1}^{m} p_{ij}(t) + \gamma \sum_{j=1}^{m} q_{ij}(t) < -\alpha(t), \quad i = 1, \ldots, m,$$

then for $t \geq t_0$

$$EV_i(t,x) \leq (\prod_{t_0<t_k\leq t} \eta_k) e^{-\int_{t_0}^{t}\beta(s)ds} \sup_{\theta\in[-\tau,0]} \sum_{i=1}^{m} EV_i(\theta),$$

where $\beta(t) := \min\{\alpha(t), \frac{\ln\gamma}{\tau}, B_k = (b_{ij}^{(k)}), D_k = (d_{ij}^{(k)}),$ $\eta_k = \max\{1, \|B_k\|_1 + \|D_k\|_1 e^{\int_{t_k}^{t_k-\tau}\beta(s)ds}\}$.

Proof. Let $h > 0$, $x(t) = (x_1(t), x_2(t), \ldots, x_m(t))$ be the solution process of (1). Then $x(t)$ satisfies (2) when $[t, t+h] \subset [t_{k-1}, t_k)$. Employing Itô formula, we have

$$V_i(t, x(t+h)) = V_i(x(t, x(t))) + \int_{t}^{t+h} LV_i(s, x(s))ds \quad (6)$$

$$+ \int_{t}^{t+h} \frac{\partial V_i(s, x(s))}{\partial x} \sigma(s, x(s), x(s-\nu))dw(s). \quad (7)$$

Hence

$$EV_i(t, x(t+h)) = EV_i(t, x(t)) + \int_{t}^{t+h} ELV_i(s, x(s))ds,$$

which implies

$$D^+(EV_i(t, x(t))) = ELV_i(t, x(t)) \leq \sum_{j=1}^{n} (p_{ij}EV_j(t, x(t)) + a_{ij}E\widetilde{V_j}(t, x(t))).$$

On the other hand,

$$x_i(t_k) - x_i(t_k - h) = \int_{t_k-h}^{t_k} [f_i(s, x_i(s)) + g_i(s, x(s), x(s-\mu))ds + \int_{t_k-h}^{t_k} \sigma_i(s, x(s), x(s-\nu))dw_i(s) + \int_{t_k-h}^{t_k} I_{ik}(s, x(s), x(s-\varsigma))\delta(s-t_k)ds.$$

Let $h \to 0^+$, we have

$$x_i(t_k) = x_i(t_k^-) + I_{ik}(, x(t_k^-), x((t_k - \varsigma))).$$

Then,

$$EV_i(t_k, x(t_k)) = EV_i(t_k^-, x_i(t_k^-) + I_{ik}(t_k^-, x(t_k^-), x((t_k - \varsigma)))) \leq \|B_k\|V(t_k^-, x(t_k^-)) + \|D_k\|\widetilde{V}(t_k^-, x(t_k^-)). \quad (8)$$

Since $\beta(t) = \min\{\alpha(t), \frac{\ln\gamma}{\tau}\} \geq 0$, we have

$$\sup_{\theta\in[-\tau,0]}\{\int_{t+\theta}^{t} \beta(s)ds\} \leq \int_{t-\tau}^{t} \frac{\ln\gamma}{\tau}ds = \ln\gamma.$$

Thus,

$$\sum_{j=1}^{m} p_{ij}(t) + \sum_{j=1}^{m} q_{ij}(t) e^{\sup_{\theta\in[-\tau,0]}\{\int_{t+\theta}^{t}\beta(s)ds\}} \leq \sum_{j=1}^{m} p_{ij}(t) + \gamma\sum_{j=1}^{m} q_{ij}(t) < -\alpha(t) \leq -\beta(t), \quad t \geq t_0, i = 1, \ldots, m. \quad (9)$$

Let $\eta_0 = 1$, $u_i(t) = EV_i(t, x(t))$, and $u_0 = \sup_{\theta\in[-\tau,0]} \sum_{i=1}^{m} V_i(\theta)$. Next, we shall prove that for any $k \in N$

$$u_i(t) \leq \eta_0\eta_1 \ldots \eta_{k-1} u_0 e^{-\int_{t_0}^{t}\beta(s)ds}, \quad t_{k-1} \leq t < t_k.$$

