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Robust controller synthesis for disturbance filter uncertainty described by dynamic integral quadratic constraints.

S.G. Dietz, C.W. Scherer and H. Köroğlu *

* Delft Center for Systems and Control, Delft University of Technology, 2628 CD Delft, The Netherlands, s.g.dietz@tudelft.nl

Abstract: Robust controller synthesis is considered for disturbances generated by an uncertain filter. The uncertainties are characterized by an integral quadratic constraint (IQC) with general frequency dependent multipliers. By exploiting the problem structure originating from the fact that the uncertainty enters the disturbance filter but not the plant, it is shown how to derive LMI-synthesis conditions for an a priori specified $\mathcal{L}_2$-induced gain. For a specific sinusoidal disturbance rejection problem, it is shown that specifying a bound on the rate-of-variation of an uncertain parameter can improve performance if compared to earlier results based on static scalings.

Keywords: Robust synthesis, disturbance model uncertainty, rate-of-variation, IQC, LMI

1. INTRODUCTION

In all sorts of dynamical systems, in particular in the field of mechatronics, it is well understood that the effect of external disturbances acting on the system can be eliminated by using active control. The precise layout of controlled systems strongly depends on the available knowledge of these disturbances. Roughly speaking, feed-forward controllers are suitable in case the disturbance signal is known a priori or measurable online and an accurate model of the plant is available. In other cases, feed-back control is needed in order to maintain stability and guarantee performance. During the past decades, optimal $\mathcal{H}_\infty$- and $\mathcal{H}_2$-control have proven to be successful tools in handling disturbance attenuation problems for linear time-invariant systems. Both methods have a clear interpretation in terms of input/output signals. An $\mathcal{H}_\infty$-norm performance level bounds the worst-case induced energy gain caused by all finite energy input signals, while the $\mathcal{H}_2$-norm indicates the amplification of the process in terms of the variance, assuming the input is generated by a white noise source.

The $\mathcal{H}_\infty$- or $\mathcal{H}_2$-synthesis procedure amounts to adding suitable transfer functions at the plant input/output so that norm minimization of the weighted plant results in satisfactory performance of the closed-loop system. In case the nature of disturbances can be nicely captured by a single filter, this filter typically acts as the input weighting. The modelling power of using a single filter to characterize disturbance classes is limited though. For example, in applications containing rotational mechanics such as helicopters, CD players or disk drives, sinusoidal disturbances prevail, see Lee and Chung [1998] and references therein. The periods/frequencies of these sinusoidal disturbance change in time, leading to non-stationary sinusoidal signals. Such signals can in fact be modelled as the output of a parameter dependent oscillator, in which the parameter is not constant but varies in time. Despite the fact that various parameterized disturbance models are available, e.g. for describing wind turbulence acting on aircraft, Hoblit [1988], water waves acting on ships, Lloyd [1989], or models of the road roughness used in ride quality analysis of land vehicles, Hać [1985], most controller synthesis techniques do not exploit this knowledge as already recognized by some authors, see Davison [1995].

Whereas robust controller synthesis in general appears to be a non-convex problem, recent developments by Dietz et al. [2007] have shown that in case the uncertain perturbation affects the disturbance filter but not the plant, convex synthesis conditions can be derived. There, parametric uncertainties were treated using the so-called D/G scales. With these static multipliers, the resulting $\mathcal{L}_2$-gain performance level is guaranteed for arbitrary fast parameter variations.

A powerful framework for handling various types of uncertainties is based on integral quadratic constraints, see Megretski and Rantzer [1997]. By a suitably chosen set of multipliers one can describe non-linearities, parametric or dynamic linear time-varying uncertainties. Despite this modelling power of dynamic IQCs, only few results are available on robust controller synthesis using IQCs, see for example Apkarian and Noll [2006]. As shown in Jönsson and Rantzer [1994], Helmersson [1997], a time-varying parameter with rate-of-variation bounds can be characterized in terms of an IQC by using the so-called 'swapping lemma'. Recent extensions of these results can be found in Köroğlu and Scherer [2007, 2006]. Let us point the reader to an alternative approach for handling time-varying parameters based on parameter dependent Lyapunov functions, see Wu et al. [1996], Haddad and

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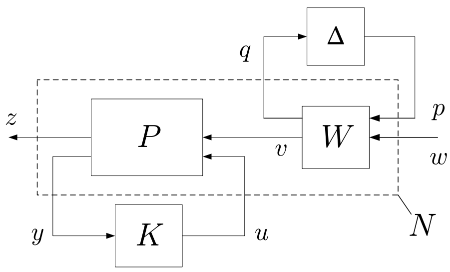

Fig. 1. Systems interconnection with uncertain filter W.

Kapila [1998], Apkarian and Tuan [2000]. In this method, the $\mathcal{L}_2$-gain analysis problem is transformed into a parameter dependent LMI that must hold on a specified parameter region. This technique allows for controller synthesis for linear parameter varying systems, provided that the parameter is on-line measurable. To the best of our knowledge, it is unknown how to design robust robust controllers in case the time-varying parameters cannot be measured online.

In the next section the main problem is described. Then, LMI synthesis conditions are derived in Section 3. In the numerical example of Section 4 we consider sinusoidal disturbances with slowly varying frequency. By making use of the recent work in Köroğlu and Scherer [2006] we are able to improve closed-loop performance by specifying parameter rate-of-variation bounds.

