Proceedings of the 17th World Congress
The International Federation of Automatic Control
Seoul, Korea, July 6-11, 2008

Separation Principle for a Class of Nonlinear Feedback Systems Augmented with Observers ⋆

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Abstract: The paper suggests conditions for presence of quadratic Lyapunov functions for nonlinear observer based feedback systems with an 'input nonlinearity' in the feedback path. Provided that the system using state feedback satisfies the *circle criterion* (*i.e.,* when all states can be measured), we show that stability of the extended system with output feedback control from a (full state) Luenberger-type observer may be concluded using the *circle criterion*. As another result, we state a *separation principle* for a class of feedback systems with an input nonlinearity. When only local stability results can be stated, our method provides an estimate of the region of attraction.

1. INTRODUCTION

The *separation principle* plays a key role in the design of output feedback controllers, which instead of a full-state feedback use signals reconstructed by an observer from available measurements (Simon, 1956; Wonham, 1968). Essentially, it states that to achieve stability of the overall system one can design the controller and the observer *separately*—*i.e.*, a controller is developed to stabilize the plant as if all states were measured, while an observer is solely designed to reconstruct the state vector from available measurements. The *separation principle* is said to hold if the system resulting from interconnection of the state feedback controller and the observer is stable for all admissible combinations. The main class of dynamical systems that satisfy the *separation principle* consists of linear systems—*e.g.,* (Simon, 1956; Wonham, 1968). In general, however, the *separation principle* is not valid for nonlinear systems—*e.g.,* Example 1 in (Arcak and Kokotovic, 2001, p.1926). A few extensions of the *separation principle* to classes of nonlinear systems can be found in (Atassi and Khalil, 1999), (Arcak and Kokotovic, 2001), (Johansson and Robertsson, 2002), (Arcak, 2002).

In a series of papers, Arcak and Kokotović studied observer-based nonlinear feedback systems using the *circle criterion* for observer design and for robustness analysis. For certain classes of nonlinear systems—*e.g.,* output feedback stabilization of systems related to the Moore-Greitzer compressor model (1986)—observer-based feedback turns out to be difficult. One instability-prone example of observer-based feedback control is the following dynamical system (Arcak, 2002)

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\left(u - x_2^5\right) \tag{1}$$

$$y = x_1 \tag{2}$$

[1] The work was partly supported by the Swedish Research Council under the grants: 2005-4182, 2006-5243.

Whereas the finite-escape time problems of observer-based output feedback can be traced to violation of the Lipschitz condition, dynamic output feedback stabilization can be accomplished. As shown by Shiriaev *et al.* (2003), robust stabilization can be accomplished without resorting to explicit observers. Consider a dynamical output feedback controller of the form

$$\begin{aligned} \frac{d}{dt}z &= \lambda_3 x_1 + \lambda_4 z \\ u &= \lambda_1 x_1 + \lambda_2 z + (c_1 x_1 + c_3 z)^5 \end{aligned} \tag{3}$$

where λ_i, c_j are real constants to be defined. With such a controller, the dynamics of the closed-loop system are

$$\begin{aligned} \frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ z \end{bmatrix} &= \begin{bmatrix} 0 & 1 & 0 \\ \lambda_1 & 1 & \lambda_2 \\ \lambda_3 & 0 & \lambda_4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ z \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} w \\ w &= (c_1 x_1 + c_3 z)^5 - x_2^5 \end{aligned} \tag{4}$$

A key observation is that the nonlinearity w of Eq. (4) and the linear virtual output of the closed-loop system (4)

$$v = c_1 x_1 - x_2 + c_3 z$$

satisfies a passivity relationship for any x_1, x_2, z

$$v \cdot w = (c_1 x_1 - x_2 + c_3 z)[(c_1 x_1 + c_3 z)^5 - x_2^5] \geq 0 \tag{5}$$

Thus, the problem to design a controller that renders the origin of the system (1) asymptotically stable can be solved (Shiriaev *et al.* 2003). Whereas such dynamic output feedback control of Eq. (4) can be given an observer interpretation, the *separation principle* is obviously irrelevant. From a control design point of view, however, the decomposition of dynamic output feedback into problems of state-feedback control and observer design is attractive. Thus, the problem how to re-use a calculated state-feedback control in cases without full access to state measurement prompts further research for the *separation principle* for classes of nonlinear system without restriction to Lipschitz-bounded nonlinearities. In this paper, we consider dynamic output feedback in the context of the *circle criterion* with attention to absolute stability and quadratic constraints as tools

