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Harmonic Analysis of Pulse-Width Modulated Systems ⋆

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Abstract: The paper considers the so-called dynamic phasor model as a basis for harmonic analysis of a class of switching systems. The dynamic phasor model is a powerful tool for exploring cyclic properties of dynamic systems. It is shown that there is a connection between the dynamic phasor model and the harmonic transfer function of a linear time periodic system and this connection is used to extend the notion of harmonic transfer function to describe periodic solutions of non-periodic systems.

1. INTRODUCTION

The paper considers harmonic analysis of a class of switching systems based on the so-called *dynamic phasor model*.

The systems considered are a class of pulse-width modulated (PWM) systems that switch between subsystems in a cyclic manner. The analysis covers both open and closed loop systems. In the open loop case the switching is periodic and the PWM systems are linear time periodic (LTP). In the closed loop case the switching instants are determined by the state and the systems are no longer periodic. However, the pulses that excite the systems begin at periodically repeated time instants and the non-periodic systems retain a "cyclic" property which is explored in the analysis.

The analysis is motivated mainly by switched mode power converters. Such devices are a common source of electromagnetic pollution and may cause excessive harmonics in power systems. To be able to predict such phenomena is important, see e.g. Möllerstedt and Bernhardsson [2000].

The dynamic phasor model is a powerful tool for exploring cyclic properties of dynamic systems. It is obtained from a Fourier series expansion of the system state over a moving time-window. It was to our knowledge introduced for power electronics applications as a means to model the transients of the harmonics generated by the switching dynamics, see e.g. Caliskan et al. [1999]. The dynamic phasor model is conceptually appealing but poses some mathematical difficulties. The main problem is that the Fourier series expansion of the system state over a given time-window in general does not converge uniformly. These convergence problems were revealed in Tadmor [2002].

For analysis of the open loop (LTP) systems we review some results on the so-called harmonic transfer function (HTF), see Wereley and Hall [1990], Möllerstedt [2000] and provide a new interpretation. The HTF generalizes the concept of transfer function to LTP system and is an efficient tool for illustrating the frequency coupling between input and output. We show that the HTF and the dynamic phasor model are connected in the sense that the dynamic phasor model yields an explicit expression for the HTF.

When applied to the (closed loop) PWM systems, the dynamic phasor model yields an equivalent infinite dimensional system in the frequency domain. We approximate this system with a truncated averaged system and consider harmonic balance equations to determine the effect of a periodic disturbance. An approximate solution to the harmonic balance equations is stated in terms of a transfer function which is analogous to the HTF of a LTP system.

The approximation methodology is applied to a realistic example and the results are verified by simulation. It is shown that the approximated and simulated responses correspond well and that the approximation captures the nonlinear phenomena caused by switching. We also use the schauder fixed point theorem to provide conditions under which the approximation is justified.

Notation

In this paper l_2 denotes the set of square summable sequences $\hat{x} = \{x_k\}_{k=-\infty}^{\infty}$ where $x_k \in \mathbb{C}^q$ satisfies $\bar{x}_k = x_{-k}$ where $\bar{x}_k$ is the complex conjugate of x_k. Note that here, l_2 is used to denote a smaller set than usual. We also define a finite version of l_2; $l_{2,N}$ denotes the set of sequences $x = \{x_k\}_{k=-N}^{N}$ where $x_k \in \mathbb{C}^q$ satisfies $\bar{x}_k = x_{-k}$. π_N is used to denote a truncation as follows: $\pi_N : l_2 \to l_{2,N}$ is defined by the relation

$$(\pi_N \hat{x})_k = x_k, \quad -N \le k \le N.$$

The transformation $\mathcal{T}$ maps a sequence $\hat{\xi} = \{\xi_k\}_{k=-\infty}^{\infty}$ in l_2 to a doubly infinite dimensional block Toeplitz matrix according to

$$\mathcal{T}[\hat{\xi}] := \begin{bmatrix} \ddots & & \vdots & & \iddots \\ & \check{\xi}_0 & \check{\xi}_1 & \check{\xi}_2 & \\ \cdots & \check{\xi}_{-1} & \check{\xi}_0 & \check{\xi}_1 & \cdots \\ & \check{\xi}_{-2} & \check{\xi}_{-1} & \check{\xi}_0 & \\ \iddots & & \vdots & & \ddots \end{bmatrix}$$

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where $\check{\xi}_k = \xi_k I_n$ if ξ_k is scalar ($q = 1$) and $\check{\xi}_k = \xi_k$ otherwise. $\mathcal{T}_N[\hat{\xi}]$ is a finite dimensional matrix consisting of the $2N+1$ central blocks of $\mathcal{T}[\hat{\xi}]$. If f is a periodic function we let $\mathcal{T}[f] := \mathcal{T}[\hat{f}]$ where $\hat{f} \in l_2$ is the sequence of Fourier coefficients of f. I is used for the identity operator on both finite and infinite dimensional spaces and $\bar{\sigma}$ denotes the maximum singular value of a matrix and finally $\otimes$ is the Kronecker product.

2. A CLASS OF SWITCHING SYSTEMS

We consider a class of systems that switch between subsystems in a given order. The systems are of the form

$$\begin{aligned} \dot{\xi}(t) &= (A_0 + s(t)A_1)\,\xi(t) + B_0 + s(t)B_1 \\ &\quad + (D_0 + s(t)D_1)\,w(t) \\ \zeta(t) &= C(t)\xi(t) \end{aligned} \tag{1}$$

where $\xi \in \mathbb{R}^n$, $w \in \mathbb{R}^p$ is an external disturbance, A_i, B_i, D_i are constant matrices, C is a T_s-periodic matrix and s is the PWM function

$$s(t) = \begin{cases} 1, & t \in [kT_s, (k+d_k)T_s) \\ 0, & t \in [(k+d_k)T_s, (k+1)T_s). \end{cases} \tag{2}$$

Here, $T_s > 0$ is the period time, $k \in \mathbb{N}$ and $d_k \in [0, 1]$ is the so-called duty cycle. The duty cycle determines the fraction of each time period each mode is active and thus controls the system dynamics.

