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An Observer that Converges in Finite Time Due to Measurement-based State Updates

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Abstract: This paper presents a new observer that estimates the exact state of a linear continuous-time system in predetermined finite time. The finite convergence time of the proposed observer is achieved by updating the observer state based on the difference between the measured output and the estimated output at discrete time instants. Simulation results are presented to illustrate the convergence behavior and the applicability of the proposed observer.

1. INTRODUCTION

In the last decades different approaches, see e.g. [Medvedev and Toivonen (1994); Han et al. (2001); Engel and Kreisselmeier (2002); Byrski (2003); Raff and Allgöwer (2007)], have been proposed to design observers that estimate the state of a linear continuous-time system in predetermined finite time. Note that the convergence time of these observers, distinct from the observer [Haskara et al. (1996)], is independent of the magnitude of the initial estimation error. Unfortunately, these observers are expensive from a computational point of view because they require a large (infinite) amount of memory due to the storage of trajectory pieces [Han et al. (2001); Byrski (2003); Medvedev and Toivonen (1994); Engel and Kreisselmeier (2002)], and/or the instantaneous solution of convolution integrals over a finite time horizon [Han et al. (2001); Byrski (2003); Medvedev and Toivonen (1994)], and/or an increased order of the observer dynamics [Engel and Kreisselmeier (2002); Raff and Allgöwer (2007)].

This paper presents a new observer with predetermined finite convergence time for linear continuous-time systems which avoids (reduces) the above mentioned implementation problems. The finite convergence time of the proposed observer is simply achieved by updating the observer state based on the difference between the measured output and the estimated output at discrete time instants. These updates of the observer state, which can be easily realized if the observer is implemented e.g. on a microcontroller, cause impulsive behavior of the observer dynamics. Compared to the observer [Raff and Allgöwer (2007)], which also exhibits impulsive dynamical behavior due to the update of the observer state (observer order $2n$) at one particular time instant, the proposed observer achieves the finite convergence time with a sequence of updates of the observer state (observer order n). Finally, the convergence behavior of the observer is illustrated via an example, the mass spring system [Geromel and de Oliveira (2001)].

The remainder of the paper is organized as follows: In Section 2 the proposed observer, that converges in predetermined finite time, is presented. Simulation results are shown in Section 3 and Section 4 concludes the paper with a summary and an outlook.

2. MAIN RESULT

Consider the linear continuous-time system

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(t_0) = x_0 \tag{1a}$$
$$y(t) = Cx(t), \tag{1b}$$

where $x \in \mathbb{R}^n$ is the system state, $x_0 \in \mathbb{R}^n$ the initial condition, $u \in \mathbb{R}^p$ the input, $y \in \mathbb{R}^q$ the output, and $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times p}$, $C \in \mathbb{R}^{q\times n}$ are constant matrices. It is assumed that system (1) is observable. The proposed observer for system (1) is

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + L(y(t) - C\hat{x}(t)),\ t \neq t_k, \tag{2a}$$
$$\hat{x}(t_k^+) = \hat{x}(t_k) + K_k(y(t_k) - C\hat{x}(t_k)) \qquad t = t_k, \tag{2b}$$
$$\hat{x}(t_0^+) = \hat{x}_0, \qquad k = 1, 2, .. \tag{2c}$$

