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On the Fractional PID Control of a Laboratory Servo System

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Abstract: In this paper are investigated several types of fractional-order PID controllers in the velocity control of a servo system. The fractional controller is more flexible and gives the possibility of adjusting carefully the dynamical properties of a control system. The servo system is controlled by using a real-time digital control system based on MATLAB/Simulink. Results are compared with those obtained from classical PID controllers. Experimental responses are presented and analyzed, showing the effectiveness of the proposed fractional-order algorithms.

1. INTRODUCTION

Nowadays, the fractional calculus (FC) is applied in science and engineering, being recognized its ability to yield a superior modeling and control in many dynamical systems. We may cite its adoption in areas such as viscoelasticity and damping, diffusion and wave propagation, electromagnetism, chaos and fractals, heat transfer, biology, electronics, signal processing, robotics, system identification, traffic systems, genetic algorithms, percolation, modeling and identification, telecommunications, chemistry, irreversibility, physics, control, economy and finance (Oldham and Spanier, 1974; Podlubny, 1999a).

In what concerns the area of control systems, the fractional controllers are now extensively investigated (Machado, 1997; Barbosa *et al.*, 2004; Podlubny, 1999a, 1999b). The advantages of this type of controllers is well acknowledged, particularly, in the control of fractional-order systems. Ma and Hori (2003) use a PI$^{\alpha}$D controller for the speed control of two-inertia system. The superior robustness performance against input torque saturation and load inertia variation are shown by comparison with integer order PID control. Feliu-Batlle *et al.* (2007) apply fractional algorithms in the control of main irrigation canals, which reveals to be robust to changes in the time delay and the gain. Valério and Sá da Costa (2004) introduce a fractional controller in a two degree of freedom flexible robot, achieving a stable response for the position of its tip. However, in spite of the increasing number of publications related to this subject, simple and effective tuning rules, such as those for classical PID controllers, are still lacking.

In this article we investigate the use of fractional PID controllers in the velocity control of a laboratory servo system. The method used for the tuning of the fractional controllers is based on the well-known Ziegler-Nichols tuning rules (Ziegler and Nichols, 1942). The authors believe that these rules constitute a good starting point to tune a fractional PID controller and to analyze the effect of the fractional orders upon the real-system control performance. The Ziegler-Nichols rules are used to tune the conventional PID controller and the final tuning of the fractional-order PID controller is obtained by adjusting the fractional orders and the controller gain in order to yield a satisfactory control.

This paper is organized as follows. Section 2 presents the fundamentals of fractional calculus. Section 3 introduces the fractional-order systems and fractional PID controllers. Section 4 outlines the Oustaloup's frequency approximation method used to implement the fractional-order operators. Section 5 describes the laboratory modular servo system setup used in the experiments. Section 6 gives the Ziegler-Nichols tuning rules based on oscillatory behaviour. The controller settings obtained with this method will serve as basis for the tuning of the fractional PID controllers. Section 7 illustrates the experimental results obtained from the different fractional-order controllers used. Finally, section 8 draws the main conclusions and addresses perspectives of future developments.

2. FUNDAMENTALS OF FRACTIONAL CALCULUS

Fractional calculus (FC) is the area of mathematics that extends derivatives and integrals to an arbitrary order (real or, even, complex order) and emerged at the same time as the classical differential calculus. FC generalizes the classical differential operator $D_t^n \equiv d^n/dt^n$ to a fractional operator D_t^{α}, where α is a real number (Spanier and Oldham, 1974; Samko *et al.*, 1993; Podlubny, 1999a). However, its inherent complexity delayed the application of the associated concepts.

