Proceedings of the 17th World Congress
The International Federation of Automatic Control
Seoul, Korea, July 6-11, 2008

Static output feedback sliding mode control for time-varying delay systems with time-delayed nonlinear disturbances

Xing-Gang Yan * **Sarah K. Spurgeon** * **Christopher Edwards** *

* *Department of Engineering, University of Leicester, University Road, Leicester, LE1 7RH, UK (e-mail: xy3@le.ac.uk, eon@le.ac.uk, ce14@le.ac.uk)*

Abstract: In this paper, a robust stabilization problem for a class of linear time-varying delay systems with disturbances is studied using sliding mode techniques. Both matched and mismatched disturbances, involving time-varying delay, are considered. The disturbances are nonlinear and have nonlinear bounds. A sliding surface is designed and the stability of the corresponding sliding motion is analysed based on the Razumikhin Theorem. Then a static output feedback sliding mode control with time-delay is synthesized to drive the system to the sliding surface in finite time. Simulation results show the effectiveness of the proposed approach.

1. INTRODUCTION

Since Karasovski extended Lyapunov theory to time-delay systems (Krasovskii 1963) and Razuminkin proposed a method to avoid the functional in the Lyapunov stability analysis (Razuminkhin 1960), great progress has been made on time-delay systems, but most of the early work focused on the analysis of unforced time delay systems. In recent years, increasing interest in the control of systems with time-delay has considered cases where the delay may appear in the system state, input, output and disturbances experienced by the systems. A variety of control approaches such as sliding mode control, H_∞ control, back-stepping techniques and adaptive control etc. have been applied to the control of systems with time delay and many results have been achieved (Gouaisbaut, Blanco & Richard 2004, Fridman, Shaked & Suplin 2005).

It is well known that sliding mode control is completely robust to so-called matched disturbances ((Utkin 1992, Edwards & Spurgeon 1998)). This has motivated the application of sliding mode techniques to time delay systems with disturbances ((Fridman et al. 2005, Niu, Ho & Lam 2005, Luo, De La Sen & Rodellar 1997, Gouaisbaut et al. 2004)). Most of the existing results are observer-based or based on the fact that all of the system state variables are accessible (Fridman et al. 2005, Gouaisbaut et al. 2004, Jafarov 2005, Niu et al. 2005). However, system state variables are often not fully available. It is possible to establish an observer to estimate the delayed system state variables and then apply an observer-based scheme (Niu et al. 2005, Jafarov 2005). However, this will require more hardware and increase system dimension. Therefore, static output feedback control may be preferable. Compared with state feedback, the static output feedback control problem is much more difficult–even for systems without delay ((Syrmos, Abdallah, Dorato & Grigoriadis 1997)). Much less attention has been paid to time delay systems with delayed disturbance using static output feedback sliding mode control and only very limited literature is available (Luo et al. 1997). Luo *et al* studied a class of time-delay systems where static and dynamic output feedback strategies are both considered (Luo et al. 1997) but it is assumed that all the uncertainty is matched. In all the existing results for time-delay systems, it is required that the bounds on the uncertainties satisfy linear growth conditions (i.e. linear functions of $\|x\|$ and/or $\|x_d(t)\|$). Since bounds on uncertainties may have nonlinear forms in reality (Chen & Pandey 1990), it is pertinent to consider the case when the bounds on the disturbances are nonlinear.

In this paper, a static output feedback sliding mode control strategy is proposed to stabilize a class of time-varying delay systems with time delayed nonlinear disturbances. Both matched and mismatched uncertainties are considered where the bounds on the uncertainties involving time-delay are employed in the control design. A memoryless sliding surface is designed and the system structure is analyzed and employed in the stability analysis of the sliding motion by using the Razumikhin-Lypunov approach. Then, a sliding mode control with time-delay based on only output information is proposed to drive the system to the sliding surface in finite time and maintain a sliding motion on it thereafter. The robustness is enhanced by fully using the nonlinear bounds on the disturbances. The conservatism is reduced by employing the system structure and the feature that the sliding mode is of reduced dynamical order.

Notation: In this paper, $\mathcal{R}^+$ denotes the nonnegative number set $\{t \mid t \geq 0\}$. The symbol $\mathcal{C}_{[a,b]}$ represents the set of $\mathcal{R}^n$-valued continuous functions on the interval $[a, b]$. For a matrix $A \in \mathcal{R}^{n\times m}$, $R(A)$ denotes the range space of A. The expression $A > 0$ means that A is symmetric positive definite and $\lambda_{\max}(A)$ ($\lambda_{\min}(A)$) represents its maximum (minimum) eigenvalue. For a matrix $A > 0$, $A^{\frac{1}{2}}$ denotes a symmetric positive definite matrix such that $A^{\frac{1}{2}}A^{\frac{1}{2}} = A$. Finally, $\|\cdot\|$ denotes the Euclidean norm or its induced norm.

2. PRELIMINARIES

Firstly, recall some basic linear systems theory. Consider a linear system

$$\dot{x} = Ax + Bu \tag{1}$$

$$y = Cx \tag{2}$$

where $x \in \mathcal{R}^n$, $u \in \mathcal{R}^m$, $y \in \mathcal{R}^p$ are the states, inputs and outputs respectively with $m \le p < n$. The triple (A, B, C) comprises constant matrices of appropriate dimensions where both B and C are of full rank.