where $\hat{x} \in \mathbb{R}^n$ is the observer state, $\hat{x}_0 \in \mathbb{R}^n$ the observer initial condition, $L \in \mathbb{R}^{n\times q}, K_k \in \mathbb{R}^{n\times q}$ are observer matrices, and t_k is a time sequence satisfying $0 \leq t_0 < t_1 < \ldots < t_k < t_{k+1} < \ldots$, $\lim_{k\to\infty} t_k = \infty$, $\delta = t_k - t_{k-1} =$ constant, and $\delta > 0$. Moreover, $\hat{x}(t_k^+) = \lim_{h\to 0} \hat{x}(t_k + h)$ and, without loss of generality [Yang (2001)], it is assumed that $\hat{x}(t_k) = \hat{x}(t_k^-) = \lim_{h\to 0} \hat{x}(t_k - h)$ and $y(t_k) = y(t_k^-) = \lim_{h\to 0} y(t_k - h)$ $(h > 0)$. Observer (2) differs from a classical Luenberger observer [Luenberger (1966)] by the additional equation (2b). Due to the updates of the observer state via equation (2b), the dynamics of observer (2) exhibits impulsive dynamical behavior. These updates of the observer state can be easily realized if observer (2) is (approximately) implemented e.g. on a digital computer or on a microchip since in that case the observer state can be set to any value at any time instant t_k. The next theorem states how the observer parameters in (2), that are L, K_k, and δ, have to be chosen such that observer (2) converges in finite time.

Theorem 1. Let the matrices $L,\ P \in \mathbb{R}^{n\times q}$ and the constant δ, that fixes the convergence time, be chosen such that the matrix $Q = A - LC$ is Hurwitz and that the matrix $R = \exp(Q\delta) - PC\exp(Q\delta)$ has all its eigenvalues at zero. Then observer (2) with observer matrices L and

$$K_k = P,\ 1 \leq k \leq n, \quad K_k = 0,\ k > n, \tag{3}$$

estimates the state of system (1) in predetermined finite time $\tau = t_n - t_0 = n\delta$, i.e. $\hat{x}(t) = x(t)\ \forall t > t_n$.

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Proof. Consider the dynamics

$$\begin{aligned} \dot{e}(t) &= (A - LC)e(t) && t \neq t_k, \\ e(t_k^+) &= (I - K_k C)e(t_k) && t = t_k, \\ e(t_0^+) &= e_0, && k = 1, 2, \dots \end{aligned} \tag{4}$$

of the estimation error $e = x - \hat{x}$ in order to study the convergence behavior of observer (2). From (4) it follows that the estimation error can be expressed as

$$e(t) = \exp(Q(t - t_k))e(t_k^+) \tag{5}$$

for $t \in (t_k, t_{k+1}]$ and, for $t = t_k$ as

$$e(t_k^+) = (I - K_k C)\exp(Q\delta)e(t_{k-1}^+). \tag{6}$$

Therefore, it follows from (5) and (6) that the solution of (4) with initial condition e_0 is

$$\begin{aligned} e(t) &= \exp(Q(t - t_k)) \\ &\times \prod_{i=1}^{k}(I - K_i C)\exp(Q\delta)e_0 \end{aligned} \tag{7}$$

for $t \in (t_k, t_{k+1}]$. Hence, for $t = t_n$ the estimation error is

$$e(t_n^+) = \left((I - PC)\exp(Q\delta)\right)^n e_0 = R^n e_0 = 0 \tag{8}$$

because $K_k = P$ for $1 \leq k \leq n$ and $\lambda_i(R) = 0, i = 1, \dots, n$, i.e R is a nilpotent matrix with the property $R^n = 0$ [Chen (1999)]. Finally, one can conclude from (8) that $e(t) = 0$ for $t > t_n$. This proves that observer (2) estimates the exact system state in finite time $\tau = t_n - t_0 = n\delta$. ■

In the following some system theoretical properties and possible extensions of observer (2) are discussed before its convergence behavior is illustrated via an example.

Remark 1. Observer (2) converges in finite time τ if matrix R has all its eigenvalues at zero. The eigenvalues of R can be placed at the origin if the discrete-time system $\eta(k+1) = \exp(Q\delta)\eta(k),\ y_\eta(k) = C\exp(Q\delta)\eta(k)$, where $\eta \in \mathbb{R}^n$ is the state and $y_\eta \in \mathbb{R}^q$ the output, is observable. The observability condition for this system is

$$\operatorname{rank}\begin{bmatrix} C\exp(Q\delta) \\ C\exp(Q\delta)^2 \\ \vdots \\ C\exp(Q\delta)^n \end{bmatrix} = n. \tag{9}$$