978-1-1234-7890-2/08/$20.00 © 2008 IFAC

10.3182/20080706-5-KR-1001.2435

for analysis and design. To this purpose, consider a nonlinear system

$$\frac{d}{dt}x = Ax + Bu, \quad u = -\psi(z,t), \quad z = \text{Input}$$
$$y = Cx \tag{6}$$

where a state vector $x \in \mathbb{R}^n$, a control action $u \in \mathbb{R}^m$, and A, B are constant matrices of appropriate dimensions. Suppose that a feedback controller is defined as

$$u = -\psi(z,t), \quad z = z(x) \tag{7}$$

where the function ψ satisfies the quadratic constraint

$$0 \geq \psi^T(z,t)\Big(\psi(z,t) - \kappa x\Big) \tag{8}$$

$$= \begin{bmatrix} x(t) \\ \psi(z,t) \end{bmatrix}^T \begin{bmatrix} 0 & -\frac{1}{2}\kappa^T \\ -\frac{1}{2}\kappa & I_m \end{bmatrix} \begin{bmatrix} x(t) \\ \psi(z,t) \end{bmatrix} \tag{9}$$

for all time t and $x \in \mathbb{R}^n$, where $\kappa \in \mathbb{R}^{m\times n}$.

Molander and Willems (1980) made a characterization of the conditions for stability with a high gain margin of feedback systems of the structure

$$\dot{x} = Ax + Bu;\ z = Lx;\ u = -\psi(z,t) \tag{10}$$

with $\psi(\cdot,\cdot)$ enclosed in a sector $[K_1, K_2]$. The following procedure was suggested to find a state-feedback vector L such that the closed-loop system will tolerate any $\psi(\cdot,\cdot)$ enclosed in a sector $[K_1,\infty)$ follows from

- Pick a matrix $Q = Q^T > 0$ such that (A,Q) is observable;
- Solve the Riccati equation
$$PA + A^T P - K_1 PBB^T P + Q = 0 \tag{11}$$
for $P > 0$. Take $L = B^T P$ and formulate the Lyapunov function $V(x) = x^T Px$ to be used for the proof of robust stability.

The algorithm provides a robustness result which fulfills an FPR condition—*i.e.*, the stability condition will be that of an SPR condition on $L(sI - A + K_1 BL)^{-1}B$, the state-feedback design procedure being based on a circle-criterion proof and involving a solution of a Riccati equation.

In (Johansson and Robertsson 2002), the Molander-Willems procedure was extended to observer-based feedback. Moreover, it was shown that the nominal pole assignment for control and for the observer dynamics can be made independently, a property identified with that of the separation principle.

In this paper, we will show that the separation principle holds for a class of nonlinear feedback systems described by the Lur'e problem with input nonlinearities and dynamic output feedback using observer-augmented design.

2. PROBLEM FORMULATION

Prior to formulation of the problem treated in the paper, let us consider the motivating example

$$\begin{aligned}\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} &= \begin{bmatrix}0 & 1\\ -3 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ -1\end{bmatrix} u = Ax + Bu\\ y &= [1\ \ 0]x = Cx\\ u &= \text{sat}(z\cdot(2+\sin^2(t))), \qquad z = x_2\end{aligned} \tag{12}$$

with a time-varying multiplicative disturbance on the input of the control signal which is saturated at level one and furthermore depends on the unmeasured state x_2.

The nonlinearity u satisfies the inequality

$$ux_2 \leq 3x_2^2, \quad \forall x_2. \tag{13}$$

At the same time, the sector of linear stability for the linear part of (12) for some $\tilde{u} = \kappa x_2$ so that

$$\kappa \in \Big(1, +\infty\Big)$$

Therefore, one can hope for robust stability of the closed loop system (12) only if the second constraint

$$ux_2 > x_2^2 \tag{14}$$

with $\kappa > 1$, is valid. This and the saturation in the nonlinearity of (12) imply that the state x_2, where (14) holds, is between ± 1. Combining (13) and (14), we get the quadratic constraint (QC)

$$0 \leq (3x_2 - u)(u - x_2) = \begin{bmatrix}x_1\\ x_2\\ u\end{bmatrix}^T \begin{bmatrix}\begin{pmatrix}0 & 0\\ 0 & -3\end{pmatrix} & \begin{pmatrix}0\\ 2\end{pmatrix}\\ (0\ \ 2) & -1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ u\end{bmatrix} \tag{15}$$

that remains valid provided $|x_2| < 1$. Later, this QC is to be used to solve an associated Riccati equation and to identify the Lyapunov function for any system which satisfies the QC (15) and the linear dynamics of (12).

Note, however, that the linear subsystem (12) is unstable and has an input saturation. Hence we can only hope to show *local* stability results for this example.