We assume [1] that for the unperturbed system (where $w \equiv 0$) there is at least one number $d^0 \in [0, 1]$ such that the system has a periodic solution $\xi^0(t) = \xi^0(t+T_s)$ when the duty cycle is fixed at d^0 ($d_k = d^0\ \forall k$). We define the deviation from ξ^0 as $x := \xi - \xi^0$ and in the sequel we consider the error dynamics

$$\begin{aligned} \dot{x}(t) &= (A_0 + s(t)A_1)\,x(t) + \big(s(t) - s^0(t)\big) \\ &\quad \times \big(A_1\xi^0(t) + B_1\big) + (D_0 + s(t)D_1)\,w(t) \\ &=: A(t)x(t) + B(t) + D(t)w(t) \\ y(t) &= C(t)x(t) \end{aligned} \tag{3}$$

where s^0 is defined according to (2) but with the duty cycle fixed at d^0 ($d_k = d^0\ \forall k$). Note that s^0 is a periodic function but s need not be periodic. Since the function s is of bounded variation, it can be shown that every solution of (3) is absolutely continuous.

In the open loop case where the duty cycle is constant and equal to d^0, the affine term $B(t)$ disappears and (3) reduces to a LTP system. However, we also consider closed loop systems. In this case the duty cycle is determined by sampling a weighted average of the state. The feedback is of the form

$$d_k = \mathrm{sat}_{[0,1]}\left(d^0 + \frac{1}{T_s}\int_{(k-1)T_s}^{kT_s} F(\tau)x(\tau)\mathrm{d}\tau\right) \tag{4}$$

where for $a < b$, $\mathrm{sat}_{[a,b]}(\cdot) := \min(\max(\cdot, a), b)$ denotes saturation between a and b and where the T_s-periodic feedback vector F is of the form

$$F(t) = \sum_{k=-N}^{N} e^{jk\omega_s t}F_k$$

where $\omega_s = 2\pi/T_s$ and where $F_k \in \mathbb{C}^{1\times n}$ satisfy $\overline{F_k} = F_{-k}$. Note that when $F_k = 0\ \forall k \neq 0$, the integral in (4) gives the average value of x over the past switch period.

3. THE DYNAMIC PHASOR MODEL

We use the idea of Caliskan et al. [1999] to represent the solution of (3) in the frequency domain where we can distinguish how the various harmonics develop over time. The n^{th} phasor (Fourier coefficient) of x is defined as

$$\langle x\rangle_n(t) := \frac{1}{T_s}\int_{t-T_s}^{t} x(\tau)e^{-jn\omega_s\tau}\mathrm{d}\tau$$

where $\omega_s = 2\pi/T_s$. Note that the phasors are defined over a moving time-window and are thus time dependent. Note also that if x is periodic with period T_s, then $\langle x\rangle_n(t)$ is constant. For brevity, the time dependence of the phasors will often be suppressed.

Using partial integration it can be shown that the phasor coefficients satisfy

$$\frac{d}{dt}\langle x\rangle_n = \left\langle \frac{d}{dt}x\right\rangle_n - jn\omega_s\langle x\rangle_n .$$

Let

$$\hat{x} := \big[\ldots\ \langle x\rangle_1^*\ \langle x\rangle_0^*\ \langle x\rangle_{-1}^*\ \ldots\big]^*$$

be an infinite vector containing the phasor coefficients of x and let $\hat{\xi}^0$, $\hat{w}$, $\hat{s}$, $\hat{s}^0$ and $\hat{y}$ be defined similarly (so that they contain the phasors of ξ^0, w, s, s^0 and y respectively). Using this notation, the phasor dynamics are written in the compact form

$$\begin{aligned} \frac{d}{dt}\hat{x} &= (-j\omega_s\hat{E}_n + I\otimes A_0 + (I\otimes A_1)\mathcal{T}[\hat{s}])\hat{x} \\ &\quad + \Big(I\otimes B_1 + (I\otimes A_1)\mathcal{T}[\hat{\xi}^0]\Big)\left(\hat{s} - \hat{s}^0\right) \\ &\quad + (I\otimes D_0 + (I\otimes D_1)\mathcal{T}[\hat{s}])\hat{w} \\ &=: (-j\omega_s\hat{E}_n + \hat{A}(\hat{s}))\hat{x} + \hat{B}(\hat{s} - \hat{s}^0) + \hat{D}(\hat{s})\hat{w} \\ \hat{y} &= \hat{C}\hat{x} \\ d_k &= \mathrm{sat}_{[0,1]}(d^0 + \mathcal{F}\pi_N\hat{x}(kT_s)). \end{aligned} \tag{5}$$

where

$$\begin{aligned} \hat{E}_n &:= \mathrm{blkdiag}(\ldots, 2I_n, I_n, 0, -I_n, -2I_n, \ldots) \\ \mathcal{F} &:= (F_{-N}, \ldots, F_0, \ldots, F_N) \end{aligned}$$

and $\hat{C} := \mathcal{T}[C(t)]$. Note that the feedback in (3) corresponds to sampling the $2N+1$ low order phasors in $\hat{x}$ and that $\hat{s}$ depends on these samples.

If the duty cycle is constant and equal to d^0 then $\mathcal{T}[\hat{s}]$ is constant and the affine term $\hat{B}$ disappears. In this case the periodically switched system (3) is represented by a linear time invariant system in the frequency domain. However, when s is determined by the feedback (4), the harmonic equations (5) are not time invariant. In this case, (5) is an infinite dimensional, non-autonomous system that depends on the sampled state with a delay. To obtain a tractable model we introduce an approximation in two steps:

In the first step we replace the phasor coefficients $\langle s\rangle_n$ with the nonlinear averaged approximation

[1] The assumption is natural for applications in power electronics as such systems are typically designed to have periodic solutions. The assumption can be verified using harmonic balance techniques.

$$s_{\mathrm{av},n}(d) = \begin{cases} d, & n = 0 \\ \dfrac{j}{n2\pi}(e^{-jn2\pi d} - 1), & n \neq 0. \end{cases} \tag{6}$$

Note that if the duty cycle is fixed so that $d_k = d\ \forall k$, then $\langle s \rangle_n (t) = s_{\mathrm{av},n}(d)$. This implies that if the duty cycle varies slowly (compared to the switch period T_s), then $s_{\mathrm{av},n}(d)$ will be a good approximation of $\langle s \rangle_n$. See Remark 2 below.