Note that condition (9), that can be also written as $\operatorname{rank}(S\exp(Q\delta)) = n$ with $S = [C^T\ (C\exp(Q\delta))^T \dots (C\exp(Q\delta)^{n-1})^T]^T$, is satisfied if matrix S has full rank due to $\operatorname{rank}(\exp(Q\delta)) = n$ for any matrix $Q\delta$, i.e. $\operatorname{rank}(S\exp(Q\delta)) = \operatorname{rank}(S)$. Furthermore, S is the observability matrix for the discrete-time system

$$\theta(k+1) = \exp(Q\delta)\theta(k), \quad y_\theta(k) = C\theta(k), \tag{10}$$

where $\theta \in \mathbb{R}^n$ is the state and $y_\theta \in \mathbb{R}^q$ is the output, is obtained from the continuous-time system $\dot{\xi}(t) = Q\xi(t),\ y_\xi(t) = C\xi(t)$ by sampling with sampling period δ. System (10) is observable if and only if the pair (A, C) is observable [Chen (1999)], which is the case by assumption. Given observability of (A, C), a sufficient condition for observability of (10) is

$$(\lambda_k - \lambda_l)\delta \neq r2\pi j, \quad r = \pm 1, \pm 2, \dots \tag{11}$$

for each pair (λ_k, λ_l) of eigenvalues of Q [Kalman et al. (1963)]. If the observer matrix L is chosen such that all eigenvalues of Q are real, (11) is satisfied for any value of δ and thus the convergence time $\tau = n\delta$ of observer (2) can be chosen arbitrarily.

Remark 2. Based on Ackermann's Formula [Dorf and Bishop (2007)], the observer matrix P can be computed for systems (1) with a single output, i.e. $q = 1$, via

$$P^T = [0\ 0\ \dots\ 1]\begin{bmatrix} (\exp(Q\delta))^T C^T \\ \vdots \\ ((\exp(Q\delta))^T)^n C^T \end{bmatrix}^{-1} \left((\exp(Q\delta))^T\right)^n.$$

Remark 3. It is also possible to design an observer with finite convergence time for system (1) which uses only equation (2b) with a constant matrix K_k, i.e. $K_k = P\ \forall\ k \geq 1$ and $L = 0$ in (2). Hence, observer (2) becomes

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t), \qquad t \neq t_k, \tag{12a}$$

$$\hat{x}(t_k^+) = \hat{x}(t_k) + P(y(t_k) - C\hat{x}(t_k)) \quad t = t_k, \tag{12b}$$

$$\hat{x}(t_0^+) = \hat{x}_0, \qquad k = 1, 2, \dots \tag{12c}$$

which is in this form closely related to the dead-beat observer for discrete-time systems. An advantage of observer (12) over observer (2) is a decreased bandwidth usage due to the reduced amount of measured information (only samples at time instants t_k) transmitted from the system to the observer.

Remark 4. Note that the proposed observer design can be extended to nonlinear systems of the form

$$\dot{x}(t) = Ax(t) + \rho(u(t), y(t)), \quad x(t_0) = x_0 \tag{13a}$$

$$y(t) = Cx(t), \tag{13b}$$

where $\rho : \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}^n$ is a locally Lipschitz function which depends on known arguments.

Remark 5. From a computational point of view the proposed observer (2) is more attractive than the finite time convergent observers [Medvedev and Toivonen (1994); Han et al. (2001); Byrski (2003); Engel and Kreisselmeier (2002); Raff and Allgöwer (2007)] since the memory requirements and the on-line computations are reduced. The reason for this is that these observers use information of the past, which have to be stored, and/or an increased order of the observer dynamics, and/or the on-line computation of convolution integrals in order to achieve finite convergence time whereas the finite convergence time of observer (2) is only achieved via n observer state updates.