Consider a Lyapunov function candidate

$$V = \begin{bmatrix}x_1\\ x_2\end{bmatrix}^T P \begin{bmatrix}x_1\\ x_2\end{bmatrix}, \quad P = P^T > 0 \tag{16}$$

Along the solutions of (12) we have

$$\begin{aligned}\frac{dV}{dt} &= \begin{bmatrix}x_1\\ x_2\end{bmatrix}^T \big(A^T P + PA\big)\begin{bmatrix}x_1\\ x_2\end{bmatrix} + 2\begin{bmatrix}x_1\\ x_2\end{bmatrix}^T PBu\\ &\leq \begin{bmatrix}x_1\\ x_2\end{bmatrix}^T \big(A^T P + PA\big)\begin{bmatrix}x_1\\ x_2\end{bmatrix} + 2\begin{bmatrix}x_1\\ x_2\end{bmatrix}^T PBu\\ &+ \begin{bmatrix}x_1\\ x_2\\ u\end{bmatrix}^T \begin{bmatrix}\begin{pmatrix}0 & 0\\ 0 & -3\end{pmatrix} & \begin{pmatrix}0\\ 2\end{pmatrix}\\ (0\ \ 2) & -1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ u\end{bmatrix}\end{aligned} \tag{17}$$

where we have used the sector condition (15). Rewriting (17) as

$$\begin{aligned}\frac{dV}{dt} &= -(u - h^T[x_1, x_2]^T)^2\\ &= -u^2 + 2\begin{bmatrix}x_1\\ x_2\end{bmatrix}^T hu - \begin{bmatrix}x_1\\ x_2\end{bmatrix}^T hh^T \begin{bmatrix}x_1\\ x_2\end{bmatrix}\end{aligned} \tag{18}$$

we get the following equations for P, h

$$\begin{aligned}&\begin{bmatrix}x_1\\ x_2\end{bmatrix}^T\left(A^T P + PA + \begin{bmatrix}0 & 0\\ 0 & -3\end{bmatrix} + hh^T\right)\begin{bmatrix}x_1\\ x_2\end{bmatrix} = 0\\ &2\begin{bmatrix}x_1\\ x_2\end{bmatrix}^T\left(PB + \begin{bmatrix}0\\ 2\end{bmatrix}\right)u = 2\begin{bmatrix}x_1\\ x_2\end{bmatrix}^T hu\\ &\Rightarrow \quad P = \begin{bmatrix}3 & 0\\ 0 & 1\end{bmatrix}, \quad h = PB + \begin{bmatrix}0\\ 2\end{bmatrix}\end{aligned} \tag{19}$$

To estimate the area of attraction, we could use the fact that the set $V(x_1, x_2) < c$ is invariant w.r.t. the closed-loop system dynamics (12). To find the critical value c, we have that the sector condition is satisfied for $|x_2| \le 1$, so that

$$c_{max} = \begin{bmatrix}0\\1\end{bmatrix}^T \begin{bmatrix}3 & 0\\0 & 1\end{bmatrix} \begin{bmatrix}0\\1\end{bmatrix} = 1 \quad (20)$$

Therefore, the estimated area of attraction is

$$\begin{bmatrix}x_1\\x_2\end{bmatrix}^T \begin{bmatrix}3 & 0\\0 & 1\end{bmatrix} \begin{bmatrix}x_1\\x_2\end{bmatrix} = 3x_1^2 + x_2^2 \le 1 \quad (21)$$

Unfortunately the suggested feedback controller in (12) is not directly applicable because the variable x_2 used in the controller is not accessible through measurements, and only the first component x_1 is measured. Therefore, some modification for the controller of Eq. (12) is needed together with some method for estimating the area of attraction for the closed-loop system. This motivates the problem statement given in the next section.

2.1 Problem Formulation

This paper deals with robust output-feedback stabilization of a nonlinear system of the form

$$\dot{x} = Ax + Bu, \quad y = Cx, \quad z = Lx \quad (22)$$

$$u = -\psi(z,t) \quad (23)$$

where $x \in \mathbb{R}^n$ is the state, $y \in \mathbb{R}^k$ is the vector of measured variables, $z \in \mathbb{R}^l$ is the vector of linear combinations of the state used in the feedback controller, $u \in \mathbb{R}^m$ is the control signal; the matrices A, B, C, L have appropriate dimensions; $\psi(\cdot,\cdot)$ is the nonlinearity in the closed-loop system. It is assumed that

[A1] The nonlinearity (23) satisfies the QC

$$(\kappa_2 z - u)^T (u - \kappa_1 z) \ge 0, \quad \forall z,\ u, \quad (24)$$

where κ_1, κ_2 are constant $m \times l$ matrices;

[A2] The system (22), (23) satisfies the *circle criterion* based on the QC (24) that ensures asymptotic stability of (22), (23).