For system (1)-(2), it is assumed that $\text{rank}(CB) = m$. Then, from (Edwards & Spurgeon 1998) it can be shown that a coordinate transformation $\widetilde{x} = \widetilde{T}x$ exists such that the system triple (A, B, C) with respect to the new coordinates $\widetilde{x}$ has the following structure

$$\widetilde{A} = \begin{bmatrix} \widetilde{A}_{11} & \widetilde{A}_{12} \\ \widetilde{A}_{21} & \widetilde{A}_{22} \end{bmatrix}, \quad \widetilde{B} = \begin{bmatrix} 0 \\ B_2 \end{bmatrix}, \quad \widetilde{C} = \begin{bmatrix} 0 & \breve{T} \end{bmatrix} \tag{3}$$

where $\widetilde{A}_{11} \in \mathcal{R}^{(n-m)\times(n-m)}$, $B_2 \in \mathcal{R}^{m\times m}$ is nonsingular and $\breve{T} \in \mathcal{R}^{p\times p}$ is orthogonal. Further, it is assumed that system $(\widetilde{A}_{11}, \widetilde{A}_{12}, \widetilde{C}_1)$ with $\widetilde{C}_1$ defined by

$$\widetilde{C}_1 = [\, 0_{(p-m)\times(n-p)} \quad I_{p-m} \,] \tag{4}$$

is output feedback stabilizable i.e. there exists a matrix $K \in \mathcal{R}^{m\times(p-m)}$ such that $\widetilde{A}_{11} - \widetilde{A}_{12}K\widetilde{C}_1$ is stable. It is shown in (Edwards & Spurgeon 1998) that a necessary condition for $(\widetilde{A}_{11}, \widetilde{A}_{12}, \widetilde{C}_1)$ to be stabilizable is that the invariant zeros of (A, B, C) lie in the open left half-plane.

Let

$$F = F_2 \,[\, K \quad I_m \,]\, \breve{T}^T \tag{5}$$

where $F_2 \in \mathcal{R}^{m\times m}$ is any nonsingular matrix. If a further nonsingular transformation $z = \widehat{T}\widetilde{x}$ with $\widehat{T}$ defined by

$$\widehat{T} = \begin{bmatrix} I_{n-m} & 0 \\ K\widetilde{C}_1 & I_m \end{bmatrix}$$

is introduced, then in the new coordinates z, system (1)–(2) has the following form

$$\widehat{A} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad \widehat{B} = \begin{bmatrix} 0 \\ B_2 \end{bmatrix}, \quad \widehat{C} = [\, 0 \quad C_2 \,] \tag{6}$$

where $A_{11} = \widetilde{A}_{11} - \widetilde{A}_{12}K\widetilde{C}_1$ is stable, $C_2 \in \mathcal{R}^{p\times p}$ is nonsingular and $\widehat{C}$ satisfies

$$F\widehat{C} = [\, 0 \quad F_2 \,] \tag{7}$$

where $F_2 \in \mathcal{R}^{m\times m}$ nonsingular. In the new coordinate system z, the system output is described by

$$y = \widehat{C}z \tag{8}$$

Definition 1. The matrix triple (A, B, C) or linear system (1)–(2) is called normalizable if there exists a nonsingular transformation $z = Tx$ such that in the new coordinate system z, the system (1)–(2) has the following form

$$\dot{z}_1 = A_{11}z_1 + A_{12}z_2 \tag{9}$$

$$\dot{z}_2 = A_{21}z_1 + A_{22}z_2 + B_2 u \tag{10}$$

$$y = [\, 0 \quad C_2 \,]\, z, \tag{11}$$

where $z_1 \in \mathcal{R}^{n-m}$, $z_2 \in \mathcal{R}^m$, A_{11} is stable, and $B_2 \in \mathcal{R}^{m\times m}$ and $C_2 \in \mathcal{R}^{p\times p}$ are nonsingular. Then, (9)–(11) is called the canonical form of system (1)–(2).

Lemma 1. System (1)–(2) is normalizable if i) $\text{rank}(CB) = m$. ii) for the triple $(\widetilde{A}_{11}, \widetilde{A}_{12}, \widetilde{C}_1)$ defined in (3) and (4), there exists a matrix K such that $\widetilde{A}_{11} - \widetilde{A}_{12}KC_1$ is stable.

Proof: By letting $T = \hat{T}\tilde{T}$, the conclusion follows directly from the analysis above. Δ

Definition 2 ((Gu, Kharitonov & Chen 2003)) A continuous function $\alpha : \ [0, a) \mapsto [0, \infty)$ is said to belong to class $\mathcal{K}$ if it is strictly increasing and $\alpha(0) = 0$. Further, it is said to belong to class $\mathcal{K}_\infty$ if $a = \infty$ and $\lim_{r\to\infty} \alpha(r) = \infty$.

Consider a time-delay system

$$\dot{x}(t) = \tilde{f}(t, x(t - d(t)) \tag{12}$$

with an initial condition $x(t) = \phi(t), \quad t \in [-\overline{d}, 0]$ where $\tilde{f} : \mathcal{R}^+ \times \mathcal{C}_{[-\overline{d},0]} \mapsto \mathcal{R}^n$ takes $\mathcal{R}\times$(bounded sets of $\mathcal{C}_{[-\overline{d},0]}$) into bounded sets in $\mathcal{R}^n$; $d(t) > 0$ is the time-varying delay and $\overline{d} := \sup_{t\in\mathcal{R}^+}\{d(t)\} < \infty$.

Lemma 2. (Razumikhin Theorem (Gu et al. 2003)) If there exist class $\mathcal{K}_\infty$ functions $\gamma_i(\cdot)$ with $i = 1, 2$, a class $\mathcal{K}$ function $\gamma_3(\cdot)$ and a continuous function $V_1(\cdot) : [-\overline{d}, \infty] \times \mathcal{R}^n \mapsto \mathcal{R}^+$ satisfying

$$\gamma_1(\|x\|) \le V_1(t, x) \le \gamma_2(\|x\|), \qquad t \in [-\overline{d}, \infty], \quad x \in \mathcal{R}^n$$

such that the time derivative of V_1 along the solution of system (12) satisfies

$$\dot{V}_1(t, x) \le -\gamma_3(\|x\|)$$

whenever $V_1(t+\theta, x(t+\theta)) \le V_1(t, x(t))$ for any $\theta \in [-\overline{d}, 0]$, then the system (12) is uniformly asymptotically stable.