3. EXAMPLE

In the following observer (2) is applied to estimate the state of a spring mass system [Geromel and de Oliveira (2001)] with system matrices

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & 1 & -1 & 0 \\ 2 & -2 & 0 & -2 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, C^T = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{14}$$

The convergence time is chosen as $\tau = 1$. Hence, one has to design the matrices L, P such that the conditions of Theorem 1 are satisfied with $\delta = \tau/4 = 0.25$. For example, one can derive the following observer matrices from these conditions: $L^T = [14.0\ 94.0\ 56.0\ -90.0]$, $K_k^T = [1.0\ 111.3\ 17.3\ 150.3]$ for $1 \leq k \leq 4$, and $K_k = 0$ for $k > 4$. The simulation results with $u(t) = \sin(t)$, that are plotted in Figure 1, show that the state of the mass spring system is estimated in finite time $\tau = 1$ via observer (2).

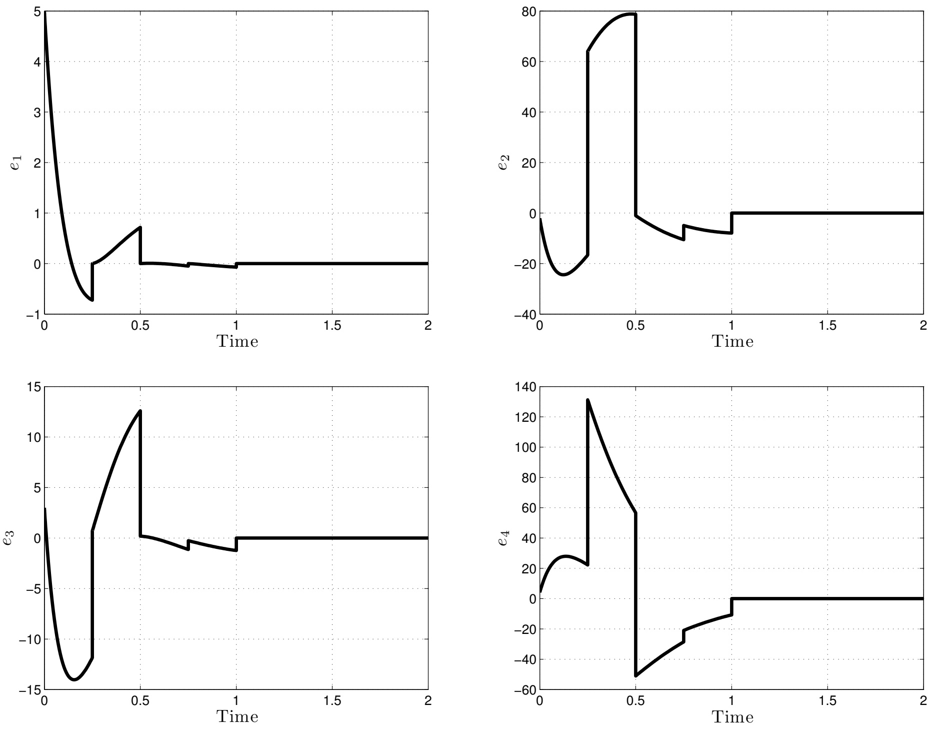

Fig. 1. Trajectories of the estimation errors $e_1(t), e_2(t), e_3(t), e_4(t)$ resulting from the mass spring system with initial condition $x_0 = [5\ -2\ 3\ 4]^T$ and proposed observer (2) with initial condition $\hat{x}_0 = [0\ 0\ 0\ 0]^T$. The proposed observer reconstructs the exact state of the mass spring system in finite time $\tau = 1$.

4. SUMMARY AND OUTLOOK

In this paper a new observer with finite convergence time and impulsive dynamical behavior, that stems from the updates of the observer state at discrete time instants, has been developed for linear continuous-time systems. Conditions have been given which ensure the finite convergence time of the proposed observer and some system theoretical properties, e.g. the low computational complexity, as well as an extension to nonlinear systems have been discussed. Finally, the applicability of the proposed observer has been demonstrated via an example. Similar to [Menold (2004)], future work intends to use the proposed observer structure to design observers with finite convergence time for linear time-varying systems or nonlinear systems. Furthermore, future work should also study the performance of the proposed observer in presence of measurement noise or model uncertainty.

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