When y only is available to measurement, one can reconstruct the variable z by the Luenberger observer

$$\tfrac{d}{dt}\hat{x} = A\hat{x} + Bu + K(y - C\hat{x}), \quad \hat{z} = L\hat{x} \quad (25)$$

and use the estimate for the feedback control

$$u = -\psi(\hat{z},t) \quad (26)$$

Interconnection of Eqs. (25) and (26) will provide an output feedback controller for system (22), and stability becomes a challenge. One can expect that stability of the overall system (22), (25)–(26) may be dependent on the choice of the observer gain K.

The main contribution of this paper is that **for any observer gain K, s.t. $(A-KC)$ is Hurwitz, the closed-loop system (22), (25)–(26) is absolutely stable**, i.e.

- for any observer gain K, which makes $(A-KC)$ asymptotically stable, a new sector condition (*i.e.,* a new quadratic constraint, different from the given one in Eq. (24)) can be found for the closed-loop system (22), (25)–(26);
- it is verified that for this new quadratic constraint all the conditions of the *Circle criterion* applied to the closed-loop system (22), (25)–(26) still hold.

As a direct consequence of this result,

- one can infer that for any gain K, that makes $(A-KC)$ Hurwitz, the closed-loop system (22), (25)–(26) is asymptotically stable.—*i.e.,* the feedback controller (23) and the observer gain K could be chosen *separately*;
- for any gain K, that makes $(A-KC)$ Hurwitz, one can explicitly find a Lyapunov function for the closed-loop system (22), (25)–(26) as a solution of the associated Riccati equation;
- one can verify robust stability of (22), (25)–(26). Here the robustness is quantified in terms of the validity of the *frequency condition* for the newly found QC.

The paper is organized as follows. Section 3 contains the preliminaries where the formulation of the *circle criterion* and a brief description of essential steps involved in its proof are outlined. The main result of the paper is given in Sec. 4. In Sec. 4.1 the example from Sec. 2 is considered for output feedback control. In Sec. 6, conclusions are drawn.

3. PRELIMINARIES

To show the asymptotic stability of the system (22), (23) based on the *circle criterion*, consider a Lyapunov function candidate

$$V(x) = x^T P x, \qquad P = P^T > 0. \quad (27)$$

Using relation (24), one gets

$$\begin{aligned}\frac{dV}{dt} &= 2x^T P \tfrac{dx}{dt} = 2x^T P\Big(Ax + Bu\Big)\\ &\le\ 2x^T P\Big(Ax+Bu\Big) + 2\Big(\kappa_2 z - u\Big)^T\Big(u - \kappa_1 z\Big)\\ &\widehat{=}\ -\begin{bmatrix}x\\u\end{bmatrix}^T \begin{bmatrix}Q_{11} & Q_{12}\\ Q_{12}^T & Q_{22}\end{bmatrix}\begin{bmatrix}x\\u\end{bmatrix} = -\begin{bmatrix}x\\u\end{bmatrix}^T Q \begin{bmatrix}x\\u\end{bmatrix}\end{aligned} \quad (28)$$

where $-Q_{11} = A^T P + PA - L^T \kappa_2^T \kappa_1 L - L^T \kappa_1^T \kappa_2 L$

$$-Q_{12} = PB + L^T(\kappa_1^T + \kappa_2^T), \quad -Q_{22} = -2I_m$$

Question: *Under what conditions is there a $P = P^T > 0$ such that Q in (28) is positive definite?* The answer is given by:

Theorem 1. (Frequency Theorem (Yakubovich, 1963)). Let the pair (A,B) be stabilizable, then there is a $P = P^T$ such that Q of Eq. (28) fulfills $Q > 0$ if and only if there is $\varepsilon > 0$ such that for any vectors $\tilde{x} \in \mathscr{C}^n$, $\tilde{u} \in \mathscr{C}^m$ related by

$$j\omega\tilde{x} = A\tilde{x} + B\tilde{u}, \quad \forall\omega \in \mathbb{R}^1 \quad (29)$$

the following inequality holds

$$\operatorname{Re}\Big\{(\kappa_2 L\tilde{x} - \tilde{u})^* (\tilde{u} - \kappa_1 L\tilde{x})\Big\} < -\varepsilon\left(|\tilde{x}|^2 + |\tilde{u}|^2\right). \quad (30)$$

If $\det(j\omega - A) \ne 0\ \forall\,\omega \in \mathbb{R}^1$, the inequality (30) is equivalent to new QC valid for all $\omega \in \mathbb{R}^1$

$$\operatorname{Re}\Big\{\Big(\kappa_2 L(j\omega I_n - A)^{-1}B - I_m\Big)^* \cdot \Big(I_m - \kappa_1 L(j\omega I_n - A)^{-1}B\Big)\Big\} < 0 \quad \blacksquare \quad (31)$$