Notation. $\mathcal{L}_{2+}$ denotes the space of vector-valued square integrable functions defined on $[0, \infty)$ equipped with the inner product $\langle ., . \rangle$.

2. PROBLEM FORMULATION

We consider the problem of designing a robust optimal controller for the interconnection of Figure 1 in order to obtain a guaranteed $\mathcal{L}_2$-gain performance level for all disturbance signals that are given by an uncertain filter. Let us be given a minimal realization of the linear time-invariant (LTI) plant P as

$$P := \left[\begin{array}{c|cc} A & B_v & B_u \\ \hline C_z & D_{zv} & D_{zu} \\ C_y & D_{yv} & 0 \end{array}\right]$$

where $A \in \mathbb{R}^{n \times n}$. Assume that the disturbance filter W is perturbed by an uncertain element Δ of dimension $n_q \times n_p$ that is allowed to be any element in a given set of linear operators $\mathbf{\Delta}$. The dependence on Δ is modelled by a linear fractional transformation, written as $\Delta \star W = W_{vw} + W_{vp}\Delta(I - W_{qp}\Delta)^{-1}W_{qw}$, in which the nominal filter W has the realization

$$W := \begin{pmatrix} W_{qp} & W_{qw} \\ W_{vp} & W_{vw} \end{pmatrix} = \left[\begin{array}{c|cc} A_W & B_p & B_w \\ \hline C_q & D_{qp} & D_{qw} \\ C_v & D_{vp} & D_{vw} \end{array}\right]$$

where $A_W \in \mathbb{R}^{n_W \times n_W}$. All eigenvalues of A_W are assumed to lie in the left half plane since the controller is unable to change the dynamics of W. The uncertain element Δ can consist of various types of gain bounded non-linearities or norm bounded dynamic time-varying operators.

p

The problem that we consider is to design an LTI controller denoted by

$$K := \left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array}\right] \tag{1}$$

that, once interconnected with P, leads to a guaranteed a priori specified $\mathcal{L}_2$-induced performance level of w to z.

3. MAIN RESULT

Let us start with the required analysis results by merging the dynamics of W with those of plant P into generalized plant N

$$N = \begin{pmatrix} M_{qp} & M_{qw} & 0 \\ M_{zp} & M_{zw} & N_{zu} \\ N_{yp} & N_{yw} & N_{yu} \end{pmatrix} \tag{2}$$

as indicated by the dashed box in Figure 1. Note that $N_{qu} = 0$ is a consequence of the fact that uncertainty only affects W. We first focus on the analysis problem of the uncontrolled plant after which we will discuss the controller synthesis problem in Section 3.3.

3.1 Robust performance analysis with dynamic multipliers

Adopting the IQC methodology, let uncertainty Δ be described in terms of a dynamic multiplier Π, Hermitian valued and essentially bounded on the imaginary axis, such that

$$\int_{-\infty}^{\infty} \begin{pmatrix} \hat{q}(i\omega) \\ \widehat{\Delta q}(i\omega) \end{pmatrix}^* \Pi(i\omega) \begin{pmatrix} \hat{q}(i\omega) \\ \widehat{\Delta q}(i\omega) \end{pmatrix} \geq 0 \quad \forall \hat{q} \in \mathcal{L}_{2+} \tag{3}$$

holds, in which $\hat{q}$ indicates the Fourier transform of a finite energy signal q. Once a suitable set of parameterized multipliers $\mathbf{\Pi}$ is known, investigating the stability and performance of the loop Δ, M as shown in Figure 1 is done as follows. Under the assumption that M is stable and for every $\tau \in [0, 1]$ and $\Delta \in \mathbf{\Delta}$ the IQC (3) is satisfied for $\tau\Delta$ as well as $(I - M_{qp}\tau\Delta)^{-1}$ is well-posed, we have that the $\mathcal{L}_2$-gain of $w \to z$ is smaller than γ if there exists a multiplier Π that satisfies (3) as well as

$$(\,..\,)' \left(\begin{array}{cc|c} \Pi_{11} & \Pi_{12} & 0 \\ \Pi_{21} & \Pi_{22} & 0 \\ \hline 0 & 0 & J_\gamma \end{array}\right) \left(\begin{array}{cc} M_{qp} & M_{qw} \\ I & 0 \\ \hline M_{zp} & M_{zw} \\ 0 & I \end{array}\right) \prec 0 \tag{4}$$

where

$$J_\gamma = \begin{pmatrix} I & 0 \\ 0 & -\gamma^2 I \end{pmatrix}. \tag{5}$$

Here M_{qp} is the transfer function from input p to output q and M_{zp}, M_{zw}, M_{qw} are defined in a similar fashion. Details on results on IQC analysis can be found in Megretski and Rantzer [1997]. Very often the dynamic multiplier Π is restricted to RH_∞. We will fix some basis transfer matrix Ψ and parameterize Π as

$$\Pi = \Psi^* Q \Psi, \qquad Q \in \mathcal{Q}. \tag{6}$$

For a given set $\mathbf{\Delta}$ it is assumed that for every $Q \in \mathcal{Q}$ the IQC (3) defined by $\Pi = \Psi^* Q \Psi$ is satisfied for all $\Delta \in \mathbf{\Delta}$. In a later section, we will construct Ψ and $\mathcal{Q}$ for the case of a time-varying parameter with rate-bounds.