For the definition of the generalized operator ${}_aD_t^{\alpha}$, where *a* and *t* are the limits and α the order of operation, one often adopts the Riemann-Liouville (RL) and the Grünwald-Letnikov (GL) definitions. The RL definition is given by $(\alpha > 0)$:

$$ {}_aD_t^\alpha x(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{x(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau\,,\ n-1<\alpha<n \qquad (1) $$

where $\Gamma(z)$ represents the Gamma function of z. The GL definition is $(\alpha \in \Re)$:

$$ {}_aD_t^\alpha x(t) = \lim_{h\to 0} \frac{1}{h^\alpha} \sum_{k=0}^{\left[\frac{t-a}{h}\right]} (-1)^k \binom{\alpha}{k} x(t-kh) \qquad (2) $$

where h is the time increment and $[v]$ means the integer part of v. Definitions (1) and (2) show that the fractional-order operators are *global* operators having a memory of all past events, making them adequate for modeling hereditary and memory effects in most materials and systems.

Another useful definition is given through the Laplace transform. It is shown that the Laplace transform (L) of a fractional derivative of a signal $x(t)$ is given by:

$$ L\{D^\alpha x(t)\} = s^\alpha X(s) - \sum_{k=0}^{n-1} s^k D^{\alpha-k-1} x(t)\Big|_{t=0} \qquad (3) $$

where $X(s) = L\{x(t)\}$. Considering null initial conditions, (3) reduces to the simple form ($\alpha \in \Re$):

$$ L\{D^\alpha x(t)\} = s^\alpha X(s) \qquad (4) $$

Expression (4) is a direct generalization of the integer-order scheme with the multiplication of the signal transform $X(s)$ by the Laplace s-variable raised to a non-integer value α. The Laplace transform is a valuable tool for the analysis and synthesis of fractional-order control systems.

3. FRACTIONAL-ORDER SYSTEMS AND FRACTIONAL PID CONTROLLERS

In general, a fractional-order system can be described by a Linear Time Invariant (LTI) fractional-order differential equation of the form:

$$ \begin{aligned} &a_n D_t^{\beta_n} y(t) + a_{n-1} D_t^{\beta_{n-1}} y(t) + \ldots + a_0 D_t^{\beta_0} y(t) \\ &\quad = b_m D_t^{\alpha_m} u(t) + b_{m-1} D_t^{\alpha_{m-1}} u(t) + \ldots + b_0 D_t^{\alpha_0} u(t) \end{aligned} \qquad (5) $$

or by a continuous transfer function of the form:

$$ G(s) = \frac{b_m s^{\alpha_m} + b_{m-1} s^{\alpha_{m-1}} + \ldots + b_0 s^{\alpha_0}}{a_n s^{\beta_n} + a_{n-1} s^{\beta_{n-1}} + \ldots + a_0 s^{\beta_0}} \qquad (6) $$

where $\beta_k, \alpha_k\ (k = 0, 1, 2, \ldots)$ are real numbers, $\beta_k > \cdots > \beta_1 > \beta_0$, $\alpha_k > \cdots > \alpha_1 > \alpha_0$ and $a_k, b_k\ (k = 0, 1, 2, \ldots)$ are arbitrary constants.

A discrete transfer function of (6) can be obtained by using a discrete approximation of the fractional-order operators, yielding (Vinagre *et al.*, 2000):

$$ G(z) = \frac{b_m \left[w(z^{-1})\right]^{\alpha_m} + b_{m-1} \left[w(z^{-1})\right]^{\alpha_{m-1}} + \ldots + b_0 \left[w(z^{-1})\right]^{\alpha_0}}{a_n \left[w(z^{-1})\right]^{\beta_n} + a_{n-1} \left[w(z^{-1})\right]^{\beta_{n-1}} + \ldots + a_0 \left[w(z^{-1})\right]^{\beta_0}} \qquad (7) $$

where $w(z^{-1})$ denotes the discrete equivalent of the Laplace operator s, expressed as a function of the complex variable z or the shift operator z^{-1}.

The fractional-order controllers were introduced by Oustaloup (1991), who developed the so-called *Commande Robuste d'Ordre Non Entier* (CRONE) controller. Oustaloup demonstrated the superior performance of the CRONE controller over the conventional PID controller. More recently, Podlubny (1999b) proposed a generalization of the PID controller, the fractional-order $PI^\lambda D^\mu$-controller, involving an integrator of order λ and a differentiator of order μ. The transfer function $G_c(s)$ of such a controller has the form:

$$ G_c(s) = \frac{U(s)}{E(s)} = K_P + K_I s^{-\lambda} + K_D s^{\mu}, \quad \lambda, \mu > 0 \qquad (8) $$

where $E(s)$ is the error signal and $U(s)$ the controller's output. The constants (K_P, K_I, K_D) are the proportional, integral, and derivative gains of the controller, respectively.