Since $u_i(t) \leq u_0 e^{-\int_{t_0}^{t}\beta(s)ds}$ for $t_0 - \tau \leq t \leq t_0$, by (8), (9) and Lemma 1, we can get

$$u_i(t) \leq u_0 e^{-\int_{t_0}^{t}\beta(s)ds}, t_0 \leq t < t_1.$$

Suppose that for $k = 1, \ldots, l$

$$u_i(t) \leq \eta_0\eta_1 \ldots \eta_{k-1} u_0 e^{-\int_{t_0}^{t}\beta(s)ds}, t_{k-1} \leq t < t_k. \quad (10)$$

Then, from (8),

$$u_i(t_l) \leq \|B_k\|u_i(t_l^-) + \|D_k\|\widetilde{u}_i(t_l^-) \leq \|B_k\|\eta_0 \ldots \eta_{l-1} u_0 e^{-\int_{t_0}^{t_l}\beta(s)ds} + \|D_k\|\eta_0 \ldots \eta_{l-1} u_0 e^{\int_{t_l}^{t_l-\tau}\beta(s)ds} e^{-\int_{t_0}^{t_l}\beta(s)ds} \leq \eta_0 \ldots \eta_{l-1}\eta_l u_0 e^{-\int_{t_0}^{t_l}\beta(s)ds},$$

and so

$$u_i(t) \leq \eta_0 \ldots \eta_{l-1}\eta_l u_0 e^{-\int_{t_0}^{t}\beta(s)ds}, \quad t_l - \tau \leq t \leq t_l.$$

Using Lemma 1 again, we obtain

$$u_i(t) \leq \eta_0 \ldots \eta_{l-1}\eta_l u_0 e^{-\int_{t_0}^{t}\beta(s)ds}, \quad t_l \leq t < t_{l+1}.$$

By the induction, the conclusion holds. Q.E.D.

With the L-operator inequalities, we then discuss the stability of hybrid systems (1).

Theorem 2. Let $\sup_{k\in N}\{t_k - t_{k-1}\} < \infty$. Assume that for $t \in R, x_i, y_i \in R^{n_i}, x, y \in R^n, i = 1, \ldots, m,$

(H_1) there exist bounded functions $a_i(t)$ such that

$$x_i^T f_i(t, x_i) \leq a_i(t) x_i^T x_i;$$

(H_2) there exist bounded functions $b_{ij}(t), \bar{b}_{ij}(t)$ such that

$$g_i^T(t, x, y) g_i(t, x, y) \leq \sum_{j=1}^{m} b_{ij}(t) x_j^T x_j + \sum_{j=1}^{m} \bar{b}_{ij}(t) y_j^T y_j;$$

(H_3) there exist bounded functions $c_{ij}(t), \bar{c}_{ij}(t)$ such that

$$\text{trace } \sigma_i^T(t,x,y)\sigma_i(t,x,y) \leq \sum_{j=1}^m c_{ij}(t)x_j^T x_j + \sum_{j=1}^m \bar{c}_{ij}(t)y_j^T y_j;$$

(H_4) there exist bounded functions $d_{ij}(t), \bar{d}_{ij}(t)$ such that

$$I_i^T(t,x,y)I_i(t,x,y) \leq \sum_{j=1}^m d_{ij}(t)x_j^T x_j + \sum_{j=1}^m \bar{d}_{ij}(t)y_j^T y_j;$$

(H_5) there are constants $\epsilon_i, \delta_i > 0, \gamma > 1$ and function $\alpha \in PC[R, R_+], i = 1, 2, \ldots, m$ such that

$$a_i(t) + \frac{1}{\epsilon_i} + \sum_{j=1}^m [\epsilon_i b_{ij}(t) + c_{ij}(t)] + \gamma \sum_{j=1}^m [\epsilon_i \bar{b}_{ij}(t) + \bar{c}_{ij}(t)] < -\alpha(t); \tag{11}$$

(H_6) let

$$\sup_{k\in N}\{\ln\theta_k + \int_{t_{k-1}}^{t_k} \beta(s)ds\} < 0, \tag{12}$$

where $\beta(t) := \min\{\alpha(t), \frac{\ln\gamma}{\tau}\}$ and

$$\theta_k = \max\{1, \rho_k + \varrho_k e^{\int_{t_k}^{t_k-\tau}\beta(s)ds}\},$$
$$\rho_k = \max_{1\leq i\leq m}\{(1+\frac{1}{\delta_i}) + (1+\delta_i)\sum_{j=1}^m d_{ij}(t)\},$$
$$\varrho_k = \max_{1\leq i\leq m}\{(1+\delta_i)\sum_{j=1}^m \bar{d}_{ij}(t)\}.$$

Then the origin of the hybrid impulsive systems (1) is globally exponentially stable in mean square.