Lemma 3. If there exist a constant γ and a function $V_2(t, x(t)) = x^T\tilde{P}x$ with $\tilde{P} > 0$ such that the time derivative of $V_2(\cdot)$ along the solution of system (12) satisfies

$$\dot{V}_2(t, x(t)) \le -\gamma \left\| \tilde{P}^{\frac{1}{2}} x(t) \right\|^2 \tag{13}$$

whenever $V_2(t+\theta, x(t+\theta)) \le V_2(t, x(t))$ for any $\theta \in [-\overline{d}, 0]$, then, the system (12) is uniformly asymptotically stable.

Proof: From the definition of $V_2(\cdot)$ it follows that

$$\lambda_{\min}(\tilde{P})\|x\|^2 \le V_2(t, x(t)) \le \lambda_{\max}(\tilde{P})\|x\|^2$$

Furthermore, from (13)

$$\dot{V}_2(t, x(t)) \le -\gamma x(t)^T\tilde{P}x(t) \le -\gamma\lambda_{\min}(\tilde{P})\|x\|^2$$

Then from Lemma 2 and exploiting the fact $\tilde{P} > 0$, the conclusion follows by letting $\gamma_1(\tau) = \lambda_{\min}(\tilde{P})\tau^2$, $\gamma_2(\tau) = \lambda_{\max}(\tilde{P})\tau^2$ and $\gamma_3(\tau) = \gamma\lambda_{\min}(\tilde{P})\tau^2$ in Lemma 2. Δ

The lemmas presented in this section will be used in the following analysis. The symbols $x_d(t)$, $y_d(t)$ and $z_d(t)$ will be used to denote $x(t - d(t))$, $y(t - d(t))$ and $z(t - d(t))$ respectively, throughout the paper.

3. SYSTEM DESCRIPTION

Consider a time-varying delay system with time-delayed disturbances described by

$$\dot{x}(t) = Ax(t) + A_0 x_d(t) + B(u(t) + g(t, x(t), x_d(t))) + f(t, x(t), x_d(t)) \quad (14)$$

$$y(t) = Cx(t), \qquad i = 1, 2, \ldots, N, \quad (15)$$

where $x \in \Omega \subset \mathcal{R}^n$ (where Ω is a neighborhood of the origin), $u \in \mathcal{R}^m$ and $y \in \mathcal{R}^p$ are system states, inputs and outputs respectively with $m \le p < n$. The matrices A, A_0, B and C represent constant matrices of appropriate dimension with B and C of full rank. The vectors $g(\cdot)$ and $f(\cdot)$ represent the matched and mismatched disturbances affecting the system respectively. The known function $d(t)$ is a time-varying delay which is assumed to be continuous, nonnegative and bounded in R^+, that is $\overline{d} := \sup_{t \in \mathcal{R}^+}\{d(t)\} < \infty$. The initial condition for the system is given by

$$x(t) = \phi(t), \qquad t \in [-\overline{d}, 0]$$

where $\phi(\cdot)$ is continuous in $[-\overline{d}, 0]$. It is assumed that the nonlinear functions $g(\cdot)$ and $f(\cdot)$ are smooth enough in their domain of definition such that the system has unique continuous solutions for the initial condition.

Assumption 1. The triple (A, B, C) is normalizable, and $R(A_0) \subset R(B)$.

Remark 1. Assumption 1 is a limitation on the linear part of system (14)–(15). It guarantees that the triple (A, B, C) can be transformed to a canonical form (9)–(11). The assumption $R(A_0) \subset R(B)$ means that the time delay term $A_0 x_d(t)$ is matched and thus it will not affect the sliding motion.

Assumption 2. There exist known continuous nonnegative functions $\rho_i(\cdot) : \mathcal{R}^+ \times \mathcal{R}^p \times \mathcal{R}^p \mapsto \mathcal{R}^+$ with $i = 1, 2$ and $\varpi(\cdot) : \mathcal{R}^+ \times \mathcal{R}^p \times \mathcal{R}^p \mapsto \mathcal{R}^+$ such that for $t \in \mathcal{R}^+$, and $x(t),\ x_d(t) \in \Omega$

$$\|f(t, x(t), x_d(t))\| \le \rho_1(t, y(t), y_d(t))\|x(t)\| + \rho_2\big(t, y(t), y_d(t)\big)\|x_d(t)\| \quad (16)$$

$$\|g(t, x(t), x_d(t))\| \le \varpi(t, y(t), y_d(t)) \quad (17)$$

In this paper, the objective is to design a static output feedback control with time-delay of the form

$$u = u(t, y(t), y_d(t)) \quad (18)$$

based-on sliding mode techniques such that the closed-loop system formed by the control law in (18) and the system (14)–(15) is uniformly stable in a domain of the origin even in the presence of the disturbances.

4. SLIDING MOTION ANALYSIS AND CONTROL DESIGN

In this section, the main results will be presented. From $R(A_0) \subset R(B)$ in Assumption 1, there exists a matrix $D \in \mathcal{R}^{m \times n}$ such that $A_0 = BD$. Then, from Section 2, it follows that under Assumption 1 there exists a coordinate transformation $z = Tx$ such that in the new coordinate system z, system (14)–(15) is described by

$$\dot{z}_1 = A_{11} z_1 + A_{12} z_2 + f_1(t, z(t), z_d(t)) \quad (19)$$

$$\dot{z}_2 = A_{21} z_1 + A_{22} z_2 + B_2 D T^{-1} z_d(t) + B_2(u + g(t, T^{-1}z(t), T^{-1}z_d(t)) + f_2(t, z(t), z_d(t)) \quad (20)$$

$$y = [\,0 \quad C_2\,]\, z, \quad (21)$$

where $A_{11} \in \mathcal{R}^{(n-m)\times(n-m)}$ is stable, $B_2 \in \mathcal{R}^{m \times m}$ and $C_2 \in \mathcal{R}^{p \times p}$ are nonsingular, and

$$\begin{bmatrix} f_1(t, z(t), z_d(t)) \\ f_2(t, z(t), z_d(t)) \end{bmatrix} := T\,[f(t, x(t), x_d(t))]_{x = T^{-1}z} \quad (22)$$

where $f_1(\cdot) \in \mathcal{R}^{n-m}$ and $f_2(\cdot) \in \mathcal{R}^m$.