In the second step we truncate the infinite dimensional system to obtain an approximation of the lower order phasors. We also describe the dynamics as a function of the deviation $\delta = d - d^0$ from the stationary duty cycle. For a fixed integer $N \geq 0$, the approximation of the first $2N + 1$ phasors is given by the system

$$\begin{aligned} \frac{d}{dt}z &= (-j\omega_s\mathcal{N} + \mathcal{A}(\delta))z + \mathcal{B}\mathcal{S}(\delta) + \mathcal{D}(\delta)w \\ y &= \mathcal{C}z \\ \delta &= \mathrm{sat}_{[-d^0,1-d^0]}\left(\mathcal{F}z\right) \end{aligned} \tag{7}$$

where $\mathcal{C} = \pi_N \hat{C} \pi_N$, $\mathcal{B} = \pi_N \hat{B} \pi_N$, $w = \pi_N \hat{w}$ and

$$\begin{aligned} \mathcal{N} &:= \mathrm{blkdiag}(NI_n, \ldots, I_n, 0, -I_n, \ldots, -NI_n) \\ \mathcal{A}(\delta) &= \pi_N \hat{A}(S_{\mathrm{av}}(\delta))\pi_N \\ \mathcal{D}(\delta) &= \pi_N \hat{D}(S_{\mathrm{av}}(\delta))\pi_N \\ \mathcal{S}(\delta) &= \pi_N \left(S_{\mathrm{av}}(\delta) - S_{\mathrm{av}}(0)\right) \end{aligned}$$

where

$$S_{\mathrm{av}}(\delta) = [\ldots, s_{\mathrm{av},1}(d^0 + \delta), s_{\mathrm{av},0}(d^0 + \delta), s_{\mathrm{av},-1}(d^0 + \delta), \ldots]'$$

and where $\hat{A}(\cdot)$, $\hat{B}(\cdot)$, $\hat{C}(\cdot)$ and $\hat{D}(\cdot)$ are defined in (5).

Remark 1. It should be noted that z is an approximation of the $2N + 1$ low order phasors $\langle x \rangle_{-N}, \ldots, \langle x \rangle_N$ in (5) and thus, $z \in \mathbb{C}^{(2N+1)n}$. The k^{th} approximate phasor is denoted $z[k] \in \mathbb{C}^n$, $k = -N, \ldots, N$.

Remark 2. Let $\hat{x}$ be a solution of the dynamic phasor model (5) and let z be a solution of the truncated approximate model (7) with initial conditions $\hat{x}(t_0) = z(t_0) = 0$. Using an assumption on exponential stability of (7) one can show (see Almér and Jönsson [2007]) that for small disturbances $\hat{w}$ the distance between $\hat{x}(t)$ and $z(t)$ can be made arbitrarily small over infinite time intervals if the truncation N is large enough and T_s is small enough.

The system (7) is an autonomous nonlinear differential equation and therefore tractable for analysis. The important distinctions from (5) is that the state space is finite dimensional and that $s_{\mathrm{av},n}$ is a continuous function of d whereas $\langle s \rangle_n$ is determined by samples $d_k = d(kT_s)$ of d.

4. HARMONIC ANALYSIS

In the section below we consider harmonic analysis of (3) in both open and closed loop.

In the open loop case (where the duty cycle is constant and equal to d^0), the system (3) reduces to a LTP system. To investigate this characteristic we review recent work on the so-called harmonic transfer function (HTF), see Wereley and Hall [1990], Möllerstedt [2000], Sandberg et al. [2005], and provide a new interpretation.

It is shown that the HTF also describes the harmonic coupling between the phasor coefficients of input and output of the LTP system. It follows that the dynamic phasor model provides an explicit formula for the HTF.

For analysis of the closed loop case we use the above mentioned connection between the HTF and the dynamic phasor model to derive an approximate counterpart of the HTF for the closed loop system. The new transfer function maps periodic inputs to the corresponding (approximate) periodic solutions of the closed loop system.

4.1 The harmonic transfer function; a review

LTP systems do not have the property of frequency separation which is characteristic for LTI systems. If the input to a LTP system is a sinusoid with angular frequency ω, then the steady state output will be a sum of sinusoids with angular frequencies $\omega + k\omega_s$ where $k \in \mathbb{Z}$ and ω_s is the angular frequency of the system. The HTF generalizes the concept of transfer function to LTP systems and is thus a powerful tool for illustrating the coupling between the frequencies in the input and output signals.

Consider the LTP system

$$\begin{aligned} \dot{x}(t) &= A_p(t)x(t) + D_p(t)w(t) \\ y(t) &= C_p(t)x(t) \end{aligned} \tag{8}$$

where A_p, D_p and C_p are T_s-periodic matrices. Under loose assumptions, see Sandberg et al. [2005], the impulse response h of (8) can be expanded in a Fourier series and the response y to the input w can be expressed as the convolution

$$\begin{aligned} y(t) &= \int_0^t \sum_{k=-\infty}^{\infty} h_k(t-\tau)e^{jk\omega_s t}w(\tau)\mathrm{d}\tau \\ &= \sum_{k=-\infty}^{\infty} \left(h_k(\cdot)e^{jk\omega_s(\cdot)} * w(\cdot)e^{jk\omega_s(\cdot)}\right)(t) \end{aligned} \tag{9}$$

where h_k are the Fourier coefficients of h and $*$ denotes the convolution operation. Let $Y(\omega) := (\mathfrak{F}y)(\omega)$, $W(\omega) := (\mathfrak{F}w)(\omega)$ be the Fourier transform of the output and input respectively. By applying the Fourier transform to (9) one can express $Y(\omega + n\omega_s)$, $n \in \mathbb{Z}$ as a function of $W(\omega + k\omega_s)$, $k \in \mathbb{Z}$ as shown in (10) below. Here, the doubly infinite matrix $\mathcal{H}(\omega)$ is the harmonic transfer function and the entries $H_k(\omega) = (\mathfrak{F}h_k)(\omega)$ are the Fourier transform of the Fourier coefficients of h.