Proof. Let $x(t) = (x_1(t), \ldots, x_m(t))^T$ be a solution through (t_0, ϕ), where $x_i \in R^{n_i}$. Define $V_i(t) = V_i(t,x) = x_i^T(t)x_i(t), i = 1, 2, \ldots, m$. From (H_1)-(H_3), for $t \neq t_k, k \in N$, we calculate the derivative LV_i along the solution $x(t)$ of (2)

$$\begin{aligned}LV_i(t) &= 2x_i^T(t)f_i(t,x_i(t)) + 2x_i^T(t)g_i(t,x(t),x(t-\mu)) \\ &\quad +\text{trace } (\sigma_i^T(t,x,x(t-\nu))\sigma_i(t,x,x(t-\nu))) \\ &\leq 2a_i(t)x_i^T x_i + \frac{1}{\epsilon_i}x_i^T x_i \\ &\quad +\epsilon_i g_i^T(t,x(t),x(t-\mu))g_i(t,x(t),x(t-\mu)) \\ &\quad +\sum_{j=1}^m c_{ij}(t)x_j^T x_j + \sum_{j=1}^m \bar{c}_{ij}(t)x_j^T(t-\nu)x_j(t-\nu) \\ &\leq (a_i(t) + \frac{1}{\epsilon_i})V_i(t) + \sum_{j=1}^m[\epsilon_i b_{ij}(t) + c_{ij}(t)]V_j(t) \\ &\quad +\sum_{j=1}^m[\epsilon_i \bar{b}_{ij}(t) + \bar{c}_{ij}(t)]\widetilde{V}_j(t).\end{aligned}$$

For $t = t_k, k \in N$,

$$\begin{aligned}&V(t_k, x_i + I_{ik}(t_k,\cdot)) \\ &= (x_i(t_k^-) + I_{ik}(t_k^-, x(t_k^-), x(t_k - \varsigma)))^T \\ &\quad \times(x_i(t_k^-) + I_{ik}(t_k^-, x(t_k^-), x(t_k - \varsigma))) \\ &= x_i^T x_i + 2x_i^T I_{ik}(t_k^-, x(t_k^-), x(t_k - \varsigma)) \\ &\quad +I_{ik}^T(t_k^-,\cdot)I_{ik}(t_k^-,\cdot) \\ &\leq (1+\frac{1}{\delta_i})x_i^T x_i + (1+\delta_i)\sum_{j=1}^m[d_{ij}(t)x_j^T x_j \\ &\quad +\bar{d}_{ij}(t)x_j^T(t-\varsigma(t))x_j(t-\varsigma\varsigma(t))] \\ &\leq (1+\frac{1}{\delta_i})V_i + (1+\delta_i)\sum_{j=1}^m d_{ij}(t)V_j \\ &\quad +(1+\delta_i)\sum_{j=1}^m \bar{d}_{ij}(t)\widetilde{V}_j(t)).\end{aligned}$$

By Theorem 1, we have, for $t \geq t_0$

$$EV_i(t,x) \leq (\prod_{t_0<t_k\leq t}\theta_k)e^{-\int_{t_0}^t \beta(s)ds}V_0,$$

where $V_0 = \sup_{s\in[-\tau,0]}\sum_{j=1}^m EV_i(s),\ i = 1, 2, \ldots, m.$

From the strict inequality (12), there must be a $r > 0$ such that for $k \in N$

$$\ln\theta_k + \int_{t_{k-1}}^{t_k}\beta(s)ds \leq -r < 0.$$

Let $0 < T := \sup_{k\in N}\{t_k - t_{k-1}\} < \infty$. Then, for $k \in N$

$$\theta_k \leq e^{-\int_{t_{k-1}}^{t_k}\beta(s)ds - r} \leq e^{-\int_{t_{k-1}}^{t_k}[\beta(s)+r/T]ds}.$$