Consider the following sliding surface for system (14)–(15)

$$S = \{x \mid FCx = 0\} \quad (23)$$

with F defined in (5). Then from (7) and (8), it follows that

$$FCx = Fy = F\widehat{C}z = [0 \;\; F_2]z = F_2 z_2$$

Since F_2 is nonsingular, it follows that in the z coordinate system the sliding surface (23) can be described by the equation $z_2 = 0$. Then from the canonical form (19)–(21), it follows that the sliding dynamics associated with the sliding surface (23) are described by

$$\dot{z}_1 = A_{11} z_1 + [f_1(t, z(t), z_d(t)]_{z_2(t)=0} \quad (24)$$

where $z_1 \in \mathcal{R}^{n-m}$ are the sliding mode state variables and A_{11} is stable. It is clear that the mismatched disturbance affects the sliding motion directly. Obviously system (24) which describes the sliding motion involves time delay.

Assumption 3 There exist known continuous functions $\phi_1(\cdot)$ and $\phi_2(\cdot)$ such that

$$\big\|[f_1(t, z(t), z_d(t))]_{z_2(t)=0}\big\| \le \phi_1(t, z_1(t), \|z_{1d}(t)\|)\|z_1(t)\| + \phi_2(t, z_1(t), \|z_{1d}(t)\|)\|z_{1d}(t)\| \quad (25)$$

where the function $\phi_1(t, r_1, r_2)$ and $\phi_2(t, r_1, r_2)$ are both nondecreasing about variables r_2.

Remark 2. Assumption 3 is a limitation on the mismatched disturbance. It implies that when a sliding motion takes place, the uncertainty f_1 can be bounded by a known continuous function of variables $z_1(t)$ and $z_{1d}(t)$. Note Assumption 3 is unnecessary if $f(\cdot)$ in (14) does not include time-delay (Yan, Edwards & Spurgeon 2004).

Since the matrix A_{11} in (24) is stable, it follows that for any $Q > 0$ ($Q \in \mathcal{R}^{m \times m}$), there exists an unique matrix $P > 0$ such that

$$A_{11}^T P + P A_{11} = -Q \quad (26)$$

Lemma 4. If Assumption 3 holds, then there exist known continuous functions $\psi_1(t, z_1(t), \|z_{1d}(t)\|)$ and $\psi_2(t, z_1(t), \|z_{1d}(t)\|)$ such that

$$\left\|P^{\frac{1}{2}}[f_1(t, z(t), z_d(t))]_{z_2(t)=0}\right\| \le \psi_1(\cdot)\|P^{\frac{1}{2}} z_1(t)\| + \psi_2(\cdot)\|P^{\frac{1}{2}} z_{1d}(t)\| \quad (27)$$

where the functions $\psi_1(t, r_1, r_2)$ and $\psi_2(t, r_1, r_2)$ are both nondecreasing about variables r_2.

Proof: It follows from the fact

$$\|z_1(t)\| \le \lambda_{\max}(P^{-\frac{1}{2}})\|P^{\frac{1}{2}} z_1(t)\| \quad (28)$$

$$\|z_{1d}(t)\| \le \lambda_{\max}(P^{-\frac{1}{2}})\|P^{\frac{1}{2}} z_{1d}(t)\| \quad (29)$$

that under Assumption 3

$$\begin{aligned}&\left\|P^{\frac{1}{2}}[f_1(t,z(t),z_d(t))]_{z_2(t)=0}\right\|\\&\le \lambda_{\max}(P^{\frac{1}{2}})\Big(\phi_1(t,z_1(t),\|z_{1d}(t)\|)\|z_1(t)\|\\&\quad+\phi_2(t,z_1(t),\|z_{1d}(t)\|)\|z_{1d}(t)\|\Big)\\&\le \lambda_{\max}(P^{\frac{1}{2}})\Big(\phi_1\left(t,z_1(t),\|z_{1d}(t)\|\right)\lambda_{\max}(P^{-\frac{1}{2}})\|P^{\frac{1}{2}}z_1(t)\|\\&\quad+\phi_2\left(t,z_1(t),\|z_{1d}(t)\|\right)\lambda_{\max}(P^{-\frac{1}{2}})\|P^{\frac{1}{2}}z_{1d}(t)\|\Big)\end{aligned}\quad(30)$$

Let

$$\psi_1(t,r_1,r_2) = \lambda_{\max}(P^{\frac{1}{2}})\lambda_{\max}(P^{-\frac{1}{2}})\phi_1(t,r_1,r_2)$$
$$\psi_2(t,r_1,r_2) = \lambda_{\max}(P^{\frac{1}{2}})\lambda_{\max}(P^{-\frac{1}{2}})\phi_2(t,r_1,r_2)$$

Then it follows that (25) is true and the functions ψ_1 and ψ_2 are both nondecreasing about variables r_2 since $\phi_1(t,r_1,r_2)$ and $\phi_2(t,r_1,r_2)$ are both nondecreasing about variables r_2. Hence the conclusion follows. □

The following theorem gives a sufficient condition under which the sliding motion is stable:

Theorem 1. Under Assumption 3, the sliding mode dynamics (24) are asymptotically uniformly stable if there exists a domain $\Omega_0=\{z_1 \mid z_1\in\mathcal{R}^{n-m}\}$ of the origin in $T(\Omega)$ such that for any $z_1(t)\in\Omega_0$ and $t\in\mathcal{R}^+$

$$\gamma := \lambda_{\min}(P^{-\frac{1}{2}}QP^{-\frac{1}{2}}) - \sup_{t\in\mathcal{R}^+,z_1(t)\in\Omega_0}\{\Theta(t,z_1(t))\} > 0 \quad (31)$$

where

$$\begin{aligned}\Theta(t,z_1(t)) :&= \psi_1\left(t,z_1(t),\lambda_{\max}(P^{-\frac{1}{2}})\|P^{\frac{1}{2}}z_1(t)\|\right)\\&+\psi_2\left(t,z_1(t),\lambda_{\max}(P^{-\frac{1}{2}})\|P^{\frac{1}{2}}z_1(t)\|\right)\end{aligned}\quad(32)$$

where the functions $\psi_1(\cdot)$ and $\psi_2(\cdot)$ satisfy (27), and the matrices P and Q satisfy (26).

Proof: For system (24), consider as a Lyapunov function candidate $V(z_1(t))=(z_1(t))^TPz_1(t)$. It follows from (25) and (26) that the time derivative of V along the trajectories of system (24) is given as

$$\begin{aligned}\dot V(z_1(t))\mid_{(24)} &= -(z_1(t))^TP^{\frac{1}{2}}\left(P^{-\frac{1}{2}}QP^{-\frac{1}{2}}\right)P^{\frac{1}{2}}z_1(t)\\&\quad+2(z_1(t))^TP^{\frac{1}{2}}P^{\frac{1}{2}}[f_1(t,z(t),z_d(t))]_{z_2(t)=0}\\&\le -\lambda_{\min}(P^{-\frac{1}{2}}QP^{-\frac{1}{2}})\|P^{\frac{1}{2}}z_1(t)\|^2\\&\quad+\|P^{\frac{1}{2}}z_1\|\Big(\psi_1(t,z_1(t),\|z_{1d}(t)\|)\|P^{\frac{1}{2}}z_1(t)\|\\&\quad+\psi_2(t,z_1(t),\|z_{1d}(t)\|)\|P^{\frac{1}{2}}z_{1d}(t)\|\Big)\end{aligned}\quad(33)$$

where Lemma 4 has been used to obtain the inequality above. Since $\psi_1(t,r_1,r_2)$ and $\psi_2(t,r_1,r_2)$ are both nondecreasing about variables r_2, it follows from (28)–(29) that

$$\psi_1(t,z_1(t),\|z_{1d}(t)\|) \le \psi_1(t,z_1(t),\zeta(t)) \quad (34)$$
$$\psi_2(t,z_1(t),\|z_{1d}(t)\|) \le \psi_2\left(t,z_1(t),\zeta(t)\right) \quad (35)$$

where $\zeta(t):=\lambda_{\max}(P^{-\frac{1}{2}})\|P^{\frac{1}{2}}z_{1d}(t)\|$.

From the definition of $V(\cdot)$, it follows that $V(z_1(t+\theta))\le V(z_1(t))$ for any $\theta\in[-\overline{d},0]]$ is equivalent to $V(z_1(t-d))\le V(z_1(t))$ for any $-d\in[-\overline{d},0]]$ which is equivalent to

$$\|P^{\frac{1}{2}}z_{1d}(t)\| \le \|P^{\frac{1}{2}}z_1(t)\| \quad (36)$$

Therefore, by substituting (36), (34) and (35) into (33),

$$\begin{aligned}&\dot V(z_1(t))\mid_{(24)}\\&\le -\lambda_{\min}(P^{-\frac{1}{2}}QP^{-\frac{1}{2}})\|P^{\frac{1}{2}}z_1(t)\|^2\\&\quad+\psi_1\left(t,z_1(t),\zeta(t)\right)\|P^{\frac{1}{2}}z_1(t)\|^2\\&\quad+\psi_2\left(t,z_1(t),\zeta(t)\right)\|P^{\frac{1}{2}}z_1(t)\|\,\|P^{\frac{1}{2}}z_{1d}(t)\|\\&\le -\lambda_{\min}\left(P^{-\frac{1}{2}}QP^{-\frac{1}{2}}\right)\|P^{\frac{1}{2}}z_1(t)\|^2\\&\quad+\psi_1\left(t,z_1(t),\lambda_{\max}(P^{-\frac{1}{2}})\|P^{\frac{1}{2}}z_1(t)\|\right)\|P^{\frac{1}{2}}z_1(t)\|^2\\&\quad+\psi_2\left(t,z_1(t),\lambda_{\max}(P^{-\frac{1}{2}})\|P^{\frac{1}{2}}z_1(t)\|\right)\|P^{\frac{1}{2}}z_1(t)\|^2\\&= -\left(\lambda_{\min}(P^{-\frac{1}{2}}QP^{-\frac{1}{2}})-\Theta(t,z_1(t))\right)\|P^{\frac{1}{2}}z_1(t)\|^2\\&\le -\gamma\|P^{\frac{1}{2}}z_1(t)\|^2\end{aligned}\quad(37)$$

Since $\gamma>0$ and $P>0$, the conclusion follows directly from Lemma 3. △

Remark 3. Theorem 1 shows that the stability of the sliding motion is completely robust to the matched uncertainty $g(\cdot)$ but is affected by the mismatched uncertainty $f(\cdot)$. Since the sliding mode is a reduced-order system, it is clear that only $f_2(\cdot)$ affects the sliding mode and thus the limitation on the mismatched uncertainty is weaker than in other work ((Wu 1999)) where a similar limitation is imposed on $f(\cdot)$ instead of $f_2(\cdot)$.