Once the multiplier set $\mathbf{\Pi}$ is fixed, the robustness analysis problem amounts to satisfying (4) for some $\Pi \in \mathbf{\Pi}$, a frequency domain inequality that can be recast as a genuine LMI by using the KYP Lemma. As discussed in Balakrishnan [2002], the resulting "KYP certificate" (i.e. the Lyapunov matrix) need not be positive definite even if M is stable, as a consequence of using dynamic multipliers. Since closed-loop stability is a vital aspect in the controller design process, a new characterization of nominal stability is needed for the case of using dynamic IQC uncertainty descriptions. This result has recently been derived in Scherer and Köse. Adopting their notation, let $\Psi = (\Psi_1 \quad \Psi_2) \in RH_\infty^{n_Q \times (n_q+n_p)}$ be partitioned according to the columns/rows of M_{qp}, with minimal realization

$$\Psi = (\ \Psi_1 \ \ \Psi_2\ ) = \left[\begin{array}{cc|cc} A_{11} & A_{12} & B_{11} & B_{12} \\ 0 & A_{22} & 0 & B_{22} \\ \hline C_1 & C_2 & D_1 & D_2 \end{array}\right] \tag{7}$$

and let (A_{11}, B_{11}) be controllable. Denote the dimensions of matrices A_{11}, A_{22} by n_1, n_2 respectively. Moreover, introduce the composite transfer matrix

$$\left(\begin{array}{cc|c|c} \Psi_1 & \Psi_2 & 0 & 0 \\ \hline 0 & 0 & I & 0 \\ \hline 0 & 0 & 0 & I \end{array}\right) \left(\begin{array}{ccc} M_{qp} & M_{qw} & 0 \\ I & 0 & 0 \\ \hline M_{zp} & M_{zw} & N_{zu} \\ \hline 0 & I & 0 \\ \hline N_{yp} & N_{yw} & N_{yu} \end{array}\right) =$$

$$\left(\begin{array}{ccc} \Psi_1 M_{qp} + \Psi_2 & \Psi_1 M_{qw} & 0 \\ \hline M_{zp} & M_{zw} & N_{zu} \\ 0 & I & 0 \\ \hline N_{yp} & N_{yw} & N_{yu} \end{array}\right), \tag{8}$$

with realization shown in (9) at the bottom of the next page. For the current discussion, the following more compact realization is preferred.

$$\left[\begin{array}{c|c|c|c} \tilde{A} & \tilde{B}_p & \tilde{B}_w & \tilde{B} \\ \hline \tilde{C}_q & \tilde{D}_{qp} & \tilde{D}_{qw} & 0 \\ \hline \tilde{C}_z & \tilde{D}_{zp} & \tilde{D}_{zw} & D_{zu} \\ 0 & 0 & I & 0 \\ \hline \tilde{C} & \tilde{D}_{yp} & \tilde{D}_{yw} & 0 \end{array}\right]. \tag{10}$$

The symbol $\tilde{}$ is used to indicate that the matrices are constructed from the realization matrices of the subsystems P, W and the realization of Ψ. The following theorem characterizes (upper bounds on) the worst-case $\mathcal{L}_2$ gain when the uncertainties are described in terms of dynamic IQCs.

Theorem 1. Consider Figure 1, let M in (2) be stable and $\Pi = \Psi^* Q \Psi$ be defined as in (6). Moreover, construct realization (9),(10) of the composite plant (8). Then, the $\mathcal{L}_2$-induced gain of $w \to z$ is bounded by γ if for some $Q \in \mathcal{Q}$ there exist solutions $\mathcal{X}$ (partitioned according to $\tilde{A}$) and $\check{X}$ satisfying

$$(\,..\,)' \left(\begin{array}{cc|c|c} 0 & \mathcal{X} & 0 & 0 \\ \mathcal{X} & 0 & 0 & 0 \\ \hline 0 & 0 & Q & 0 \\ \hline 0 & 0 & 0 & J_\gamma \end{array}\right) \left(\begin{array}{c|c|c} I & 0 & 0 \\ \tilde{A} & \tilde{B}_p & \tilde{B}_w \\ \hline \tilde{C}_q & \tilde{D}_{qp} & \tilde{D}_{qw} \\ \hline \tilde{C}_z & \tilde{D}_{zp} & \tilde{D}_{zw} \\ 0 & 0 & I \end{array}\right) \prec 0, \tag{11}$$

$$(\,..\,)' \begin{pmatrix} 0 & \check{X} & 0 \\ \check{X} & 0 & 0 \\ 0 & 0 & Q \end{pmatrix} \begin{pmatrix} I & 0 \\ A_{11} & B_{11} \\ C_1 & D_1 \end{pmatrix} \succ 0, \tag{12}$$

and the coupling condition

$$\begin{pmatrix} \mathcal{X}_{11} & \mathcal{X}_{12} & \mathcal{X}_{14} \\ \mathcal{X}_{12}^T & \mathcal{X}_{22} - \check{X} & \mathcal{X}_{24} \\ \mathcal{X}_{14}^T & \mathcal{X}_{24}^T & \mathcal{X}_{44} \end{pmatrix} \succ 0. \tag{13}$$

Proof. The stability characterization can be found in Scherer and Köse and can be extended to $\mathcal{L}_2$-gain analysis in a straightforward manner. ■

Remark 2. It is important to note that for static multipliers, i.e. $\Psi = I$, condition (12) vanishes and (13) reduces to $\mathcal{X} \succ 0$, the (standard) stability characterization as it was used in Dietz et al. [2007].