Since the functions $a_i, b_{ij}, c_{ij}, d_{ij}, \bar{b}_{ij}, \bar{c}_{ij}, \bar{d}_{ij}$ are bounded, there is a constant $b \geq |\beta(t)|$. For $t_{k-1} \leq t < t_k$, we have

$$\begin{aligned}EV_i(t) &\leq \theta_1\ldots\theta_{k-1}e^{\int_{t_0}^t\beta(s)ds}V_0 \\ &\leq e^{-\int_{t_0}^{t_{k-1}}[\beta(s)+r/T]ds}e^{\int_{t_0}^t\beta(s)ds}V_0 \\ &\leq c\, e^{-(r/T)(t-t_0)}V_0,\end{aligned}$$

in which $e^{\int_{t_{k-1}}^t[\beta(s)+r/T]ds} \leq c := e^{bT+r}$. Thus, the origin of (1) is globally exponentially stable in mean square. Q.E.D.

Then, we consider a linear hybrid impulsive system in the following form:

$$\begin{aligned}dx_i(t) &= A_i x_i(t)dt + \sum_{j=1}^m[B_{ij}x_j(t)dt + \bar{B}_{ij}x_j(t-\mu)]dt \\ &\quad +\sum_{j=1}^m C_{ij}x_j(t)dw_{ij}(t) + \sum_{j=1}^m \bar{C}_{ij}x_j(t-\nu)d\bar{w}_{ij}(t) \\ &\quad +\sum_{k=1}^\infty[D_{ik}x_i(t) + \bar{D}_{ik}x_i(t-\varsigma)]\delta(t-t_k),\end{aligned} \tag{13}$$

with matrices $A_i, B_{ij}, C_{ij}, D_{ij}, \bar{B}_{ij}, \bar{C}_{ij}, \bar{D}_{ij} \in R^{n_i\times n_j}, i = 1, \ldots, m.$

Applying the above results to system (13), we have the following theorem.

Theorem 3. Let $\sup_{k\in N}\{t_k - t_{k-1}\} < \infty$. Suppose that there are positive constants $\epsilon_i, \delta_i, \zeta_i, \lambda$ and positive definite matrices $P_i, i = 1, 2, \ldots, m$ such that

$$[\epsilon_i + \delta_i + \frac{\lambda_M(A_i^T P_i + P_i A_i)}{\lambda_m(P_i)}] + \frac{1}{\lambda_m(P_i)}\sum_{j=1}^{m}[\frac{m}{\epsilon_i}\lambda_M(B_{ij}^T P_i B_{ij}) + \lambda_M(C_{ij}^T P_i C_{ij})] + \frac{e^{\lambda\tau}}{\lambda_m(P_i)}\sum_{j=1}^{m}[\frac{m}{\delta_i}\lambda_M(\bar{B}_{ij}^T P_i \bar{B}_{ij}) + \lambda_M(\bar{C}_{ij}^T P_i \bar{C}_{ij})] < -\lambda.$$

If

$$\sup_{k\in N}\{\frac{\ln\vartheta_k}{t_k - t_{k-1}}\} < \lambda, \qquad (14)$$

where

$$\vartheta_k = \max\{1, \rho_k + \varrho_k e^{\lambda\tau}\},$$
$$\rho_k = \max_{1\le i\le m}\{(1+\zeta_i)\frac{\lambda_M((I+D_{ik})^T P_i (I+D_{ik}))}{\lambda_m(P_i)}\},$$
$$\varrho_k = \max_{1\le i\le m}\{\frac{\lambda_M(\bar{D}_{ik}^T P_i \bar{D}_{ik})}{\zeta_i\lambda_m(P_i)}\},$$

then the origin of the stochastic systems (13) is globally exponentially stable in mean square.