The $PI^\lambda D^\mu$-controller is represented by a fractional integro-differential equation of type:

$$ u(t) = K_P e(t) + K_I D^{-\lambda} e(t) + K_D D^\mu e(t) \qquad (9) $$

Clearly, depending on the values of the orders λ and μ, we get an infinite number of choices for the controller's type (defined continuously on the (λ, μ)-plane). For instance, taking $(\lambda, \mu) \equiv (1, 1)$ gives a classical PID controller, $(\lambda, \mu) \equiv (1, 0)$ gives a PI controller, $(\lambda, \mu) \equiv (0, 1)$ gives a PD controller and $(\lambda, \mu) \equiv (0, 0)$ gives a P controller. All these classical types of PID controllers are the particular cases of the fractional $PI^\lambda D^\mu$-controller (8). Therefore, the $PI^\lambda D^\mu$-controller is more flexible and gives the possibility of adjusting more carefully the dynamical properties of a control system.

4. OUSTALOUP'S APPROXIMATION METHOD

In order to implement the term s^α ($\alpha \in \Re$) of the fractional controller, a frequency-band limited approximation is used by cutting out both high and low frequencies of transfer $(s/\omega_u)^\alpha$ to a given frequency range $[\omega_b, \omega_h]$, distributed geometrically around the unit gain frequency $\omega_u = (\omega_b\omega_h)^{1/2}$ (Oustaloup, 2000). The resulting continuous transfer function of such approximation is given by the formula:

$$ D_N(s) = \left(\frac{\omega_h}{\omega_b}\right)^\alpha \prod_{k=-N}^{N} \frac{1 + s/\omega'_k}{1 + s/\omega_k} \qquad (10) $$

where the zero and pole of rank k can be evaluated, respectively, as:

$$\omega'_k = \left(\frac{\omega_h}{\omega_b}\right)^{\frac{k+N+\frac{1}{2}-\frac{\alpha}{2}}{2N+1}} \omega_b\,, \quad \omega_k = \left(\frac{\omega_h}{\omega_b}\right)^{\frac{k+N+\frac{1}{2}+\frac{\alpha}{2}}{2N+1}} \omega_b \qquad (11)$$

Taking N, ω_b, ω_h, and α, permits the determination of the values of the set of zeros and poles of (11) and, consequently, the synthesis of the desired transfer function (10). This algorithm is easily implemented in a language like MATLAB.

5. THE LABORATORY MODULAR SERVO SYSTEM

The laboratory Modular Servo System (MSS) consists of the Inteco (http://www.inteco.com.pl) digital servomechanism and open-architecture software environment for real-time control experiments. The MSS supports the real-time design and implementation of advanced control methods, using MATLAB and Simulink tools. The MSS setup (Fig. 1) consists of a several modules mounted at the metal rail and coupled with small clutches. The modules are arranged in the chain. The DC motor with the generator module is at the front of the chain and the gearbox with the output disk is placed at the end of the chain.

The DC motor can be coupled with the following modules: the inertia module, magnetic brake module, backlash and gearbox ($M = 100$) modules with the output disk. The angle of rotation of the DC motor shaft is measured using an incremental encoder. The generator is connected directly to the DC motor and generates voltage proportional to the angular velocity.

The servomechanism is connected to a computer where a control algorithm is implemented based on the measurement of the angular position and velocity. The accuracy of measurement of the position is 0.1% while the accuracy of measured velocity is 5%. The armature voltage of the DC motor is controlled by a PWM signal $v(t)$ excited by a dimensionless control signal in the form $u(t) = v(t)/v_{max}$. The admissible controls satisfy $|u(t)| \le 1$ and $v_{max} = 12$ [V] (Manual Inteco, 2006).