3.2 Robust analysis against rate-bounded parameters.

In order to illustrate the power of dynamic IQC multipliers of the form (6) in describing uncertainty classes, let us assume $\Delta = \delta(t) I_r$ where $\delta(t)$ is a time-varying parameter satisfying

$$(\delta(t), \dot{\delta}(t)) \in \mathcal{R} \subset \mathbb{R}^2 \quad \text{for all } t \geq 0. \tag{14}$$

The IQC multiplier theorem for this type of uncertainty is taken from Köroğlu and Scherer [2006] and leads to an extended uncertainty structure depending on both the parameter and its rate-of-variation by applying the swapping Lemma, Köroğlu and Scherer [2006], Jönsson and Rantzer [1994]. For the given region of variation $\mathcal{R}$, let $\mathcal{Q}$ be a parameterized set of matrices such that for any $Q \in \mathcal{Q}$,

$$\left(\begin{array}{cc} I & 0 \\ 0 & I \\ \hline \delta I_l & 0 \\ 0 & \nu I_k \end{array}\right)' Q \left(\begin{array}{cc} I & 0 \\ 0 & I \\ \hline \delta I_l & 0 \\ 0 & \nu I_k \end{array}\right) \succeq 0 \quad \text{for all } (\delta, \nu) \in \mathcal{R}. \tag{15}$$

Then, the class of dynamic multiplier is constructed by first fixing a (stable) basis matrix H as

$$H = I_r \otimes \begin{pmatrix} 1 \\ (s+\beta)^{-1} \\ \vdots \\ (s+\beta)^{-\alpha} \end{pmatrix} = \left[\begin{array}{c|c} A_H & B_H \\ \hline C_H & D_H \end{array}\right], \tag{16}$$

after choosing the pole $\beta > 0$ and the order α. This defines the dimensions k, l in (15) as $l = (\alpha+1)r$ and $k = r\alpha$. Assume that the realization of H is minimal. Next, form the extended transfer matrices

$$H_1 = \left[\begin{array}{c|c} A_H & B_H \\ \hline C_H & D_H \\ I & 0 \end{array}\right], \quad H_2 = \left[\begin{array}{c|cc} A_H & B_H & I \\ \hline C_H & D_H & 0 \\ 0 & 0 & I \end{array}\right]. \tag{17}$$

Then the structure of the multiplier class becomes

$$\Pi = \Psi^* Q \Psi \text{ with } \Psi = (\ \Psi_1 \ |\ \Psi_2\ ) = \left(\begin{array}{c|c} H_1 & 0 \\ 0 & H_2 \end{array}\right). \tag{18}$$

For the next IQC analysis result, we need to extend the generalized plant N by adding k zero columns as follows

$$N_e := \left(\begin{array}{cc|c|c} M_{qp} & 0 & M_{qw} & 0 \\ \hline M_{zp} & 0 & M_{zw} & N_{zw} \\ \hline N_{yp} & 0 & N_{yw} & N_{yu} \end{array}\right).$$

which corresponds to the following substitutions

$$\begin{aligned} B_p &\to (B_p \ \ 0), \\ D_{qp} &\to (D_{qp} \ \ 0), \\ D_{vp} &\to (D_{vp} \ \ 0). \end{aligned}$$

We adopt the notation in (10) and introduce $\tilde{B}_{p,e}, \tilde{D}_{qp,e}$, $\tilde{D}_{zp,e}$ and $\tilde{D}_{yp,e}$ to indicate that (9)-(10) have been constructed with these modified matrices B_p, D_{qp}, D_{vp}. Moreover, the restriction of N_e to the first two input/output channels is denoted by M_e.

Theorem 3. Consider the interconnection of Figure 1. Let $\Delta = \delta(t)I_r$ where $\delta(t)$ satisfies (14) for some specified region $\mathcal{R}$ and let multiplier Π be of the form (18), where H_1, H_2 have been constructed after fixing the parameter α, β in (16). Assume M_e is stable. Then, the L_2-induced gain of $w \to z$ is bounded by γ for all parameter trajectories $\delta(t)$ with $(\delta(t), \dot{\delta}(t)) \in \mathcal{R}$ for all $t \geq 0$ if there exist solutions $\mathcal{X}, \check{X}$ and $Q \in \mathcal{Q}$ satisfying (15) for which

$$(\,..\,)' \left(\begin{array}{cc|c|c} 0 & \mathcal{X} & 0 & 0 \\ \mathcal{X} & 0 & 0 & 0 \\ \hline 0 & 0 & Q & 0 \\ \hline 0 & 0 & 0 & J_\gamma \end{array}\right) \left(\begin{array}{c|c|c} I & 0 & 0 \\ \tilde{A} & \tilde{B}_{p,e} & \tilde{B}_w \\ \hline \tilde{C}_q & \tilde{D}_{qp,e} & \tilde{D}_{qw} \\ \hline \tilde{C}_z & \tilde{D}_{zp,e} & \tilde{D}_{zw} \\ 0 & 0 & I \end{array}\right) \prec 0,$$

as well as (12)-(13) hold.