Proof. Define $V_i(t) = V_i(t, x) = x_i^T(t)P_i x_i(t)$. Firstly, we calculate the derivative LV_i along the solution $x(t)$ of (13), for $t \ne t_k, k \in N$

$$\begin{aligned} LV_i(t) &= x_i^T[A_i^T P_i + P_i A_i]x_i \\ &\quad +2\sum_{j=1}^{m} x_i^T P_i[B_{ij}x_j + \bar{B}_{ij}x_j(t-\mu)] \\ &\quad +\sum_{j=1}^{m}[x_j^T C_{ij}^T P_i C_{ij} x_j \\ &\quad +x_j^T(t-\nu)\bar{C}_{ij}^T P_i \bar{C}_{ij} x_j(t-\nu)] \\ &\le \lambda_M(A_i^T P_i + P_i A_i)x_i^T x_i + \sum_{j=1}^{m}[(\frac{\epsilon_i}{m} + \frac{\delta_i}{m})x_i^T P_i x_i \\ &\quad +\frac{m}{\epsilon_i}x_j^T B_{ij}^T P_i B_{ij} x_j + x_j^T C_{ij}^T P_i C_{ij} x_j] \\ &\quad +\sum_{j=1}^{m}[\frac{m}{\delta_i}x_j^T(t-\mu)\bar{B}_{ij}^T P_i \bar{B}_{ij} x_j(t-\mu) \\ &\quad +x_j^T(t-\nu)\bar{C}_{ij}^T P_i \bar{C}_{ij} x_j(t-\nu)] \\ &\le [\epsilon_i + \delta_i + \frac{\lambda_M(A_i^T P_i + P_i A_i)}{\lambda_m(P_i)}]V_i(t) \\ &\quad +\frac{1}{\lambda_m(P_i)}\sum_{j=1}^{m}[\frac{m}{\epsilon_i}\lambda_M(B_{ij}^T P_i B_{ij}) \\ &\quad +\lambda_M(C_{ij}^T P_i C_{ij})]V_j(t) \\ &\quad +\frac{1}{\lambda_m(P_i)}\sum_{j=1}^{m}[\frac{m}{\delta_i}\lambda_M(\bar{B}_{ij}^T P_i \bar{B}_{ij}) \\ &\quad +\lambda_M(\bar{C}_{ij}^T P_i \bar{C}_{ij})]\widetilde{V}_j(t) \end{aligned}$$

Also,

$$\begin{aligned} V_i(t_k) &= [(I+D_{ik})x_i(t) + \bar{D}_{ik}x_i(t-\varsigma)]^T P_i \\ &\quad \times[(I+D_{ik})x_i(t) + \bar{D}_{ik}x_i(t-\varsigma)] \\ &\le (1+\zeta_i)\frac{\lambda_M((I+D_{ik})^T P_i (I+D_{ik}))}{\lambda_m(P_i)}V_i(t_k^-) \\ &\quad +\frac{\lambda_M(\bar{D}_{ik}^T P_i \bar{D}_{ik})}{\zeta_i\lambda_m(P_i)}\widetilde{V}_i(t_k^-). \end{aligned}$$

By a similar process of Theorem 2, we can complete the remainder of the proof. Q.E.D.

Before the end of this section, we introduce an illustrative example.

Example 1. Consider the hybrid impulsive systems (13) where $x_i = (x_{i1}, x_{i2})^T, i = 1, 2$, $x = (x_1^T, x_2^T)^T$ and

$$A_1 = \begin{pmatrix} -6 & 0 \\ 0 & -5 \end{pmatrix},\ A_2 = \begin{pmatrix} -5 & 0 \\ 0 & -6 \end{pmatrix},$$
$$B_{ij} = C_{ij} = \bar{D}_{ij} = 0, i, j = 1, 2,$$
$$\bar{B}_{11} = \begin{pmatrix} -0.5 & 0 \\ 1 & 0.8 \end{pmatrix},\ \bar{B}_{12} = \begin{pmatrix} 1 & -1 \\ 0 & 0.5 \end{pmatrix},$$
$$\bar{B}_{21} = \begin{pmatrix} 0 & -0.5 \\ 1 & 0.5 \end{pmatrix},\ \bar{B}_{22} = \begin{pmatrix} 1 & -0.5 \\ -1 & 2 \end{pmatrix},$$
$$\bar{C}_{11} = \begin{pmatrix} 0.5 & 0.1 \\ 0 & 0.5 \end{pmatrix},\ \bar{C}_{12} = \begin{pmatrix} 0.2 & -0.8 \\ 0 & 0.3 \end{pmatrix},$$
$$\bar{C}_{21} = \begin{pmatrix} 0.5 & 0 \\ -0.3 & 0.4 \end{pmatrix},\ \bar{C}_{22} = \begin{pmatrix} -0.2 & 0.5 \\ 0 & 0.3 \end{pmatrix},$$
$$D_{1k} = \begin{pmatrix} 0.1 & 0 \\ 0 & -0.1 \end{pmatrix},\ D_{2k} = \begin{pmatrix} -0.2 & 0.05 \\ 0 & 0.3 \end{pmatrix}.$$