Theorem 1 above has shown that, under appropriate conditions, the sliding motion on the sliding surface (23) is stable. The objective now is to design a controller to drive the system to the sliding surface in finite time. Comparing system (14)–(15) with (19)–(21) gives

$$CT^{-1}=[0\ \ C_2], \qquad TB=\begin{bmatrix}0\\B_2\end{bmatrix}$$

where $C_2\in\mathcal{R}^{p\times p}$ and $B_2\in\mathcal{R}^{m\times m}$ are nonsingular. From (6) and (7), it follows that

$$FCB = F\underbrace{[0\ \ C_2]}_{\widehat{C}}\begin{bmatrix}0\\B_2\end{bmatrix} = [0\ \ F_2]\begin{bmatrix}0\\B_2\end{bmatrix} = F_2B_2$$

is nonsingular since both $F_2\in\mathcal{R}^{m\times m}$ and $B_2\in\mathcal{R}^{m\times m}$ are nonsingular. Partition AT^{-1}, A_0T^{-1} and T as

$$AT^{-1}:=[\,\Lambda_1\quad\Lambda_2\,],\ A_0T^{-1}:=[\,\Upsilon_1\quad\Upsilon_2\,],\ T:=\begin{bmatrix}T_1\\T_2\end{bmatrix} \quad (38)$$

where $\Lambda_1\in\mathcal{R}^{n\times m}$ and $\Upsilon_1\in\mathcal{R}^{n\times m}$ are the first m columns of AT^{-1} and A_0T^{-1} respectively; $T_1\in\mathcal{R}^{m\times n}$ and $T_2\in\mathcal{R}^{(n-m)\times n}$ are the first m and the last $n-m$ rows of T. Then, from the analysis above,

$$Tx=\begin{bmatrix}T_1x\\T_2x\end{bmatrix}=\begin{bmatrix}z_1\\z_2\end{bmatrix}=\begin{bmatrix}T_1x\\C_2^{-1}y\end{bmatrix} \quad (39)$$

Now, consider system (14)–(15) in $\Omega_1\times\Omega_2$ where

$$\Omega_1 := \{x(t)\mid \|T_1x\|\le\mu_1\}\subset\Omega \quad (40)$$
$$\Omega_2 := \{x_d(t)\mid \|T_1x_d(t)\|\le\mu_2\}\subset\Omega \quad (41)$$

with T_1 defined in (38). Then the following output feedback sliding mode controller with time-delay is proposed for the system

$$u=-k(t,y(t),y_d(t))(FCB)^{-1}\frac{Fy(t)}{\|Fy(t)\|} \quad (42)$$

where

$$\begin{aligned} & k(t, y(t), y_d(t)) \\ & = \|\Lambda_1\|\mu_1 + \|\Lambda_2 C_2^{-1} y\| + \|FCB\|\varpi(t, y(t), y_d(t)) \\ & \quad + \|FC\|\,\|T^{-1}\|\Big(\rho_1(t, y(t), y_d(t))\,\big(\mu_1 + \|C_2^{-1} y(t))\|\big) \\ & \quad + \rho_2\big(t, y(t), y_d(t)\big)\,\big(\mu_2 + \|C_2^{-1} y_d(t))\,\|\Big) + \eta \end{aligned} \tag{43}$$

for some $\eta > 0$ where the matrices Λ_1 and Λ_2 are defined by (38), the positive constants μ_1 and μ_2 are given in (40) and (41) respectively, and the functions $\varpi(\cdot)$, $\rho_1(\cdot)$ and $\rho_2(\cdot)$ are given in Assumption 2.

Remark 4. From the analysis above, it is clear to see that the sliding mode controller (42) with $k(\cdot)$ defined by (43) is well defined since the matrix FCB is nonsingular and the functions $\varpi(\cdot)$, $\rho_1(\cdot)$ and $\rho_2(\cdot)$ are assumed to be known. Obviously, the proposed control only depends on the time t, the known time-delay $d(t)$ and system output $y(t)$.

Theorem 2. Consider system (14)–(15) in $\Omega_1 \times \Omega_2$. Under Assumptions 1 and 2, the controller (42) with the gain $k(\cdot)$ defined by (43) drives the system (14)–(15) to the sliding surface (23) in finite time and maintains a sliding motion thereafter.

Proof: Let $\sigma(x) := FCx$. Then the sliding surface (23) can be described by equation $\sigma(x) = 0$. From (14) and (42), it follows that

$$\begin{aligned} \sigma^T(x)\dot\sigma(x) \le \|\sigma(x)\| \Big(& \|FC(Ax(t) + A_0 x_d(t))\| \\ & + \|FCB\|\,\|g(t, x(t), x_d(t))\| \\ & + \|FC\|\,\|f(t, x(t), x_d(t))\|\Big) - k(\cdot)\|\sigma(x)\| \end{aligned} \tag{44}$$

From (39) it follows that in $\Omega_1 \times \Omega_2$,

$$\|Tx(t)\| \le \mu_1 + \|C_2^{-1} y(t)\| \tag{45}$$

$$\|Tx_d(t)\| \le \mu_2 + \|C_2^{-1} y_d(t)\| \tag{46}$$

From (38) and (39),

$$\begin{aligned} & FC(Ax(t) + A_0 x_d(t)) \\ & = FC\Big([\,\Lambda_1 \quad \Lambda_2\,] \begin{bmatrix} T_1 x \\ C_2^{-1} y \end{bmatrix} + [\,\Upsilon_1 \quad \Upsilon_2\,] \begin{bmatrix} T_1 x_d(t) \\ C_2^{-1} y_d(t) \end{bmatrix} \Big) \\ & = FC\Lambda_1 T_1 x + FC\Lambda_2 C_2^{-1} y + FC\Upsilon_1 T_1 x_d(t) \\ & \quad + FC\Upsilon_2 C_2^{-1} y_d(t) \end{aligned}$$