As was stated above, the HTF extends the notion of transfer function to LTP systems. From the transfer function of a LTI system one can immediately determine the response to a sinusoidal input. Next we show that the HTF has the corresponding property for LTP systems. Let the input signal be $w(t) = \sin(\omega t) = \frac{1}{2j}\left(e^{j\omega t} - e^{-j\omega t}\right)$ and assume that this signal has been applied since time $t = -\infty$. The output becomes

$$\begin{aligned} y(t) &= \sum_{k=-\infty}^{\infty} \int_0^{\infty} h_k(\tau)w(t-\tau)\mathrm{d}\tau e^{jk\omega_s t} \\ &= \sum_{k=-\infty}^{\infty} \int_0^{\infty} h_k(\tau)\frac{1}{2j}\left(e^{j\omega(t-\tau)} - e^{-j\omega(t-\tau)}\right)\mathrm{d}\tau e^{jk\omega_s t} \\ &= \sum_{k=-\infty}^{\infty} H_k(\omega)\frac{1}{2j}e^{j(\omega+k\omega_s)t} - H_k(-\omega)\frac{1}{2j}e^{-j(\omega-k\omega_s)t}. \end{aligned}$$

$$\begin{bmatrix} \vdots \\ Y(\omega+\omega_s) \\ Y(\omega) \\ Y(\omega-\omega_s) \\ \vdots \end{bmatrix} = \underbrace{\begin{bmatrix} \ddots & & \vdots & & \iddots \\ & H_0(\omega+\omega_s) & H_1(\omega) & H_2(\omega-\omega_s) & \\ \dots & H_{-1}(\omega+\omega_s) & H_0(\omega) & H_1(\omega-\omega_s) & \dots \\ & H_{-2}(\omega+\omega_s) & H_{-1}(\omega) & H_0(\omega-\omega_s) & \\ \iddots & & \vdots & & \ddots \end{bmatrix}}_{\mathcal{H}(\omega)} \begin{bmatrix} \vdots \\ W(\omega+\omega_s) \\ W(\omega) \\ W(\omega-\omega_s) \\ \vdots \end{bmatrix} \tag{10}$$

We now use the relation $H_k(-\omega) = \overline{H_{-k}(\omega)}$ and reorder the sum above. Denoting the real and imaginary parts of H_k by H_k^r and H_k^j respectively we have

$$\begin{aligned} y(t) &= \sum_{k=-\infty}^{\infty} H_k(\omega)\frac{1}{2j}e^{j(\omega+k\omega_s)t} - \overline{H_k(\omega)}\frac{1}{2j}e^{-j(\omega+k\omega_s)t} \\ &= \sum_{k=-\infty}^{\infty} H_k^r \sin((\omega+k\omega_s)t) + H_k^j \cos((\omega+k\omega_s)t) \\ &= \sum_{k=-\infty}^{\infty} |H_k(\omega)| \sin((\omega+k\omega_s)t+\phi_k) \end{aligned} \tag{11}$$

where $\phi_k = \arg H_k(\omega)$.

An explicit formula for the HTF can be obtained from the dynamic phasor model corresponding to (8). To see this, let $\langle w\rangle_n$ be the n^{th} phasor coefficient of w. One can show that the Fourier transform of $\langle w\rangle_n$ satisfies

$$(\mathfrak{F}\langle w\rangle_n)(\omega) = W(\omega+n\omega_s)\frac{1-e^{-j\omega T_s}}{j\omega T_s}.$$

Using the relation (9) we can also derive the equality

$$(\mathfrak{F}\langle y\rangle_n)(\omega) = \sum_{k=-\infty}^{\infty} H_k(\omega+(n-k)\omega_s)W(\omega+(n-k)\omega_s)\frac{1-e^{-j\omega T_s}}{j\omega T_s}.$$

By identifying $\mathfrak{F}\langle w\rangle_{n-k}$ in this expression we recognize that the relation between $\mathfrak{F}\langle w\rangle_n$ and $\mathfrak{F}\langle y\rangle_n$ is given by the HTF, i.e.,

$$\begin{bmatrix} \vdots \\ (\mathfrak{F}\langle y\rangle_1)(\omega) \\ (\mathfrak{F}\langle y\rangle_0)(\omega) \\ (\mathfrak{F}\langle y\rangle_{-1})(\omega) \\ \vdots \end{bmatrix} = \mathcal{H}(\omega) \begin{bmatrix} \vdots \\ (\mathfrak{F}\langle w\rangle_1)(\omega) \\ (\mathfrak{F}\langle w\rangle_0)(\omega) \\ (\mathfrak{F}\langle w\rangle_{-1})(\omega) \\ \vdots \end{bmatrix}.$$

The relation above implies that the HTF of a LTP system can be derived by applying the Fourier transform to the corresponding dynamic phasor model.

Let us now consider the dynamic phasor model of the open loop system (3), i.e.,

$$\begin{aligned} \frac{d}{dt}\hat{x} &= \left(-j\omega_s\hat{E}_n + \hat{A}(\hat{s}^0)\right)\hat{x} + \hat{D}(\hat{s}^0)\hat{w} \\ \hat{y} &= \hat{C}\hat{x}. \end{aligned} \tag{12}$$

By formally applying the Fourier transform to (12) we obtain an explicit formula for the HTF $\mathcal{H}(\omega)$ corresponding to the open loop system (3)

$$\mathcal{H}(\omega) = \hat{C}(j\omega I - (-j\omega_s\hat{E}_n + \hat{A}(\hat{s}^0)))^{-1}\hat{D}(\hat{s}^0). \tag{13}$$

It should be noted that the open loop system (3) does *not* satisfy the assumptions in Sandberg et al. [2005] which are sufficient to establish that the HTF is well-posed. The expression (13) is strictly formal and is merely used to show an analogy between the HTF of a LTP system and the transfer function derived in Section 4.2 below. However, it can be shown (Almér and Jönsson [2007]) that (12) is approximated arbitrarily well by square truncations. Thus, (13) can be interpreted as a limit of truncated HTFs.

In Section 4.2 we consider (3) in closed loop. In this case, it can be shown (Almér and Jönsson [2007]) that for small disturbances w, the truncated phasor model (7) approximates the dynamic phasor model arbitrarily well. The approximate model (7) can thus be used to extend the notion of HTF to describe periodic solutions of the non-periodic closed loop system (3).

4.2 A HTF approximation of the closed loop system

In the section above it was shown that the HTF of a LTP system can be obtained from the corresponding dynamic phasor model as shown in (13). In the remainder of the section we use this fact to extend the notion of harmonic transfer function to describe periodic solutions of non-periodic systems. We consider the system (3) in closed loop (then the system is no longer periodic). We use the corresponding dynamic phasor model to approximate the steady state response to a periodic disturbance by solving harmonic balance equations. A first order harmonic balance approximation where all frequencies except the zero and first order term are neglected results in a harmonic transfer function that maps periodic disturbances to the corresponding (approximate) periodic output.

We consider the system (3) with the feedback defined in (4). To estimate the effect of the disturbance on the system we consider the truncated averaged approximate phasor model (7) corresponding to the system above.