Take $\epsilon_i = 1, P_i = I, \lambda = 0.39$ in Theorem 3. By a direct calculation, we have $\vartheta_k \approx 4.32$.

If the impulsive moments satisfying $t_k - t_{k-1} \ge 4.0$, then

$$\sup_{k\in N}\{\frac{\ln\vartheta_k}{t_k - t_{k-1}}\} \le 0.3658 < \lambda = 0.39.$$

It follows from Theorem 3 that the origin of (13) is globally exponentially stable in mean square.

Figure 1 shows the stability when taking $t_k = 4k, \mu = \varsigma = \nu = 1$.

4. CONCLUSION

In this paper, we discussed a class of hybrid stochastic systems with impulses and time delays. The conditions on the stability and global exponential stability in mean square of the impulsive stochastic delayed systems were obtained. An example was also given for illustration. In fact, there are many unsolved problems for such systems. The stabilization and other control issues of these systems are under investigation.

REFERENCES

P. Baldi, and A. F. Atiya. How delays affect neural dynamics and learning. *IEEE Trans. Neur. Netw.*, 5 (3):612–621, 1994.

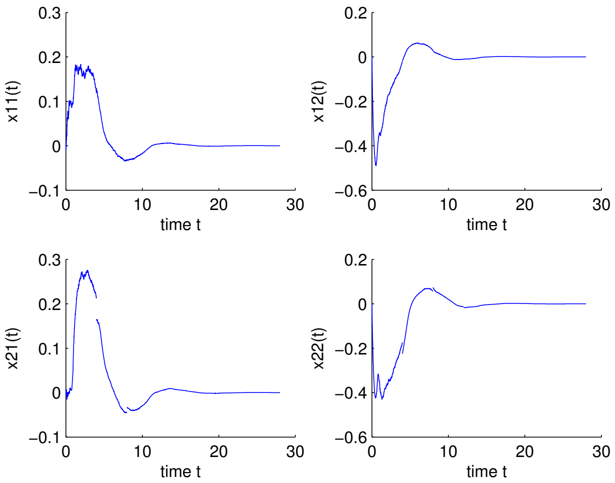

Fig. 1. Time response of hybrid impulsive systems with delays and stochastic effects in Example 1.

J. P. Richard. Time delay systems: an overview of some recent advances and open problems. *Automatica*, 39(10): 1667–1694, 2003.

S.E.A. Mohammed. *Stochastic Differential Systems with Memory.* Springer-Verlag, 2006.

V. Lakshmikantham, D.D. Bainov, and P.S. Simeonov. *Theory of Impulsive Differential Equations.* World Scientific, Singapore, 1989.

D. D. Bainov, andP.S. Simenov. *Impulsive differential equations: Periodic Solutions and Applications.* Longman Group UK. Limited, 1993.

A. N. Michel. Recent trends in the stability analysis of hybrid dynamical systems. *IEEE Trans. Circuits & Systems I*, 46(1): 120–134, 1999.

Girish N. Nair, and Robin J. Evans. Stabilizability of Stochastic Linear Systems with Finite Feedback Data Rates. *SIAM Journal on Control and Optimization*, 43: 413–436, 2004.

V. B. Kolmanovskii, and V. R. Nosov. *Stability of Functional Differential Equations.* Academic Press, New York, 1986.

D. Y. Xu. Stable criteria of large-scale systems of the neutral type. in Proc. 12th IMACS World Congress on Scientific Computations, R. Vichnevetsky, P. Borne, and J. Vignes, Eds. Paris, pages 213-215, 1988.

X. Mao. Exponential Stability in mean square of neutral stochastic differential functional equations. *Systems and Control Letters*, 26: 245–251, 1995.