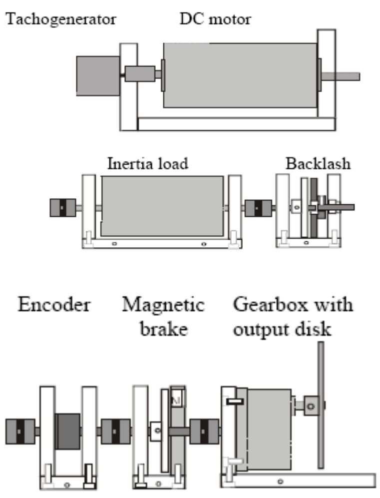

Fig. 1. The MSS setup (from Manual Inteco (2006)).

6. ZIEGLER-NICHOLS TUNING RULES

Ziegler and Nichols (1942) proposed two methods for tuning the controller parameters based on the transient response characteristics of a given plant. In the first method, the choice of the controller parameters is based on a decay ratio of approximately 0.25. In the second method, the criterion for tuning the controller parameters consists in evaluating the system at the limit of stability (ultimate sensitivity method) (Ziegler and Nichols, 1942). Therefore, the proportional gain is increased until we observe continuous oscillations, that is, until the system becomes marginally stable. The corresponding gain K_u and the period of oscillation P_u (also called *ultimate gain* and *ultimate period*, respectively) are then determined (Franklin *et al.*, 1994). In this work, we apply the oscillation method.

Ziegler and Nichols suggest to tune the proportional gain K_p, integral time T_I, and derivative time T_D according with the formulae shown in Table 1. Once the values of T_I and T_D have been obtained, the gains K_P and K_I, are computed as:

$$K_I = \frac{K_P}{T_I}, \quad K_D = K_P T_D \qquad (12)$$

In general, the controller settings according to Ziegler–Nichols rules provide a good closed-loop response for many systems.

7. EXPERIMENTAL RESULTS

For this experiment the MSS setup includes the modules of the DC motor with tacho-generator, inertia load, encoder module and gearbox module with the output disk (see Fig. 1).

All real-time experiments related to the fractional PID velocity control are performed using the MATLAB/Simulink real-time model given in Fig. 2. A fixed-step solver (Euler's integration method) of a fixed-step size set to 0.01 (sampling period of $T = 0.01$ s) is chosen.

First, a proportional controller is applied to the velocity servo system until the system shows nondecaying oscillations, as represented in Fig. 3. The ultimate gain and period yield $K_u = 0.08$ and $P_u = 0.74$ s, respectively. The controller parameters are then calculated according the Ziegler-Nichols rules illustrated in Table 1.

Table 1. Ziegler-Nichols tuning for controller $G_c(s) = K_P(1 + 1/T_I s + T_D s)$ based on oscillatory behavior

Type of controller	K_p	T_I	T_D
P	$0.5K_u$	∞	0
PI	$0.45K_u$	$\frac{1}{1.2}P_u$	0
PID	$0.6K_u$	$\frac{1}{2}P_u$	$\frac{1}{8}P_u$

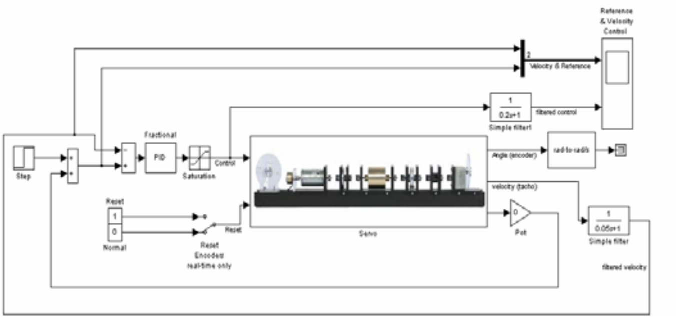

Fig. 2. Real-time model of the servo with the fractional PID controller.