In the sequel we assume that the steady state response of the truncated phasor system (7) to a periodic disturbance is also periodic. I.e., we assume that the steady state response to a periodic disturbance $\mathcal{w}(t)$ with frequency ω is periodic with period $T = 2\pi/\omega$ and given as

$$\begin{aligned} \mathcal{w}(t) &= \sum_{k=-\infty}^{\infty} \mathcal{w}_k e^{jk\omega t}, \quad \mathcal{z}(t) = \sum_{k=-\infty}^{\infty} \mathcal{z}_k e^{jk\omega t} \\ \delta(t) &= \sum_{k=-\infty}^{\infty} \delta_k e^{jk\omega t} \end{aligned} \tag{14}$$

and finally $\mathcal{y}(t) = \mathcal{C}\mathcal{z}(t)$. The assumption above is nontrivial, but in section 4.2.3 below we provide conditions under which the assumption is valid for small disturbances $\mathcal{w}$. We state the corresponding harmonic balance equations and suggest an approximate solution to the nonlinear equations. The approximate solution is represented by a transfer function from disturbance to output.

Since $\delta(t)$ is periodic, $\mathcal{A}(\delta(t))$, $\mathcal{S}(\delta(t))$ and $\mathcal{D}(\delta(t))$ are also periodic and can be represented by the Fourier series expansions

$$\mathcal{A}(\delta(t)) = \sum_{k=-\infty}^{\infty} \mathcal{A}_k e^{jk\omega t}, \quad \mathcal{S}(\delta(t)) = \sum_{k=-\infty}^{\infty} \mathcal{S}_k e^{jk\omega t}$$
$$\mathcal{D}(\delta(t)) = \sum_{k=-\infty}^{\infty} \mathcal{D}_k e^{jk\omega t} \tag{15}$$

where the (constant) Fourier coefficients $\mathcal{A}_k$, $\mathcal{S}_k$, $\mathcal{D}_k$ are functions of $\{\delta_k\}_{k=-\infty}^{\infty}$. We now ignore the saturation in (7) and consider the harmonic balance equation associated with the periodic solution (14). We introduce

$$\hat{z} = [\ldots, z_1^*, z_0^*, z_{-1}^*, \ldots]^*$$
$$\hat{\delta} = [\ldots, \delta_1^*, \delta_0^*, \delta_{-1}^*, \ldots]^*$$
$$\hat{w} = [\ldots, w_1^*, w_0^*, w_{-1}^*, \ldots]^*$$

and the harmonic balance equations can be expressed as

$$\begin{aligned} j\omega \hat{E}_q \hat{z} &= (-j\omega_s \hat{\mathcal{N}} + \hat{\mathcal{A}}(\hat{\delta}))\hat{z} + \hat{\mathcal{B}}\hat{\mathcal{S}}(\hat{\delta}) + \hat{\mathcal{D}}(\hat{\delta})\hat{w} \\ \hat{y} &= \hat{\mathcal{C}}\hat{z} \\ \hat{\delta} &= \hat{\mathcal{F}}\hat{z} \end{aligned} \tag{16}$$

where $\hat{\mathcal{A}}(\hat{\delta}) := \mathcal{T}[\mathcal{A}(\delta(t))]$, $\hat{\mathcal{D}}(\hat{\delta}) := \mathcal{T}[\mathcal{D}(\delta(t))]$ are the infinite block Toeplitz matrices determined by the Fourier coefficients in (15) and where $\hat{\mathcal{N}} := I \otimes \mathcal{N}$, $\hat{\mathcal{C}} := I \otimes \mathcal{C}$, $\hat{\mathcal{F}} = I \otimes \mathcal{F}$, $\hat{\mathcal{B}} = I \otimes \mathcal{B}$ and where

$$\hat{E}_q := \text{blkdiag}(\ldots, 2I_q, I_q, 0, -I_q, -2I_q, \ldots)$$
$$\hat{\mathcal{S}}(\hat{\delta}) := (\ldots, \mathcal{S}_1, \mathcal{S}_0, \mathcal{S}_{-1} \ldots)'$$

where $q = (2N+1)n$.

Approximate solution to the harmonic balance equations To find an approximate solution to the (highly nonlinear) harmonic balance equations (16) we use a first order approximation of $S_{\text{av}}(\delta)$. We have for $n \neq 0$

$$\begin{aligned} s_{\text{av},n}(\delta(t)) &= \frac{j}{n2\pi}\left(e^{-jn2\pi(d^0+\delta(t))} - 1\right) \\ &\approx \frac{j}{n2\pi}\left(e^{-jn2\pi d^0}\left(1 - jn2\pi \sum_{k=-\infty}^{\infty} \delta_k e^{jk\omega t}\right) - 1\right) \\ &= \frac{j}{n2\pi}\left(e^{-jn2\pi d^0} - 1\right) + e^{-jn2\pi d^0} \sum_{k=-\infty}^{\infty} \delta_k e^{jk\omega t} \\ &= s_{\text{av},n}(d^0) + e^{-jn2\pi d^0}\delta(t) \end{aligned}$$

where we used the approximation $e^x \approx 1 + x$. Since $s_{\text{av},0}(\delta(t)) = d^0 + \delta(t)$ the approximation of $S_{\text{av}}(\delta)$ is written

$$S_{\text{av}}(\delta) \approx S_{\text{av}}(0) + \Psi\delta(t) \tag{17}$$

where

$$\Psi = [\ldots, e^{-j2\pi d^0 2}, e^{-j2\pi d^0}, 1, e^{j2\pi d^0}, e^{j2\pi d^0 2}, \ldots]'.$$

The approximation above is used to derive linear approximations (linear in δ_k) of the Fourier coefficients $\mathcal{A}_k$, $\mathcal{S}_k$, $\mathcal{D}_k$. By using the linearity of $\mathcal{T}_N[\cdot]$ one can show that

$$\mathcal{A}_k(\hat{\delta}) \approx \begin{cases} \mathcal{A}(0) + \delta_0(I \otimes A_1)\mathcal{T}_N[\Psi], & k = 0 \\ \delta_k(I \otimes A_1)\mathcal{T}_N[\Psi], & k \neq 0 \end{cases}$$
$$\mathcal{S}_k(\hat{\delta}) \approx \Psi_N \delta_k$$
$$\mathcal{D}_k(\hat{\delta}) \approx \begin{cases} \mathcal{D}(0) + \delta_0(I \otimes D_1)\mathcal{T}_N[\Psi], & k = 0 \\ \delta_k(I \otimes D_1)\mathcal{T}_N[\Psi], & k \neq 0 \end{cases}.$$

where $\mathcal{A}(0) = \pi_N \hat{A}(S_{\text{av}}(0))\pi_N = \pi_N \hat{A}(\hat{s}^0)\pi_N$, $\mathcal{D}(0) = \pi_N \hat{D}(S_{\text{av}}(0))\pi_N = \pi_N \hat{D}(\hat{s}^0)\pi_N$ and $\Psi_N = \pi_N \Psi$.