Therefore, from (45)–(46),

$$\begin{aligned} \|FC(Ax(t) + A_0 x_d(t))\| \le & \|FC\Lambda_1\|\mu_1 + \|FC\Lambda_2 C_2^{-1} y(t)\| \\ & + \|FC\Upsilon_1\|\mu_2 + \|FC\Upsilon_2 C_2^{-1} y_d(t)\| \end{aligned} \tag{47}$$

From (16) and (45)–(46),

$$\begin{aligned} & \|f(t, x(t), x_d(t))\| \\ & \le \rho_1(t, y(t), y_d(t))\,\|T^{-1}\|\,\|Tx(t)\| \\ & \quad + \rho_2\big(t, y(t), y_d(t)\big)\,\|T^{-1}\|\,\|Tx_d(t)\| \\ & \le \|T^{-1}\|\,\Big(\rho_1(t, y(t), y_d(t))\,\big(\mu_1 + \|C_2^{-1} y(t)\|\big) \\ & \quad + \rho_2\big(t, y(t), y_d(t)\big)\,\big(\mu_2 + \|C_2^{-1} y_d(t)\|\big)\Big) \end{aligned} \tag{48}$$

Substituting (17), (47), (48) and (43) to (44), yields $\sigma^T(x)\dot\sigma(x) \le -\eta\|\sigma(x)\|$. This shows that the reachability condition ((Utkin 1992, Edwards & Spurgeon 1998)) is satisfied and thus the conclusion follows. Δ

Theorems 1 and 2 together show that the closed-loop system formed by applying control (42) with $k(\cdot)$ defined by (43) to system (14)–(15) is uniformly stable.

5. NUMERICAL SIMULATION

Consider the time varying delay system with delayed disturbance described by

$$\begin{aligned} \dot x = & \underbrace{\begin{bmatrix} -10 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & -5 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}}_{A_0} x_d(t) \\ & + \underbrace{\begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}}_{B} \Big(u(t) + g(t, x(t), x_d(t))\Big) \\ & + \underbrace{\begin{bmatrix} \sqrt2\beta_1(\cdot)x_1(t) + \beta_2(\cdot)x_{1d}(t) \\ 0 \\ \beta_1(\cdot)x_3(t) + \beta_2(\cdot)x_{3d}(t) \end{bmatrix}}_{f(t, x(t), x_d(t))} \end{aligned} \tag{49}$$

$$y = \underbrace{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}}_{C} x \tag{50}$$

where $x = \mathrm{col}(x_1, x_2, x_3)$, u and $y = \mathrm{col}(y_1, y_2)$ are respectively the state variables, the inputs and the outputs of the system. The unknown functions $\beta_1(\cdot)$ and $\beta_2(\cdot)$ are time-delayed disturbances which are assumed to satisfy

$$\begin{aligned} |\beta_1(t, x(t), x_d(t))| & \le (y_2(t))^2\,|\sin y_{1d}(t)| \\ |\beta_2(t, x(t), x_d(t))| & \le |y_{1d}(t)|\sin^2 y_1(t) + (y_2(t))^2 \end{aligned}$$

The disturbance $g(\cdot)$ has unknown structure but satisfies

$$\|g(\cdot)\| \le \underbrace{y_2^4(t)\sin^2 y_{1d}(t)}_{\varpi(\cdot)}$$

The domain considered here is

$$\Omega = \{(x_1, x_2, x_3) \mid x_2 \in \mathcal{R}, \quad \frac{1}{2}x_1^2 + x_3^2 < 19\}$$

Obviously

$$\begin{aligned} \|f(t, x(t), x_d(t))\| \le & \underbrace{\sqrt2 (y_2(t))^2\,|\sin y_{1d}(t)|}_{\rho_1(\cdot)}\,\|x(t)\| \\ & + \underbrace{\big(|y_{1d}(t)|\sin^2 y_1(t) + (y_2(t))^2\big)}_{\rho_2(\cdot)}\,\|x_d(t)\| \end{aligned}$$

Clearly $CB = [0 \quad -1]^T$ is full rank. According to the algorithm given in (Edwards & Spurgeon 1998), the coordinate transformation $\widetilde x = \widetilde T x$ with

$$\widetilde T = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

transforms the triple (A, B, C) into the following from

$$\begin{bmatrix} \widetilde A_{11} & \widetilde A_{12} \\ \widetilde A_{21} & \widetilde A_{22} \end{bmatrix} = \left[\begin{array}{cc|c} -10 & 0 & -1 \\ 0 & -5 & 1 \\ \hline -1 & 0 & 0 \end{array}\right] \tag{51}$$

$$\begin{bmatrix} 0 \\ \widetilde{B}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \hline -1 \end{bmatrix}, \qquad \begin{bmatrix} 0 & \breve{T}_2 \end{bmatrix} = [0 \quad I_2] \tag{52}$$

It is clear that the triple (A, B, C) is output feedback normalizable with $K = 0$ due to the stability of $\widetilde{A}_{11}$. Further $\mathcal{R}(A_0) \subset \mathcal{R}(B)$ since $A_0 = BD$ with $D = [1 \quad 0 \quad -1]$. Therefore Assumptions 1 and 2 are satisfied. Since (51)–(52) already has the canonical form (9)–(11), it follows that $T = \widetilde{T}$, $A_{11} = \widetilde{A}_{11}$, $A_{12} = \widetilde{A}_{12}$, $A_{21} = \widetilde{A}_{21}$, $A_{22} = \widetilde{A}_{22}$, $B_2 = \widetilde{B}_2$, $C_2 = \widetilde{C}_2 = I_2$. Let $Q = 10I_2$. It follows that the Lyapunov equation (26) has a unique solution $P = \text{diag}\{0.5, 1\}$ and thus $P^{\frac{1}{2}} = \text{diag}\{\frac{\sqrt{2}}{2}, 1\}$. According to (Edwards & Spurgeon 1998), choose $F = [0 \quad 1]$. The designed sliding surface from (23) is then described by

$$S(x) = \{(x_1, x_2, x_3) \mid \ y_2 = 0\}$$

By direct computation, it follows from (22) that

$$f_1 = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \beta_1(\cdot) P^{\frac{1}{2}} z_1(t) + \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 1 \end{bmatrix} \beta_2(\cdot) P^{\frac{1}{2}} z_{1d}(t) \tag{53}$$