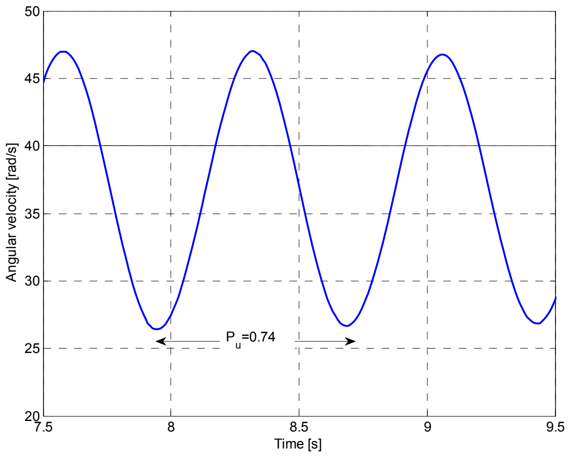

Fig. 3. Ultimate gain K_u =0.08 and ultimate period P_u.

The fractional operator s^{α} ($\alpha \in \Re$) in the PID controller transfer function (8) is implemented by using the Oustaloup's frequency approximation method described in section 4. The values used are $N = 5$, $\omega_b = 1$ rad/s and $\omega_h = 1000$ rad/s.

The fractional-order controllers are implemented in digital form by discretization of the continuous controller transfer functions. The discretization technique used consists in the bilinear (or Tustin's) approximation with a sampling period of $T = 0.01$ s.

A step input of amplitude 40 rad/s is applied to the servo system and the angular velocity *versus* time is acquired.

In the following, we apply several types of fractional-order PID controllers for the control of the angular velocity of the servo system. The experimental results are presented and analyzed.

7.1 The D^{μ}-controller

The transfer function of a D^{μ}-controller is given by ($K_P = 0$ and $K_I = 0$ in (8)):

$$G_c(s) = K_D s^{\mu}, \quad \mu > 0 \tag{13}$$

We have two parameters to be tuned, namely the gain K_D and the derivative order μ of the fractional controller.

The fractional controller is designed by adopting the proportional gain of the P-controller obtained from the Zigler-Nichols rules, that is, $K_D = 0.5K_u = 0.04$. In the following experiments (K_D, μ) are varied and the effect upon the control system performance is analyzed.

Figure 4 depicts the experimental step responses of the angular velocity for several values of derivative order $\mu = \{0.05,\ 0.15,\ 0.25,\ 0.35,\ 0.45,\ 1\}$ while maintaining the derivative gain $K_D = 0.04$. The plots for variation of gain K_D (with derivative order $\mu = 0.2$ fixed) are illustrated in Fig. 5. As expected, the system reveals a steady-sate error that diminishes as the gain K_D increases. Note also that the derivative order μ produces the same effect in the response, with an increasing steady state error as the order μ increases. However, the overshoot and settling time are more acceptable for the case where the order μ is changed. In fact, we verify that the extra degree of tuning provided by the fractional controller, in comparison to the classical P-controller, may be useful to yield a satisfactory control, as shown in Figs. 4-5.

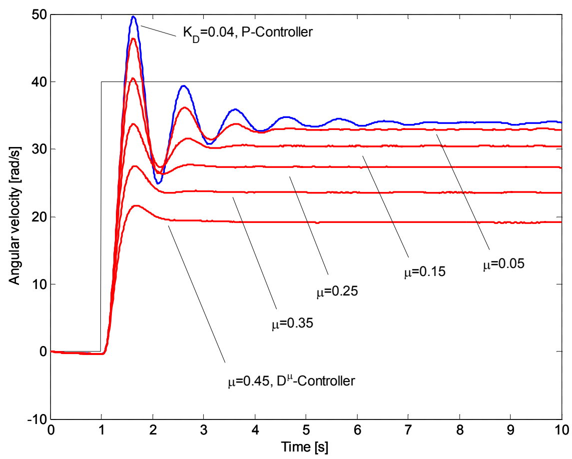

Fig. 4. Sensitivity of angular velocity to variation of order μ (with a fixed value of gain $K_D = 0.04$).