Proof. See Köroğlu and Scherer [2006, 2007]. ■

Relaxing the (generally non-tractable) condition (15) in order to implement the property $Q \in \mathcal{Q}$ is relatively simple for polytopic regions $\mathcal{R}$, by using convexity arguments. If $\mathcal{R}$ is described by polynomial inequalities, as in case of ellipsoidal regions, sum-of-squares relaxations are needed to numerically handle (15), see Dietz et al. [2006], Scherer [2006].

3.3 Robust $\mathcal{L}_2$-gain synthesis with dynamic scalings

After introducing the main IQC analysis results for dynamic IQC multipliers, let us derive the main robust synthesis result of the paper. To be particular, we restrict ourselves to the case of having a single time-varying parameter $\delta(.)$ characterized by a specified region $\mathcal{R}$ as done in (14). We emphasize that the synthesis result derived in this section holds for arbitrary dynamic multipliers of the form (6).

For notational convenience, we make use of the abbreviation in (9) as considered for the extended plant N_e. As explained in Dietz et al. [2007], a suitable congruence transformation allows to convexify the robust synthesis problem for the case of using static multipliers. There, the essential structure needed in the derivation was that the control input u is unobservable from the output q. By choosing the particular realization (9), this structure is preserved. As a consequence, the analysis conditions (11)-(13), as considered for the closed loop system, can be turned into an LMI problem.

Theorem 4. Given the interconnection in Figure 1, let $\Delta = \delta(t)I_r$ with $(\delta(t), \dot{\delta}(t)) \in \mathcal{R}$ for all $t \geq 0$, $\Pi \in \mathbf{\Pi}$ as in (18) with $Q \in \mathcal{Q}$ parameterized as in (15). Moreover, partition T, X according to the system matrix in (9) as

$$T = \left(\begin{array}{c|c} \bar{T}_{11} & \bar{T}_{12} \\ \hline \bar{T}_{12}^T & \bar{T}_{22} \end{array}\right), \quad X = \left(\begin{array}{c|c} \bar{X}_{11} & \bar{X}_{12} \\ \hline \bar{X}_{12}^T & \bar{X}_{22} \end{array}\right)$$

in which $\bar{T}_{11}, \bar{X}_{11}$ has compatible dimensions with A Further. let $\bar{K}, L, \bar{M}$ be partitioned as

$$\begin{pmatrix} \bar{M} \\ \bar{K} \end{pmatrix} = \begin{pmatrix} \bar{M}_1 & \bar{M}_2 \\ \bar{K}_1 & \bar{K}_2 \end{pmatrix}, \qquad L = \begin{pmatrix} L_1 \\ L_2 \end{pmatrix}$$

in which $\bar{K}_1, \bar{M}_1, L_1^T$ and $\bar{K}_2, \bar{M}_2, L_2^T$ have n and $n_1 + n_2 + n_W$ columns respectively. Then, there exists a controller such that the robust L_2-induced gain of $w \to z$ is smaller than γ if there exists $\{T, X, \bar{K}, L, \bar{M}, N, \check{X}, H\}$ and $Q \in \mathcal{Q}$ for which

$$(\,..\,)' \left(\begin{array}{cc|c|c} 0 & I & 0 & 0 \\ I & 0 & 0 & 0 \\ \hline 0 & 0 & Q & 0 \\ \hline 0 & 0 & 0 & J_\gamma \end{array}\right) \left(\begin{array}{c|c|c} I & 0 & 0 \\ \mathbf{A} & \mathbf{B}_{p,e} & \mathbf{B}_w \\ \hline C_e & \tilde{D}_{qp,e} & \tilde{D}_{qw} \\ \hline \mathbf{C}_z & \mathbf{D}_{zp,e} & \mathbf{D}_{zw} \\ 0 & 0 & I \end{array}\right) \prec 0, \quad (19)$$

$$(\,..\,)' \begin{pmatrix} 0 & \check{X} & 0 \\ \check{X} & 0 & 0 \\ 0 & 0 & Q \end{pmatrix} \begin{pmatrix} I & 0 \\ A_{11} & B_{11} \\ C_1 & D_1 \end{pmatrix} \succ 0 \quad (20)$$

and coupling condition

$$\left(\begin{array}{cc|cc} \bar{T}_{11} & 0 & I & \bar{T}_{12} \\ 0 & \bar{T}_{22} & 0 & \bar{T}_{22} \\ \hline I & 0 & \bar{X}_{11} & \bar{X}_{12} \\ \bar{T}_{12}' & \bar{T}_{22} & \bar{X}_{21} & \bar{X}_{22} \end{array}\right) - \left(\begin{array}{c|c} 1 & 1 \\ \hline 1 & 1 \end{array}\right) \otimes \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \check{X} & 0 & 0 \\ 0 & 0 & H & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \succ 0, \quad (21)$$

hold. Here $\mathbf{A}, \mathbf{C}_z$ are given in (22) at the bottom of the next page, J_γ as given by (5) and