The Fourier coefficients in (16) are replaced by the linear approximations above and all cross terms are dropped. We then obtain a linear system of equations which is written

$$\begin{aligned} j\omega \hat{E}_q \hat{z} &= (-j\omega_s \hat{\mathcal{N}} + \hat{\mathcal{A}}_0)\hat{z} + \hat{\mathcal{B}}\hat{\Psi}\hat{\delta} + \hat{\mathcal{D}}_0\hat{w} \\ \hat{y} &= \hat{\mathcal{C}}\hat{z} \\ \hat{\delta} &= \hat{\mathcal{F}}\hat{z} \end{aligned} \tag{18}$$

where $\hat{\Psi} := I \otimes \Psi_N$, $\hat{\mathcal{A}}_0 := I \otimes \mathcal{A}(0)$ and $\hat{\mathcal{D}}_0 := I \otimes \mathcal{D}(0)$.

Since the matrices in (18) are block diagonal, there is no coupling between the Fourier coefficients. The approximation yields a linear map from w_k to y_k which is written

$$y_k = \mathcal{H}_{cl}(k\omega)w_k. \tag{19}$$

where

$$\mathcal{H}_{cl}(\omega) := \mathcal{C}\left(j\omega I - (-j\omega_s\mathcal{N} + \mathcal{A}(0)) - \mathcal{B}\Psi_N\mathcal{F}\right)^{-1}\mathcal{D}(0).$$

is a transfer function of dimension $(2N+1)n$. As was noted above, $\mathcal{A}(0) = \pi_N \hat{A}(\hat{s}^0)\pi_N$ and $\mathcal{D}(0) = \pi_N \hat{D}(\hat{s}^0)\pi_N$ and $\mathcal{N} = \pi_N \hat{E}_n \pi_N$. In light of the expression (13) for the HTF, $\mathcal{H}_{cl}$ can be seen as a truncated version of $\mathcal{H}$ with an additional term $\mathcal{B}\Psi_N\mathcal{F}$ representing the effect of the feedback. In the section below we use the individual entries of $\mathcal{H}_{cl}$. They are indexed as

$$\mathcal{H}_{cl}(\omega) = \begin{bmatrix} \ddots & & \vdots & & ⋰ \\ & H_{cl,1,1}(\omega) & H_{cl,1,0}(\omega) & H_{cl,1,-1}(\omega) & \\ \cdots & H_{cl,0,1}(\omega) & H_{cl,0,0}(\omega) & H_{cl,0,-1}(\omega) & \cdots \\ & H_{cl,-1,1}(\omega) & H_{cl,-1,0}(\omega) & H_{cl,-1,-1}(\omega) & \\ ⋰ & & \vdots & & \ddots \end{bmatrix} \tag{20}$$

where $H_{cl,0,0}$ denotes the central block of the matrix. It should be noted that $\mathcal{H}_{cl}$ does not have the same structure as (10). This structure is lost because of the feedback and pulse modulation.

Connection to the time domain The equation (19) gives an approximate steady state response of the phasor dynamic system (7) in terms of a transfer function from input w to output y. To connect this result with the time domain system (3) we will now express the time domain representation of the (approximate) steady state response of (7) to a sinusoidal input as described by (19).

Let the disturbance in (3) be $w(t) = \sin(\omega t)$ and assume that $\omega \ll \omega_s$ the phasors may then be approximated as $\langle w \rangle_0(t) \approx \sin(\omega t)$ and $\langle w \rangle_n(t) \approx 0$ for $n \neq 0$. I.e., the truncated phasor representation of $w(t)$ is approximately

$$w(t) = w_{-1}e^{-j\omega t} + w_1 e^{j\omega t}$$

where $w_{\pm 1} = [0, \ldots, 0, \pm\frac{1}{2j}, 0, \ldots, 0]'$. The frequency separation property of (18) implies that the only nonzero coefficients in $\hat{y}$ are $\mathcal{Y}_1$ and $\mathcal{Y}_{-1}$. The approximate response of (3) to the sinusoidal disturbance is therefore given by

$$y(t) \approx \sum_{n=-N}^{N} \mathcal{Y}[n]e^{jn\omega_s t}$$
$$\approx \sum_{n=-N}^{N} \left\{\mathcal{Y}_1 e^{j\omega t} + \mathcal{Y}_{-1}e^{-j\omega t}\right\}[n]e^{jn\omega_s t}$$
$$= \sum_{n=-N}^{N} \left\{\mathcal{H}_{cl}(\omega)\mathcal{W}_1 e^{j\omega t} + \mathcal{H}_{cl}(-\omega)\mathcal{W}_{-1}e^{-j\omega t}\right\}[n]e^{jn\omega_s t}.$$

where $\mathcal{Y}[n]$ denotes the n^{th} approximate phasor coefficient of y (see remark 1). We now use that only the zero coefficient of $\mathcal{W}_{\pm 1}$ is non-zero and that $H_{cl,n,0}(-\omega) = \overline{H_{cl,-n,0}(\omega)}$ (see (20) for the definition). It follows

$$y(t) \approx$$
$$\sum_{n=-N}^{N} H_{cl,n,0}(\omega)\frac{1}{2j}e^{j(\omega+n\omega_s)t} - H_{cl,n,0}(-\omega)\frac{1}{2j}e^{-j(\omega-n\omega_s)t}$$
$$= \sum_{n=-N}^{N} H_{cl,n,0}(\omega)\frac{1}{2j}e^{j(\omega+n\omega_s)t} - \overline{H_{cl,n,0}(\omega)}\frac{1}{2j}e^{-j(\omega+n\omega_s)t}$$
$$= \sum_{n=-N}^{N} |H_{cl,n,0}(\omega)| \sin((\omega+n\omega_s)t + \phi_{n,0}) \quad (21)$$

where $H_{cl,n,k}$ is the $(n,k)-$block of H_{cl} (see (20)), where $\phi_{n,0} = \arg H_{cl,n,0}(\omega)$ and where the last equality was proved in the previous section. The expression above is analogous to the expression given in Section 4.1 for the response of a LTP system to a sinusoidal input.