Suppose that there exists a (virtual) feedback

$$\bar{u} = -Rz \quad (32)$$

that satisfies the QC (24), and such that the matrix $(A-BRL)$ is Hurwitz. Then the time derivative of V with the feedback (32) reduces to the Lyapunov inequality

$$\frac{dV}{dt} = x^T\Big((A-BRL)^T P + P(A-BRL)\Big)x < 0 \quad (33)$$

As $(A-BRL)$ is Hurwitz, it implies that the matrix P is positive definite. Note that existence of the feedback (32) satisfying the QC is an assumption for the proof and should not be confused with the real closed-loop feedback with the nonlinearity $\psi(\cdot)$. Note that completing the squares after adding the quadratic form, which is not negative due to the imposed quadratic constraint, to the derivative of a quadratic Lyapunov function is the classical idea, known as the simplest form of application of the S-procedure. Summing up the arguments, one can formulate:

Theorem 2. (Circle Criterion (Yakubovich, 1963)). Consider the system of Eqs. (22), (23), where the nonlinearity satisfies the quadratic constraint (24). Suppose that there exists a linear feedback (32) such that it satisfies the quadratic constraint (24) and makes the matrix $(A-BRL)$ Hurwitz. If the *frequency condition* (30), (29) holds, then the *nonlinear* interconnected system (22), (23) is globally exponentially stable. ■

Remark 1. The given formulation of the *circle criterion* does not pretend to be general and we refer to the original papers (Yakubovich, 1963; Zames, 1966) and textbooks such as (Khalil, 2002) and references therein for different extensions. The *frequency condition* and the *circle criterion* are shown here to highlight the ideas behind the development in the next section. Note also the result of (Molander and Willems, 1980), where the *circle criterion* was used for synthesis of state feedback controllers with specified gain and phase margins.

4. STABILITY OF THE OBSERVER-BASED SYSTEM

Consider the case when the variable z of the system (22) is not available, while only the output variable y is measured. To form the dynamic output feedback controller, form an observer and let its state substitute the true system state in the state-feedback control. Let an observer be given by (25), and the feedback controller be chosen by Eq. (26).

By assumption the nonlinearity $u=-\psi(z,t)$, see (23), satisfies the QC (24), that is

$$(\kappa_2 z-u)^T(u-\kappa_1 z)\geq 0,\quad \forall z,\ u, \tag{34}$$

where z and u are not necessarily solutions of the closed-loop system, but rather could be seen as any vectors of appropriate dimensions.

From this observation follows that for any choice of the observer gain K, the solutions of the closed-loop system (22), (25)–(26) with $u=-\psi(\hat{z},t)$ satisfy the following new QC

$$(\kappa_2\hat{z}-u)^T(u-\kappa_1\hat{z})\geq 0,\quad \forall\hat{z},\ u, \tag{35}$$

The next question arises: *Given an observer gain K, under what conditions could the circle criterion be applied to the extended nonlinear system (22), (25), (26) based on the QC (35)?*

The answer is given in the next statement:

Theorem 3. Suppose all assumptions of Theorem 2 hold. Then for any gain K that makes the matrix $(A-KC)$ Hurwitz, the closed-loop system (22), (25), (26) satisfy all conditions of the *circle criterion* applied to the quadratic constraint (35)—*i.e.,* with such choice of the observer gain the closed-loop system (22), (25), (26) is robustly globally exponentially stable. ■

Proof. The system equation (22) augmented with the observer (25) looks as follows

$$\frac{d}{dt}\begin{bmatrix}x\\ \hat{x}\end{bmatrix}=\underbrace{\begin{bmatrix}A & 0\\ KC & A-KC\end{bmatrix}}_{A_{aug}}\begin{bmatrix}x\\ \hat{x}\end{bmatrix}+\underbrace{\begin{bmatrix}B\\ B\end{bmatrix}}_{B_{aug}}u \tag{36}$$

Choose a Lyapunov function candidate as

$$W=X^T\mathscr{P}X,\quad \text{where}\quad X\hat{=}\left[x^T\ \hat{x}^T\right]^T \tag{37}$$

Its time derivative along the closed-loop system solution is

$$\frac{d}{dt}W\leq 2X^T\mathscr{P}\left(A_{aug}X+B_{aug}u\right)+2\left(\kappa_2\hat{z}-u\right)^T\left(u-\kappa_1\hat{z}\right)$$
$$\hat{=}-\begin{bmatrix}X\\ u\end{bmatrix}^T\mathscr{Q}\begin{bmatrix}X\\ u\end{bmatrix},\ \mathscr{Q}>0 \tag{38}$$

where $\mathscr{Q}$ is defined in Table 1, Eq. (39). From the *frequency theorem* , it can be concluded that there exists a matrix $\mathscr{P}=\mathscr{P}^T$ such that $\mathscr{Q}>0$ if and only if $\forall\tilde{X}\in\mathscr{C}^n$, $\forall\tilde{u}\in\mathscr{C}^m$ related by

$$j\omega\tilde{X}=A_{aug}\tilde{X}+B_{aug}\tilde{u},\quad \forall\omega\in\mathbb{R}^1 \tag{40}$$

the inequality valid

$$\text{Re}\left\{\left(\left[0_{m\times n},\ \kappa_2 L\right]\tilde{X}-\tilde{u}\right)^*\left(\tilde{u}-\left[0_{m\times n},\ \kappa_1 L\right]\tilde{X}\right)\right\}<0 \tag{41}$$