A.M. Samoilenko, and N.A. Perestyuk. *Impulsive Differential Equations.* World Scientific, Singapore, 1995.

A. Obolenskii, S. Borisenko, and V. Kosolapov. *Stability of Processes under Continuous and Discrete Perturbations* (in Russian). Kiev, Naukova Dumka, 1988.

J. Yan, and J. Shen. Impulsive stabilization of functional differential equations by Lyapunov-Razumikhin functions. *Nonlinear Analysis*, 37: 245–255, 1999.

X. Liu, and G. Ballinger. Uniform asymptotic stability of impulsive delay differential equations. *Computers Math. Applic.*, 41: 903–915, 2001.

J. Yu. Stability for nonlinear delay differential equations of unstable type under impulsive perturbations. *Applied Mathematics Letters*, 14: 849–857, 2001.

Z. Guan, G. Chen, X. Yu, and Y. Qin. Robust decentralized stabilization for a class of large-scale time-delay uncertain impulsive dynamical systems. *Automatica*, 38: 2075–2084, 2002.

J. Hespanha. Stochastic Impulsive Systems Driven by Renewal Processes. *Technical Report*, University of California, 2005.

E.K. Boukas. Stabilization of stochastic singular nonlinear hybrid systems. *Nonlinear Analysis*, 64:217–228, 2006.

Appendix A. PROOF OF LEMMA 1

Proof. We first prove that for any number $\epsilon > 0, t \in [\sigma, b)$

$$u_i(t) \le (\kappa + \epsilon)e^{-\int_\sigma^t r(s)ds} \triangleq y(t),\ i = 1, \dots, n. \quad \text{(A.1)}$$

Let

$$\mathcal{J} = \{i | u_i(t) > y(t) \text{ for some } t \in [\sigma, b)\},$$
$$\theta_i = \inf\{t \in [\sigma, b) | u_i(t) > y(t), i \in I\}.$$

If the inequality (A.1) is not true, then $\mathcal{J}$ is a nonempty set and there must exist some integer $m \in \mathcal{J}$ such that $\theta_m = \min_{i\in I}\{\theta_i\} \in [\sigma, b)$. Employing the continuity of functions $u_i(t)$ and $y_i(t)$ for $t \in [\sigma, b), i = 1, \dots, n$, from (4), we can get

$$u_i(t) \le y(t),\ \ \sigma - \tau \le t \le \theta_m,\ i = 1, \dots, n, \quad \text{(A.2)}$$
$$u_m(\theta_m) = y(\theta_m),\ \ D^+u_m(\theta_m) \ge \dot{y}(\theta_m). \quad \text{(A.3)}$$

Combining with

$$[y(\theta_m)]_\tau = (\kappa + \epsilon) \sup_{\theta\in[-\tau,0]} \{e^{-\int_\sigma^{\theta_m+\theta} r(s)ds}\}$$
$$= (\kappa + \epsilon)e^{-\int_\sigma^{\theta_m} r(s)ds} \sup_{\theta\in[-\tau,0]} \{e^{\int_{\theta_m+\theta}^{\theta_m} r(s)ds}\},$$

we have

$$\begin{aligned}
D^+u_m(\theta_m) &\le \sum_{j=1}^{n}[p_{mj}(\theta_m)u_j(\theta_m) + q_{mj}(\theta_m)[u_j(\theta_m)]_\tau] \\
&\le \sum_{j=1}^{n}[p_{mj}(\theta_m)y(\theta_m) + q_{mj}(\theta_m)[y(\theta_m)]_\tau] \\
&= \sum_{j=1}^{n}[p_{mj}(\theta_m) + q_{mj}(\theta_m) \\
&\quad \times e^{\sup_{\theta\in[-\tau,0]}\{\int_{\theta_m+\theta}^{\theta_m} r(s)ds\}}](\kappa + \epsilon)e^{-\int_\sigma^{\theta_m} r(s)ds} \\
&< -r(\theta_m)(\kappa + \epsilon)e^{-\int_\sigma^{\theta_m} r(s)ds} \\
&= \dot{y}(\theta_m),
\end{aligned}$$

which contradicts the inequality in (A.3). Then, (A.1) is true for any $\epsilon > 0$. Letting $\epsilon \to 0^+$, we obtain the estimate (5). The proof is complete.