Existence of solution to the harmonic balance equations To justify the assumption that the harmonic balance equations have a solution we show that for small disturbances $\mathcal{W}$ this is indeed the case. The harmonic balance equations (16) have a solution iff there is a solution $\hat{\delta}$ to

$$\hat{\delta} = \hat{\mathcal{F}}\mathcal{H}_1(\omega)\hat{\mathcal{D}}_0\hat{\mathcal{W}} + \hat{\mathcal{F}}\mathcal{H}_1(\omega)\left(\hat{\Delta}_1(\hat{\delta})\hat{\mathcal{W}} + \hat{\Delta}_2(\hat{\delta})\right)$$

where

$$\mathcal{H}_1(\omega) = \left(j\omega\hat{E}_q - (-j\omega_s\hat{\mathcal{N}} + \hat{\mathcal{A}}_0) - \hat{\mathcal{B}}\hat{\Psi}\hat{\mathcal{F}}\right)^{-1} \quad (22)$$
$$\hat{\Delta}_1(\hat{\delta}) = \left(I - (\hat{\mathcal{A}}(\hat{\delta}) - \hat{\mathcal{A}}_0)\mathcal{H}_1(\omega)\right)^{-1}\hat{\mathcal{D}}(\hat{\delta}) - \hat{\mathcal{D}}_0$$
$$\hat{\Delta}_2(\hat{\delta}) = \left(I - (\hat{\mathcal{A}}(\hat{\delta}) - \hat{\mathcal{A}}_0)\mathcal{H}_1(\omega)\right)^{-1}\hat{\mathcal{B}}\left(\hat{\mathcal{S}}(\hat{\delta}) - \hat{\Psi}\hat{\delta}\right).$$

From (18) it is clear that the first term $\hat{\mathcal{F}}\mathcal{H}_1(\omega)\hat{\mathcal{D}}_0\hat{\mathcal{W}}$ is the approximate solution given by the linearized harmonic balance equations while the second term is a higher order function of $\hat{\delta}$. We note that the operators $\mathcal{H}_1(\omega)$, $j\omega\hat{E}_q\mathcal{H}_1(\omega)$ and $\hat{\mathcal{F}}\mathcal{H}_1$ are block diagonal and denote the induced l_2-norms as $\|\mathcal{H}_1(\omega)\|$, $\|j\omega\hat{E}_q\mathcal{H}_1(\omega)\|$ and $\|\hat{\mathcal{F}}\mathcal{H}_1\|$.

Let

$$H(\hat{\delta},\hat{\mathcal{W}}) = \hat{\mathcal{F}}\mathcal{H}_1(\omega)\left(\left(\hat{\mathcal{D}}_0 + \hat{\Delta}_1(\hat{\delta})\right)\hat{\mathcal{W}} + \hat{\Delta}_2(\hat{\delta})\right). \quad (23)$$

There exists a solution to the harmonic balance equations (16) iff there exists a solution to the fixed point equation $\hat{\delta} = H(\hat{\delta},\hat{\mathcal{W}})$. Clearly, for $\hat{\mathcal{W}} = 0$ there is the solution $0 = H(0,0)$. One can show that three is a solution

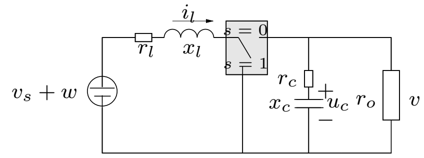

Fig. 1. Synchronous boost converter with disturbance w in the source voltage.

also for nonzero disturbances $\hat{\mathcal{W}}$, provided they are small enough. The claim is formalized in the following theorem.

Theorem 1. Let $r_w > 0$ and let $r > 0$ be such that

$$\sup_{|\delta|<r} \bar{\sigma}(\mathcal{A}(\delta) - \mathcal{A}(0)) < 1/(2\|\mathcal{H}_1\|) \quad (24)$$
$$C_1 r_w + C_3(r)r^2 - (1 - C_2(r)r_w)r < 0 \quad (25)$$

where $C_i > 0$ are defined as

$$C_1 = \|\hat{\mathcal{F}}\mathcal{H}_1\|\bar{\sigma}(\mathcal{D}(0))$$
$$C_2(r) = \|\hat{\mathcal{F}}\mathcal{H}_1\|\sqrt{2}\left(\gamma_2^2 + \left(\gamma_1(\gamma_2 r + \bar{\sigma}(\mathcal{D}(0)))\right.\right.$$
$$\left.\left.+ \frac{c_5}{\sqrt{2}} + 2\pi c_3\right)^2\right)^{1/2}$$
$$C_3(r) = \|\hat{\mathcal{F}}\mathcal{H}_1\| 2\sqrt{2}c_1\bar{\sigma}(\mathcal{B})\left(1 + \left(\frac{1}{2\sqrt{2}} + \gamma_1 r\right)^2\right)^{1/2}$$

where

$$\gamma_1 = c_4\left(\|\mathcal{H}_1\|^2 + T^2\|j\omega\hat{E}_q\mathcal{H}_1\|^2\right)^{1/2} + c_2 T\|j\omega\hat{E}_q\mathcal{H}_1\|$$
$$\gamma_2 = 2\left(c_3 + c_2\|\mathcal{H}_1\|\bar{\sigma}(\mathcal{D}(0))\right)$$

and where $c_i > 0$, $i = 1, \ldots, 5$ are constants satisfying

$$\sup_{|\delta|<r} |\mathcal{S}(\delta) - \Psi_N\delta| < c_1 r^2$$
$$\sup_{|\delta|<r} |\mathcal{S}'(\delta) - \Psi_N| < c_1 r$$
$$\sup_{|\delta|<r} \bar{\sigma}(\mathcal{A}(\delta) - \mathcal{A}(0)) < c_2 r$$
$$\sup_{|\delta|<r} \bar{\sigma}(\mathcal{D}(\delta) - \mathcal{D}(0)) < c_3 r$$
$$\sup_{|\delta|<r} \bar{\sigma}(\mathcal{A}'(\delta)) < c_4$$
$$\sup_{|\delta|<r} \bar{\sigma}(\mathcal{D}'(\delta)) < c_5.$$

Then the harmonic balance equations (16) have a solution for all $\hat{\mathcal{W}}$ such that $2(\|\hat{\mathcal{W}}\|_{l_2}^2 + T^2\|j\omega\hat{E}_q\hat{\mathcal{W}}\|_{l_2}^2) \le r_w^2$.