$$\begin{aligned} \mathbf{B}_{p,e} &= \begin{pmatrix} \tilde{B}_{e_1} + \bar{T}_{12}\tilde{B}_{e_2} + B_u N \tilde{D}_{yp,e} \\ \bar{T}_{22}\tilde{B}_{e_2} \\ \bar{X}_{11}\tilde{B}_{e_1} + \bar{X}_{12}\tilde{B}_{e_2} + L_1 \tilde{D}_{yp,e} \\ \bar{X}_{21}\tilde{B}_{e_1} + \bar{X}_{22}\tilde{B}_{e_2} + L_2 \tilde{D}_{yp,e} \end{pmatrix}, \\ \mathbf{B}_w &= \begin{pmatrix} \tilde{B}_{w_1} + \bar{T}_{12}\tilde{B}_{w_2} + B_u N D_{yw} \\ \bar{T}_{22}\tilde{B}_{w_2} \\ \bar{X}_{11}\tilde{B}_{w_1} + \bar{X}_{12}\tilde{B}_{w_2} + L_1 D_{yw} \\ \bar{X}_{21}\tilde{B}_{w_1} + \bar{X}_{22}\tilde{B}_{w_2} + L_2 D_{yw} \end{pmatrix}, \\ \mathbf{D}_{zp,e} &= \tilde{D}_{zp,e} + D_{zu} N \tilde{D}_{yp,e}, \\ \mathbf{D}_{zw} &= \tilde{D}_{zw} + D_{zu} N \tilde{D}_{yw}, \\ C_e &= \begin{pmatrix} 0 & \tilde{C}_{q_2} & 0 & \tilde{C}_{q_2} \end{pmatrix}. \end{aligned} \quad (23)$$

Note that all boldface symbols depend on the decision variables in an affine fashion. In order to reconstruct the controller matrices, let

$$Y = \begin{pmatrix} \bar{T}_{11} + \bar{T}_{12}\bar{T}_{22}^{-1}\bar{T}_{12}' & -\bar{T}_{12}\bar{T}_{22}^{-1} \\ -\bar{T}_{22}^{-1}\bar{T}_{12}' & \bar{T}_{22}^{-1} \end{pmatrix} \quad (24)$$

and find matrices U, V such that $UV^T = I - XY$. Then, with

$$\begin{pmatrix} \hat{K} \\ \hat{M} \end{pmatrix} = \begin{pmatrix} \bar{K} \\ \bar{M} \end{pmatrix} \begin{pmatrix} I & 0 \\ \bar{T}_{12}' & \bar{T}_{22} \end{pmatrix}^{-1},$$

$$\left[\begin{array}{cccc|c|c|c} A & 0 & 0 & B_v C_v & B_v D_{vp} & B_v D_{vw} & B_u \\ 0 & A_{11} & A_{12} & B_{11} C_q & B_{12} + B_{11} D_{qp} & B_{11} D_{qw} & 0 \\ 0 & 0 & A_{22} & 0 & B_{22} & 0 & 0 \\ 0 & 0 & 0 & A_W & B_p & B_w & 0 \\ \hline 0 & C_1 & C_2 & D_1 C_q & D_2 + D_1 D_{qp} & D_1 D_{qw} & 0 \\ \hline C_z & 0 & 0 & D_{zv} C_v & D_{zv} D_{vp} & D_{zv} D_{vw} & D_{zu} \\ 0 & 0 & 0 & 0 & 0 & I & 0 \\ \hline C_y & 0 & 0 & D_{yv} C_v & D_{yv} D_{vp} & D_{yv} D_{vw} & 0 \end{array}\right] = \left[\begin{array}{cc|c|c|c} A & \tilde{A}_{12} & \tilde{B}_{e_1} & \tilde{B}_{w_1} & B_u \\ 0 & \tilde{A}_2 & \tilde{B}_{e_2} & \tilde{B}_{w_2} & 0 \\ \hline 0 & \tilde{C}_{q_2} & \tilde{D}_{qp,e} & \tilde{D}_{qw} & 0 \\ \hline \tilde{C}_{z_1} & \tilde{C}_{z_2} & \tilde{D}_{zp,e} & \tilde{D}_{zw} & D_{zu} \\ 0 & 0 & 0 & I & 0 \\ \hline \tilde{C}_1 & \tilde{C}_2 & \tilde{D}_{yp,e} & \tilde{D}_{yw} & 0 \end{array}\right] \quad (9)$$

the controller matrices can be obtained as

$$
\begin{aligned}
D_K &:= N \\
C_K &:= (\hat{M} - D_K\tilde{C}X)U^{-T} \\
B_K &:= V^{-1}(L - Y\tilde{B}D_K) \\
A_K &:= V^{-1}\big(\hat{K} - VB_K\tilde{C}X \\
&\quad -Y\tilde{B}C_K U^T - Y(\tilde{A} + \tilde{B}D_K\tilde{C})X\big)U^{-T}
\end{aligned}
\tag{25}
$$

Proof. The proof is an extension of the proof given in Dietz et al. [2007] that relies on combining two transformations from the literature. Here, a sketch of the proof is given due to space limitations. Using the realization (10) for the extended generalized plant N_e, closing the loop with controller K defined in (1) leads to the closed-loop system matrix

$$
\mathcal{A}_{cl} = \begin{pmatrix} \tilde{A} + \tilde{B}D_K\tilde{C} & \tilde{B}C_K \\ B_K\tilde{C} & A_K \end{pmatrix}. \tag{26}
$$

The first congruence transformation needed is taken from Scherer et al. [1997] and resolves the bilinearity between $\mathcal{A}_{cl}$ and the Lyapunov matrix $\mathcal{X}$ as it arises in (11).