To simplify the left hand side of (42), one can use the identities of the next statement

Lemma 1. The following equalities hold

$$\left[0,I_n\right]\tilde{X}=\left[0,I_n\right]\left(j\omega I_n-A_{aug}\right)^{-1}B_{aug}\tilde{u} \tag{42}$$
$$=\left[0,I_n\right](j\omega I_n-A)^{-1}B\tilde{u}\quad ■$$

Proof of Lemma 1 comes from standard matrix computations and is omitted. Based on this fact one can rewrite the inequality (42) like the inequality (31). Therefore, from the *frequency theorem* one concludes that there exists a matrix $\mathscr{P}=\mathscr{P}^T$ such that $\frac{d}{dt}W$ evaluated along any solution of the system (22), (25), (26) is negative definite.

To check that $\mathscr{P}>0$, consider the *linear* subsystem (22), (25) with the feedback controller

$$\bar{u}=-R\hat{z} \tag{43}$$

As assumed, this feedback satisfies the QC (24), and at the same time the system (22) is stabilized by (32). It follows from the *separation principle* for linear time-invariant systems that the closed-loop system (22), (25) and (44) is asymptotically stable.

The time derivative of W with the feedback (44) is of the form

$$\tfrac{d}{dt}W=2X^T\mathscr{P}\left(A_{aug}X+B_{aug}u\right) \tag{44}$$
$$=2X^T\mathscr{P}\left(A_{aug}-B_{aug}\begin{bmatrix}0_{n\times n}\\ RL\end{bmatrix}\right)X<0$$

The last relation is again a Lyapunov inequality, and the asymptotic stability of the matrix

$$A_{aug}-B_{aug}\begin{bmatrix}0_{n\times n}\\ RL\end{bmatrix}$$

implies that $\mathscr{P}$ is positive definite. ■

Let us comment the result:

1). The main step—*i.e.,* checking the *frequency condition*—becomes trivial due to *asymptotic unbiasedness* of the estimate

$$\mathscr{Q} = \begin{bmatrix} A_{aug}^T \mathscr{P} + \mathscr{P} A_{aug} - \begin{bmatrix} 0_{n\times n} & 0_{n\times n} \\ 0_{n\times n} & -L^T \kappa_2^T \kappa_1 L - L^T \kappa_1^T \kappa_2 L \end{bmatrix} & \mathscr{P} B_{aug} + \begin{bmatrix} 0_{m\times n}, L^T(\kappa_1^T + \kappa_2^T) \end{bmatrix} \\ B_{aug}^T \mathscr{P} + \begin{bmatrix} 0_{m\times n}, (\kappa_1 + \kappa_2) L \end{bmatrix}^T & -2I_m \end{bmatrix} \tag{39}$$

Table 1. Expression for $\mathscr{Q}$ in Eq. (38)

$\hat{X}$, which is a basic property of the Luenberger observer. It implies a solvability of the corresponding Riccati equation for the system augmented by the observer. Positiveness of the solution of Riccati equation, in turn, follows from the validity of the *separation principle* for linear systems.
2). The Lyapunov functions V and W, constructed from the solutions P and $\mathscr{P}$ of the corresponding Riccati equations, have no clear connection. The constructive procedure for design of a gain K and an appropriate Lyapunov function W, based on V, could be found in (Johansson and Robertsson, 2002).

4.1 Example (cont'd)—Output Feedback

Consider observer-based feedback control for the system (12)

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} u = Ax + Bu \tag{45}$$

with

$$\begin{aligned} \begin{bmatrix} \dot{\hat{x}}_1 \\ \dot{\hat{x}}_2 \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} u + \begin{bmatrix} K_1 \\ K_2 \end{bmatrix} (y - Cz) \\ &= (A - KC)z + Bu + Ky \\ y &= [1 \;\; 0]\, x = Cx \\ u &= \mathrm{sat}(v \cdot (2 + \sin^2(t))), \qquad v = \hat{x}_2 \end{aligned} \tag{46}$$

where we have a similar local sector condition as before, now w.r.t. u and $\hat{x}_2$

$$\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ u \end{bmatrix}^T \begin{bmatrix} \begin{pmatrix} 0 & 0 \\ 0 & -3 \end{pmatrix} & \begin{pmatrix} 0 \\ 2 \end{pmatrix} \\ (0 \;\; 2) & -1 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ u \end{bmatrix} \geq 0 \tag{47}$$