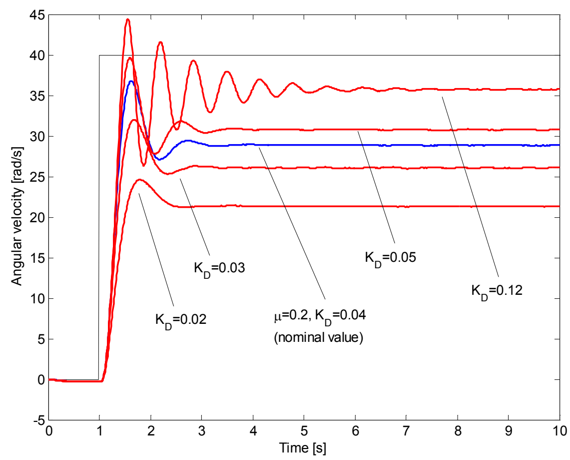

Fig. 5. Sensitivity of angular velocity to variation of gain K_D (with a fixed value of derivative order $\mu = 0.2$).

7.2 *The I^λ-controller*

The transfer function of an I^λ-controller is given by ($K_P = 0$ and $K_D = 0$ in (8)):

$$G_c(s) = \frac{K_I}{s^\lambda}, \quad \lambda > 0 \tag{14}$$

We have also two parameters to be tuned, namely the gain K_I and the integration order λ of the fractional controller.

In order to assure a good steady state error, the term $1/s^\lambda$ must be implemented by means of an integer integrator (Axtell and Bise, 1990; Franklin *et al.*, 1994). The modified fractional-order controller is then given in the form:

$$G_c(s) = K_I \frac{s^{1-\lambda}}{s}, \quad 0 < \lambda < 1 \tag{15}$$

Like in the case of the D^μ-controller, the I^λ-controller is designed by adopting the proportional gain of the P-controller obtained from the Ziegler-Nichols rules, that is, $K_I = 0.5K_u = 0.04$. In the following experiments (K_I, λ) are varied and the effect on the control system performance is analyzed.

Figure 6 shows the experimental step responses of the angular velocity for several values of integrative order $\lambda = \{0.1,\ 0.3,\ 0.5,\ 0.7\}$ while maintaining the integral gain $K_I = 0.04$. The plots for variation of gain K_I (with integrative order $\lambda = 0.5$ fixed) are illustrated in Fig. 7. As can be observed, the steady-state error is very small. Note that the real system is nonlinear and, therefore, the oscillations are damped very quickly. Once more, we verify that the fractional order λ is a very useful parameter for adjusting the dynamics of the control system. In fact, the order λ has a large influence upon the system dynamics, as illustrated in Fig. 6. The transient response of the system can be easily modified through the controller parameters. Note also that the system shows a large time delay, particularly when a weak integrator is used. One of the reasons for this phenomenon is related with the high order transfer function approximation used for the fractional controller. This aspect needs further investigation and will be addressed in future research.

7.3 *The PI^λ-controller*

The transfer function of a PI^λ-controller is given by ($K_D = 0$ in (8)):

$$G_c(s) = K_P + \frac{K_I}{s^\lambda}, \quad \lambda > 0 \tag{16}$$

In this case, we have three parameters that can be tuned, namely the proportional gain K_P, the integral gain K_I and the integration order λ of the fractional controller. The term $1/s^\lambda$ is implemented through an integer integrator $s^{1-\lambda}/s$ $(0 < \lambda < 1)$ in order to provide a good steady sate error, as mentioned in previous section.

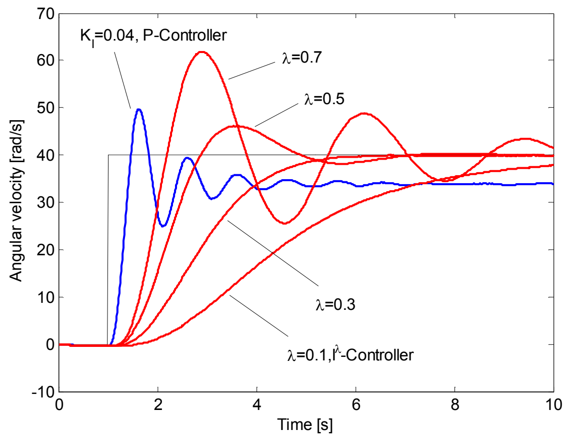

Fig. 6. Sensitivity of angular velocity to variation of order λ (with a fixed value of gain constant $K_I = 0.04$).