After applying this transformation to condition (11), as considered for the closed-loop system, as well as introducing transformed controller parameters $\{\bar{K}, L, \bar{M}, N\}$, condition (19) follows with the substitutions

$$
\begin{aligned}
\mathbf{A} &\rightarrow \begin{pmatrix} \tilde{A}Y + \tilde{B}M & \tilde{A} + \tilde{B}N\tilde{C} \\ K & X\tilde{A} + L\tilde{C} \end{pmatrix}, \\
\mathbf{B}_p &\rightarrow \begin{pmatrix} \tilde{B}_{p,e} + \tilde{B}N\tilde{D}_{yp,e} \\ X\tilde{B}_{p,e} + L\tilde{D}_{yp,e} \end{pmatrix}, \\
\mathbf{B}_w &\rightarrow \begin{pmatrix} \tilde{B}_w + \tilde{B}N\tilde{D}_{yw} \\ X\tilde{B}_w + L\tilde{D}_{yw} \end{pmatrix}, \\
\mathbf{C}_z &\rightarrow \begin{pmatrix} \tilde{C}_zY + D_{zu}M & \tilde{C}_z + D_{zu}N\tilde{C} \end{pmatrix}, \\
\mathbf{D}_{zp} &\rightarrow \tilde{D}_{zp,e} + D_{zu}N\tilde{D}_{yp,e}, \\
\mathbf{D}_{zw} &\rightarrow \tilde{D}_{zw} + D_{zu}N\tilde{D}_{yw}, \\
C_e &\rightarrow \mathbf{C}_q = \begin{pmatrix} \tilde{C}_qY & \tilde{C}_q \end{pmatrix}.
\end{aligned}
\tag{27}
$$

Due to multiplication of C_e with Q, the matrix inequality (19) involves non-linear terms. By somehow rendering C_e independent on Y this problem would be overcome. Fortunately, due to the particular structure among $\tilde{A}, \tilde{B}$ and $\tilde{C}_q$ this is possible, as worked out in detail in Dietz et al. [2007]. As a consequence, a suitable congruence transformation turns condition (11), again using closed-loop realization matrices, into (19) with substitutions as in (23). As C_e no longer depends on Y, the expression (19) becomes affine in all of the new decision variables $\Theta := \{T, X, \bar{K}, L, \bar{M}, N, \check{X}, H\}$ and $Q \in \mathcal{Q}$.

The condition (12) for the closed loop system happens to be convex in $\check{X}$ already, i.e. (20). In view of Theorem 1, the coupling condition (13) is fundamentally different from the usual $\mathcal{X} \succ 0$. Without providing the details, one can show that a suitable congruence transformation turns coupling condition (13) into an LMI in the same new variables Θ.

For the controller reconstruction formulae the reader is referred to Scherer et al. [1997], Masubuchi et al. [1998], where further details can be found.

■

Remark 5. Note that using static scalings corresponds to A_{11}, A_{22} being empty matrices and $\bar{T}_{22}$ having the dimension of A_W. The key insight for handling dynamic IQC multipliers is therefore to start with a realization that displays the fact that input u is unobservable in output q.

Remark 6. We strongly emphasize that the synthesis result holds in full generality, i.e. for any set of uncertainties described by dynamic multipliers of the form (6).

4. ILLUSTRATIVE EXAMPLE

The robust controller synthesis algorithm of Theorem 4 is applied on an academic example. As shown in Section 3.2, time-varying parameters with specified bounds on the rate-of-variation (rov) can be captured by using dynamic multipliers. We will show that the proposed synthesis algorithm can improve the results obtained with static D/G scalings.

Let G be the plant model given as

$$
G(s) = \frac{s + 0.1}{(s + 0.2)(s + 0.5)}.
$$

We adopt the well-known S/KS methodology for solving the disturbance rejection problem. The performance output shown in Figure 1 is $z = \mathrm{col}(e, u)$, in which e is the tracking error. With additional weights W_u, W_e at the control output and tracking error, the plant P becomes

$$
P = \begin{pmatrix} -W_e & -W_eG \\ 0 & W_u \\ -I & -G \end{pmatrix}.
$$

The disturbances v are (non-stationary) sinusoidal disturbances with nominal frequency ω_0 that are generated as the second state of the autonomous system

$$
\dot{\xi} = \begin{pmatrix} 0 & \omega_0(1+\delta) \\ -\omega_0(1+\delta) & 0 \end{pmatrix}\xi, \quad \xi(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{28}
$$

in which δ is the time-varying parameter bounded by $\bar{\delta}$. Since the algorithm outlined in this paper requires the filter to be robustly stable, we treat sinusoidal signals with time-invariant frequency using the filter

$$
W_\delta(s, \delta) = \delta I \star W(s) = \kappa + \frac{2\zeta\omega_\delta s}{s^2 + 2\zeta\omega_\delta s + \omega_\delta^2}, \tag{29}
$$

where $\omega_\delta = \omega_0(1 + \delta)$ and $\delta \in [-\bar{\delta}, \bar{\delta}]$. For time-varying frequency $\omega_\delta(.)$ we can rely on a realization of this filter as in the interconnection of Figure 1 with matrices

$$
\left[\begin{array}{c|c:c} A_W & B_p & B_w \\ \hline C_q & D_{qp} & D_{qw} \\ \hdashline C_v & D_{vp} & D_{vw} \end{array}\right] = \left[\begin{array}{cc|cc:c} 0 & \omega_0 & 0 & \omega_0 & 1 \\ -\omega_0 & -2\zeta\omega_0 & -\omega_0 & -2\zeta\omega_0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \hdashline 0 & 2\omega_0 & 0 & 2\omega_0 & \kappa \end{array}\right]. \tag{30}
$$