Consider the Lyapunov function candidate

$$W = X^T \mathscr{P} X, \quad \mathscr{P} = \mathscr{P}^T > 0$$

with $X = [x_1, x_2, \hat{x}_1, \hat{x}_2]^T$. The related Riccati equation is

$$\begin{aligned} & A_{aug}^T \mathscr{P} + \mathscr{P} A_{aug} + \begin{bmatrix} \mathbf{0}_{2\times 2} & \mathbf{0}_{2\times 2} \\ \mathbf{0}_{2\times 2} & \begin{pmatrix} 0 & 0 \\ 0 & 3 \end{pmatrix} \end{bmatrix} \\ & + \left(\mathscr{P} B_{aug} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 2 \end{pmatrix} \right) \left(\mathscr{P} B_{aug} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 2 \end{pmatrix} \right)^T = \mathbf{0}_{4\times 4} \end{aligned} \tag{48}$$

where

$$A_{aug} = \begin{bmatrix} A & 0 \\ KC & A - KC \end{bmatrix}, \quad B_{aug} = \begin{bmatrix} B \\ B \end{bmatrix}$$

and the solution

$$\mathscr{P} = \begin{bmatrix} 21.0774 & -3.4024 & -17.6894 & 1.8689 \\ -3.4024 & 2.6455 & 3.7056 & -1.4151 \\ -17.6894 & 3.7056 & 17.3014 & -2.1720 \\ 1.8689 & -1.4151 & -2.1720 & 1.1847 \end{bmatrix}, \lambda(\mathscr{P}) = \begin{bmatrix} 0.3182 \\ 1.1503 \\ 2.7983 \\ 37.9422 \end{bmatrix}$$

To estimate the region of attraction for $X_0 = 0_{4\times 1}$ we need to find the critical value c_{max} for the invariant set of a maximal ellipsoid

$$W = X^T \mathscr{P} X \leq c_{max} \tag{49}$$

which will lie between the two surfaces $\hat{x}_2 = \pm 1$. Obviously, such an ellipsoid exists. As in the state feedback case, we have:

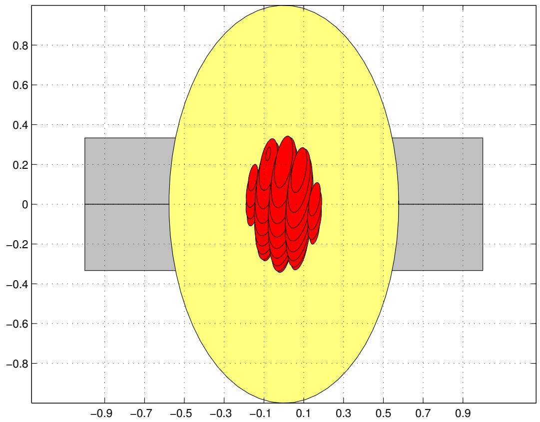

Fig. 1. Estimated region of attraction (RoA) in the $x_1 - x_2$ plane. *Yellow ellipsoid:* Estimated RoA ($3x_1^2 + x_2^2 < 1$) for state feedback case. *Red ellipsoids:* Estimated RoA (projected) for the observer based control with different initial conditions $\{\hat{x}_1(0), \hat{x}_2(0)\}$.

$$\hat{x}_2^0 = 1, \; c_{max} = [x_1^0, x_2^0, \hat{x}_1^0, 1]\, \mathscr{P}\, [x_1^0, x_2^0, \hat{x}_1^0, 1]^T \tag{50}$$

with

$$\begin{bmatrix} x_1^0 \\ x_2^0 \\ \hat{x}_1^0 \\ 1 \end{bmatrix} = \frac{1}{2y_4} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}, \quad \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \mathscr{P}^{-1} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{51}$$

Note that in inequality (50) we can set the initial conditions of the observer $\hat{x}_1$ and $\hat{x}_2$ ourselves. If the QC is not satisfied globally, but only for $x \in \Omega$, an open set containing the origin in its interior, say $\Omega = \{|x_2| < 1\}$), then estimates of the regions of attraction can be calculated from $x^T P x < c$ with the biggest c such that Ω contains this set. Finally, the stability domain can be increased using high-gain observers (Atassi and Khalil, 1999).