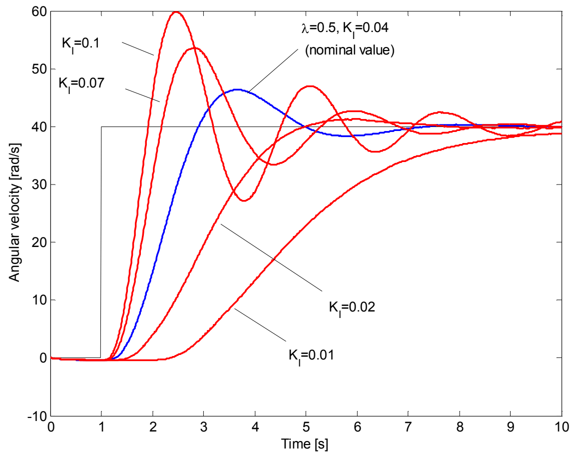

Fig. 7. Sensitivity of angular velocity to variation of gain K_I (with a fixed value of integration order $\lambda = 0.5$).

The fractional controller is designed by adopting the controller parameters of the PI-controller obtained from the Ziegler-Nichols rules, that is, $K_P = 0.45K_u = 0.0364$ and $K_I = 1.2K_P/P_u = 0.0590$. In the following experiments (K_I, λ) are varied and the resulting effect is analyzed.

Figure 8 shows the experimental step responses of the angular velocity for several values of integrative order $\lambda = \{0.1,\ 0.3,\ 0.5,\ 0.7,\ 0.9\}$ while maintaining the gains $K_P = 0.0364$ and $K_I = 0.0590$. The plots for variation of integral gain K_I (with integrative order $\lambda = 0.5$ fixed) are illustrated in Fig. 9. As in previous case, the steady-state error is very small. The steady-state behavior could be also improved by multiplying the fractional controller by a term of the form $(s+\eta)/s$, with η being a small value (Feliu-Batlle *et al.*, 2007).

Note the influence of the order λ in the system overshoot and settling time. An adequate phase margin can be easily established by a proper choice of fractional order λ. However, the output converges to its final value more slowly, as should be expected by a weak fractional integral term.

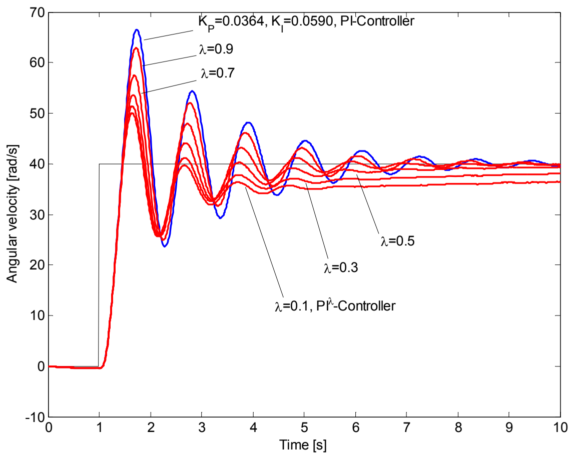

Fig. 8. Sensitivity of angular velocity to variation of integration order λ (with $K_P = 0.0364$ and $K_I = 0.0590$).

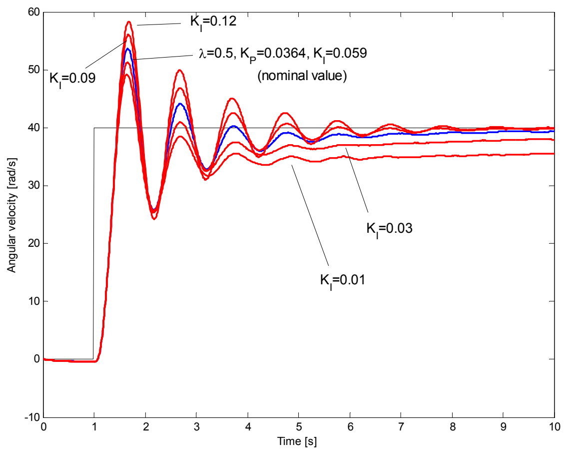

Fig. 9. Sensitivity of angular velocity to variation of integral gain K_I (with $\lambda = 0.5$ and $K_P = 0.0364$).