Proof 1. The result follows from an application of the Schauder fixed point theorem. A complete proof can be found in Almér and Jönsson [2007].

5. EXAMPLE

To illustrate the theory presented in the paper we consider a simple numerical example; the synchronous boost (step-up) converter depicted in Fig. 1. The system is considered in both open and closed loop.

The boost converter is of the form (1) with state $\xi = [i_l \ v_c]'$ where i_l is the inductor current and v_c is the capacitor voltage. The system matrices are

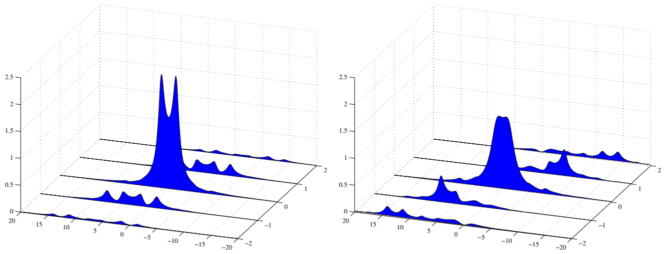

Fig. 2. Harmonic transfer function $\mathcal{H}(\omega)$ (left) and closed loop harmonic transfer function $\mathcal{H}_{cl}(\omega)$ (right) of the boost converter.

$$A_0 = \begin{bmatrix} -\dfrac{1}{x_l}\left(r_l + \dfrac{r_o r_c}{r_o + r_c}\right) & -\dfrac{1}{x_l}\dfrac{r_o}{r_o + r_c} \\ \dfrac{1}{x_c}\dfrac{r_o}{r_o + r_c} & -\dfrac{1}{x_c}\dfrac{1}{r_o + r_c} \end{bmatrix}, \; B_0 = \begin{bmatrix} v_s/x_l \\ 0 \end{bmatrix}$$

$$A_1 = \begin{bmatrix} \dfrac{1}{x_l}\dfrac{r_o r_c}{r_o + r_c} & \dfrac{1}{x_l}\dfrac{r_o}{r_o + r_c} \\ -\dfrac{1}{x_c}\dfrac{r_o}{r_o + r_c} & 0 \end{bmatrix}, \; D_0 = \begin{bmatrix} 1/x_l \\ 0 \end{bmatrix}$$

B_1 and D_1 are zero and $C(t) = [0\ 1]$. I.e., we take the capacitor voltage as the output signal. The dynamics have been scaled to obtain switch period $T_s = 1$ and the parameters are expressed in the per unit system. They are $x_l = 3/10\pi$ p.u., $x_c = 70/10\pi$ p.u., $r_l = 0.05$ p.u., $r_c = 0.005$ p.u., $r_o = 1$ p.u. and the source voltage is $v_s = 0.75$ p.u.

The reference output voltage is $v_{\text{ref}} = 1$. The stationary duty cycle d^0 is chosen to make the average output voltage equal v_{ref} and the corresponding periodic stationary solution is denoted ξ^0. The error dynamics $x := \xi - \xi^0$ is considered in both open loop (i.e., $d_k = d^0\ \forall k$) and in closed loop with the linear feedback

$$d_k = \text{sat}_{[0,1]}(d^0 + F\pi_0\hat{x}(kT)), \quad F = [-0.1021, 0.1555].$$

Note that we only use the average value $\pi_0\hat{x}$ of the state in the feedback.

In the open loop case we truncate the dynamic phasor model corresponding to (8) and apply the Fourier transform to obtain a truncated HTF. The gains $|H_k(\omega)|$ (see 10) of the truncated HTF are plotted for $k = -2, \ldots, 2$ in Fig. 2.

In the closed loop case we consider the averaged dynamic phasor system (7) and derive the corresponding closed loop harmonic transfer function $\mathcal{H}_{cl}(\omega)$. The gains $|H_{cl,k,0}(\omega)|$ (see 20) are plotted for $k = -2, \ldots, 2$ in Fig. 2. Recall that $H_{cl,k,0}$ is the $(k, 0)$ – block of $\mathcal{H}_{cl}$ used in (21).

The approximations given by the harmonic transfer functions are verified by simulations in Matlab. The boost converter is subjected to the disturbance $w(t) = a\sin(2\pi f t)$ with amplitude $a = 0.1$ and frequency $f = 0.1$. The steady state response of the open and closed loop system is shown in Fig. 3. The approximate responses given by the truncated transfer functions are also plotted and they correspond well with the simulated results. I,e., the formulas (11) and (21) yields a response which is close to the one observed in the simulation. We also note that the approximation to a large extent capture the nonlinear effects caused by the switching, see the close ups in Fig. 3.

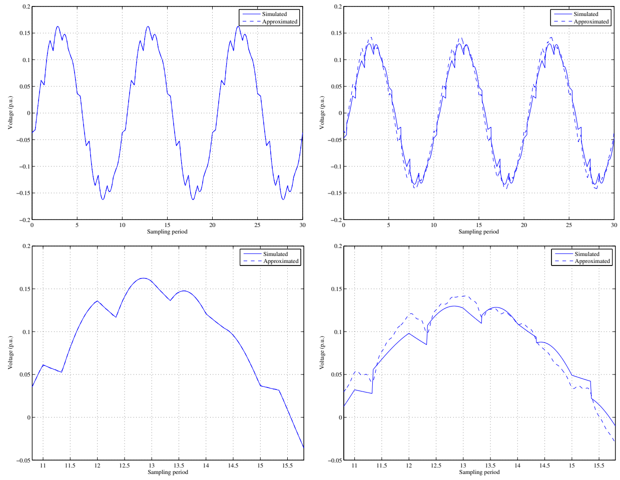

Fig. 3. Steady state response of the capacitor voltage v_c to the input disturbance $w(t) = a\sin(2\pi f t)$, $a = 0.1$, $f = 0.1$. The top left figure shows the open loop response and the top right figure shows the closed loop response. The two figures below show close ups of the responses.

6. CONCLUDING REMARKS

We have shown how the dynamic phasor model in Caliskan et al. [1999] can be used for harmonic analysis of pulse width modulated systems. A connection between the dynamic phasor model and the HTF was shown and the dynamic phasor model was used to extend the notion of HTF to closed loop, non periodic systems. The method was applied to a realistic example and it was shown that the approximation given by the HTF correspond well with simulated results.

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