Example 2—Separation Principle and Global Stability

Consider observer-based feedback control of a system with the double integrator dynamics (Johansson and Robertsson, 2002)

$$\dot{x} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \tag{52}$$

$$\dot{\hat{x}} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \hat{x} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u + K(y - C\hat{x}), \; K = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, X = \begin{bmatrix} x \\ \hat{x} - x \end{bmatrix} \tag{53}$$

$$y = Cx = [0 \;\; 2]\, x \tag{54}$$

$$u = -\mathrm{sgn}(L\hat{x}), L = [1.7321 \;\; 1.000], P = \begin{bmatrix} 1.732 & 1.000 \\ 1.000 & 1.732 \end{bmatrix} \tag{55}$$

where $W(X) = X^T \mathscr{P} X$—*cf.* Eq. (37)—and $L = B^T P$ have been calculated based on a feedback transformation with

$$A_0 = \begin{bmatrix} A - BL & -BL \\ 0 & A - KC \end{bmatrix}, B_0 = \begin{bmatrix} B \\ 0 \end{bmatrix} \tag{56}$$

$$\mathscr{P} B_0 = \mathscr{P} \begin{bmatrix} B \\ 0 \end{bmatrix} = \begin{bmatrix} L^T \\ L^T \end{bmatrix}, \; \mathscr{P} A_0 + A_0^T \mathscr{P} = -\mathscr{Q}, \tag{57}$$

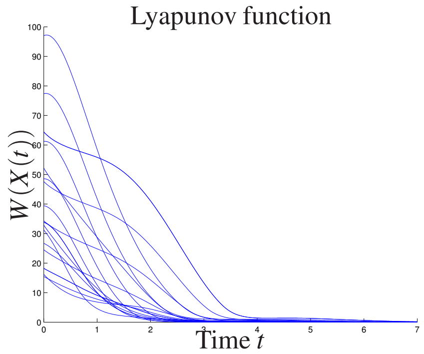

Fig. 2. Lyapunov function trajectories from switching output feedback control of double integrator dynamics.

with the weighting matrices

$$\mathscr{P} = \begin{bmatrix} 1.732 & 1.000 & 1.732 & 1.000 \\ 1.000 & 1.732 & 1.000 & 1.7321 \\ 1.732 & 1.000 & 17.266 & -5.800 \\ 1.000 & 1.732 & -5.800 & 11.979 \end{bmatrix}, \mathscr{Q} = \begin{bmatrix} 4.000 & 1.732 & 5.000 & 8.928 \\ 1.732 & 2.000 & 3.464 & 2.000 \\ 5.000 & 3.464 & 17.60 & 14.417 \\ 8.928 & 2.000 & 14.417 & 26.716 \end{bmatrix}$$

This example demonstrates the *separation principle* with global asymptotic stability in the context of observer-supported high-gain feedback (Fig. 2).

5. DISCUSSION

We have provided a constructive procedure for observers for Lur'e interconnections, exploiting that there are solutions to the Yakubovich-Kalman-Popov equation in the form of nonminimal positive real systems. Thus, the very restrictive SPR conditions relevant to observer-based feedback control systems can be significantly relaxed. Important applications of such systems can be found in nonlinear stability theory, observer design and design of feedback stabilization—*e.g.,* by means of the Popov criterion. A method for construction of Lur'e-Lyapunov functions for systems with observer-based feedback control is given. By construction, the nonlinear dynamic output feedback control accomplished exhibits SPR properties, yet non-minimal, with asymptotic stability and passivity properties guaranteed for the closed-loop system. The *circle criterion* considered here, could be seen as a particular example of the so-called *Quadratic Criterion* for absolute stability developed by Yakubovich and others, see (Yakubovich, 1967; Yakubovich, 2000). As expected, a different form of quadratic constraint, such as the Popov criterion, may also result in the validity of the *separation principle*. The main obstacle in the direct extension of Theorem 3 for the general case of quadratic constraint is checking condition for the so-called *minimal stability* of the augmented system, see (Yakubovich, 2000; Shiriaev, 2000).

Apart from its relevance to observer-based feedback control, we expect that the new method will have application to hybrid system and to high-gain feedback systems controlled by logic-based switching devices.

6. CONCLUSIONS

This paper describes a class of nonlinear control systems, for which the *separation principle* is valid. It is assumed that the system stability is determined from the *circle criterion*, while the non-linearity in the system is related to a non-stationary state feedback controller. If the state vector of the system, used as input to the feedback controller, is not available, then following the standard procedure, one can extend the system by a full-order observer, using the observer state as controller input. This leads to the question of stability of the overall system, even if it is known that the observer state converges to the true one. One can expect that stability may depend on the observer gain chosen. The main contribution of the paper states that this is not the case. For any choice of the observer gain that provides convergence of the observer state to the true one, the extended system is globally asymptotically stable, and an explicit form of a quadratic Lyapunov function for the extended system is derived.

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