When sliding motion takes place, $y_2(t) = 0$, and thus $\beta_1(\cdot) = 0$ and $|\beta_2(\cdot)\| \leq |y_{1d}(t)| \sin^2 y_1(t)$. Then,

$$\begin{aligned} &\left\| P^{\frac{1}{2}} [f_1(t, z(t), z_d(t))]_{z_2(t)=0} \right\| \\ &\leq \|z_1(t - d(t))\| (\sin y_1(t))^2 \left\| P^{\frac{1}{2}} z_d(t) \right\| \end{aligned} \tag{54}$$

By comparing (54) with (27), it follows that $\psi_1(\cdot) = 0$ and $\psi_2(\cdot) = \|z_1(t - d(t))\| (\sin y_1(t))^2$. Therefore,

$$\Theta(t, z_1(t)) = \sqrt{2} (\sin y_1(t))^2 \left\| P^{\frac{1}{2}} z_d(t) \right\|$$

By direct computation, it follows that the conditions of Theorem 1 hold in the domain $T(\Omega)$. From (42) and (43), the control is well defined and is obtained directly. For implementation purposes, choose $\mu_1 = \mu_2 = 2$ and $\eta = 1$. The time-varying delay $d(t)$ is chosen as $d(t) = 2 + \sin t$. A simulation with the initial condition $\phi(t) = (\cos(t), 1, -2\sin(t))$ is shown in Figure 1 and confirms that the proposed approach is effective.

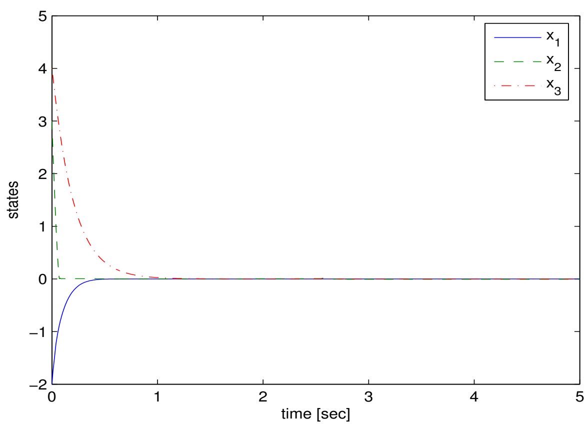

Fig. 1. The time responses of the system states

6. CONCLUSIONS

A robust static output sliding mode controller has been developed to stabilise a class of time-varying delay systems in this paper. An approach to deal with nonlinear matched and mismatched disturbances is shown when time-varying delay is involved in the nonlinear bounds on the disturbances. Compared with existing results, the nonlinear bounds are fully used in the control design. The conservatism is reduced by using the system structure and the property that the sliding mode dynamics are of reduced-order. Some remarks have shown the advantages of the approach. Simulations have shown the effectiveness of the proposed control scheme.

REFERENCES

Chen, Y. H. & Pandey, S. (1990). Uncertain bounded hybrid control for robot manipulators, *IEEE Trans. on Robotics and Automation* **6**(3): 303–311.

Edwards, C. & Spurgeon, S. K. (1998). *Sliding mode control: theory and applications*, London: Taylor and Francis Ltd.

Fridman, E., Shaked, U. & Suplin, V. (2005). Input/output delay approach to robust sampled-data H_∞ control, *Systems & Control Letters* **54**(3): 271–282.

Gouaisbaut, F., Blanco, Y. & Richard, J. (2004). Robust sliding mode control of non-linear systems with delay: A design via polytopic formulation, *Int. J. Control* **77**(2): 206–215.

Gu, K., Kharitonov, V. L. & Chen, J. (2003). *Stability of time-delay systems*, Boston: Birkhäuser.

Jafarov, E. M. (2005). Design modification of sliding mode observers for uncertain MIMO systems without and with time-delay, *Asian Journal of Control* **7**(4): 380–392.

Krasovskii, N. N. (1963). *Stability of motion (Translation form Russian)*, Stanford CA: Stanford University Press, USA.

Luo, N., De La Sen, M. & Rodellar, J. (1997). Robust stabilization of a class of uncertain time delay systems in sliding mode, *Int. J. Robust Nonlinear Control* **7**(1): 59–74.

Niu, Y., Ho, D. & Lam, J. (2005). Robust integral sliding mode control for uncertain stochastic systems with time-varying delay, *Automatica* **41**(5): 873–880.

Razuminkhin, B. S. (1960). Application of Liapunov's method to problems in the stability with a delay (in Russian), *Automat i Telemeh* **21**: 740–749.

Syrmos, V. L., Abdallah, C. T., Dorato, P. & Grigoriadis, K. (1997). Static output feedback—a survey, *Automatica* **33**(2): 125–137.

Utkin, V. I. (1992). *Sliding modes in control optimization*, Berlin: Springer-Verlag.

Wu, H. (1999). Robust output feedback controllers for dynamical systems including delayed perturbations, *Int. J. Systems Sci.* **30**(2): 211–218.

Yan, X. G., Edwards, C. & Spurgeon, S. K. (2004). Decentralised robust sliding mode control for a class of nonlinear interconnected systems by static output feedback, *Automatica* **40**(4): 613–620.