$$
\left(\frac{\mathbf{A}}{\mathbf{C}_z}\right) = \left(\begin{array}{cccc} A\bar{T}_{11} + B_u\bar{M}_1 & -A\bar{T}_{12} + \tilde{A}_{12} + \bar{T}_{12}\tilde{A}_2 + B_u\bar{M}_2 & A + B_uN\tilde{C}_1 & \tilde{A}_{12} + \bar{T}_{12}\tilde{A}_2 + B_uN\tilde{C}_2 \\ 0 & \bar{T}_{22}\tilde{A}_2 & 0 & \bar{T}_{22}\tilde{A}_2 \\ \bar{K}_1 & \bar{K}_2 & \bar{X}_{11}A + L_1\tilde{C}_1 & \bar{X}_{11}\tilde{A}_{12} + \bar{X}_{12}\tilde{A}_2 + L_1\tilde{C}_2 \\ & & \bar{X}_{21}A + L_2\tilde{C}_1 & \bar{X}_{21}\tilde{A}_{12} + \bar{X}_{22}\tilde{A}_2 + L_2\tilde{C}_2 \\ \hline \tilde{C}_{z_1}\bar{T}_{11} + D_{zu}\bar{M}_1 & -\tilde{C}_{z_1}\bar{T}_{12} + \tilde{C}_{z_2} + D_{zu}\bar{M}_2 & \tilde{C}_{z_1} + D_{zu}N\tilde{C}_1 & \tilde{C}_{z_2} + D_{zu}N\tilde{C}_2 \end{array}\right) \tag{22}
$$

Upper bounds on the worst case $\mathcal{L}_2$-gain from the weighted input u to weighted output $z = \text{col}(e, u)$ are computed using the presented synthesis algorithm. The weights are chosen to be $W_u = 1$ and

$$W_e = \frac{0.5s + 0.35}{s + 0.01}.$$

The numerical values of the filter W defined in (29) are chosen as $\kappa = 0.2$, $\bar{\delta} = 0.3$, $\omega_0 = 0.05$, $\zeta = 0.005$. The first design denoted by K_{DG} was based on static D/G scales, as done in Dietz et al. [2007]. It provides stability and performance guarantees against arbitrarily fast time-varying parameters. With a synthesis optimal value of $\gamma = 1.25$, the resulting closed-loop sensitivity is shown in Figure 2.

Now let us apply Theorem 4. Choose $\alpha = 1$ and $\beta = 1$ and define H in (16). For rov-bounds $|\dot{\delta}| < 1$ and $|\dot{\delta}| < 0.06$ the resulting controllers are denoted K_1, K_2 respectively. As seen in Figure 2, the notch of design K_1 shifts to lower frequency, as compared to K_{DG}. In case the rov-bound is further reduced, the sharp notch essentially disappears. The synthesis optimal values are $\gamma = 1.32$ and $\gamma = 1.28$ for K_1,K_2 respectively. Time-domain simulations have been performed with a non-stationary sinusoidal disturbance input v generated as the initial response of the system (30) in which $\zeta = \kappa = 0$ and collected in Figure 3.

Remark 7. Since numerical algorithms (generally) provide upper bound values on the worst case $\mathcal{L}_2$-gain, reducing the rov-bound need not always result in lower optimal values.

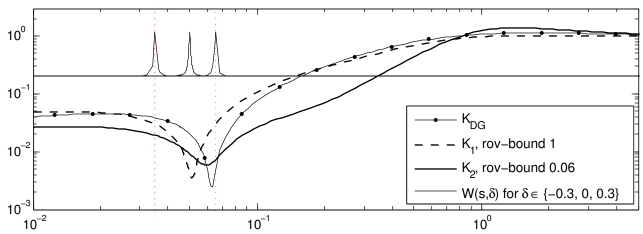

Fig. 2. Closed-loop sensitivity $v \to e$. Also shown is W_δ for parameter values $\delta \in \{-0.3, 0, 0.3\}$.

5. CONCLUSIONS

A complete solution to the robust controller synthesis has been presented in which the uncertainty affects the disturbance filter only. Adopting the IQC framework, our algorithm minimizes an upper bound on the worst-case $\mathcal{L}_2$-gain over the set of specified multipliers and all output feed-back controllers. For a single time-varying parameter a suitable IQC multiplier enables us to incorporate parameter rate bounds.

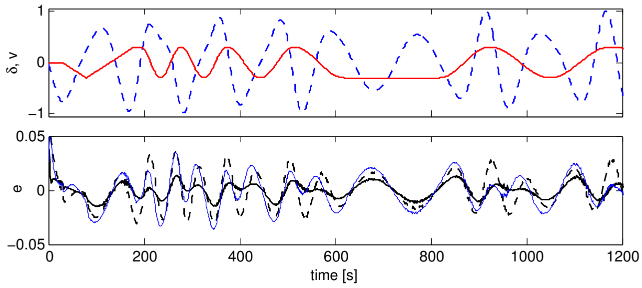

Fig. 3. Tracking error e for designs K_{DG}, K_1 (dashed) and K_2 (bold). On top, signals δ and v (dashed) are shown.

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