8. CONCLUSIONS

In this article we investigated the velocity control of a servo system by using different types of fractional-order PID controllers. For the tuning of the controllers we adopted the well-known Ziegler-Nichols rules. It was shown that the fractional controllers can effectively enhance the control system performance providing extra tuning parameters useful for the adjustment of the control system dynamics. The Zeigler-Nichols rules revealed to be simple and effective in the final tuning of the fractional-order algorithms. In fact, with this kind of controllers the users have extra design possibilities over the desired system specifications.

The results show that fractional controllers can produce (at least) the same performance as the classical PID controllers with the advantage of having less parameters to be tuned. More systematic approaches can be used to design the fractional controllers. Robustness against nonlinearities and perturbations need also to be tested. Thus, the study reported here represents only the first steps and a deeper investigation will be pursued in future research.

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REFERENCES

Axtell, M., M. E. Bise (1990). Fractional Calculus Applications in Control Systems. In: *Proc. of the IEEE 1990 National Aerospace and Electronics Conference.* New York, pp. 563-566.

Barbosa, R. S., J. A. T. Machado and I. M. Ferreira (2004). Tuning of PID Controllers Based on Bode's Ideal Transfer Function. *Nonlinear Dynamics* **38**, 305–321.

Feliu-Batlle, V., R. R. Pérez and L. S. Rodríguez (2007). Fractional Robust Control of Main Irrigation Canals with Variable Dynamic Parameters. *Control Engineering Pratice* **15**, 673–686.

Franklin, G. F., J. D. Powel and M. L. Workman (1990). *Digital Control of Dynamic Systems*. Addison-Wesley, 2nd ed. Reading, MA.

Ma, C. and Y. Hori (2003). Design of Robust Fractional Order PI$^{\alpha}$D Speed Control for Two-Inertia System. In: *Proc. of the Japan Industry Applications Society Conference.* Tokyo, Japan, August 26-28, pp. 1-4.

Machado, J. A. T. (1997). Analysis and Design of Fractional-Order Digital Control Systems. *SAMS Journal of Systems Analysis, Modelling and Simulation* **27**, 107–122.

Manual Inteco (2006). Modular Servo System (MSS), User's Manual, Krakow, Poland.

Oldham, K. B. and J. Spanier (1974). *The Fractional Calculus*. Academic Press, New York.

Oustaloup, A., (1991). *La Commande CRONE*. Éditions Hermès, Paris.

Oustaloup, A., F. Levron, B. Mathieu and F. M. Nanot (2000). Frequency-Band Complex Noninteger Differentiator: Characterization and Synthesis. *IEEE Trans. on Circuits and Systems-I: Fund. Theory and Applications* **47** (1), 25–39.

Podlubny, I. (1999a). *Fractional Differential Equations*. Academic Press, San Diego.

Podlubny, I. (1999b). Fractional-Order Systems and PID-Controllers. *IEEE Transactions on Automatic Control* **44** (1), 208–214.

Samko, S. G., A. A. Kilbas and O. I. Marichev (1993). *Fractional Integrals and Derivatives: Theory and Applications*. Gordon and Breach Science, Amsterdam.

Valério, D. and José Sá da Costa (2004). Non-Integer Order Control of a Flexible Robot. In: *Proc. of the First IFAC Workshop on Fractional Differentiation and Its Applications*, pp. 520–525, Bordeaux, France.

Vinagre, B., I. Podlubny, A. Hernandez, and V. Feliu (2000). Some Approximations of Fractional Order Operators Used in Control Theory and Applications. *Journal of Fractional Calculus & Applied Analysis*, **3** (3), 231–248.

Ziegler, J. G. and N. B. Nichols (1942). Optimum Settings for Automatic Controllers. *Transactions of the ASME* **64**, 759–765.