Proceedings of the 17th World Congress
The International Federation of Automatic Control
Seoul, Korea, July 6-11, 2008

Robust H_2 perfomance of discrete-time periodic systems: LMIs with reduced dimensions

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Abstract: Recent papers in the field of LMI-based robust control have provided extensions of known results for linear time-invariant systems to the case of periodically time varying linear systems. These results, theoretically satisfactory because formulated in terms of optimization problems of polynomial complexity, may still have limited applications in practice because the number of variables and constraints is very large. The present paper proposes a new formulation of these results that allows to reduce the computational burden both by reducing the number of decision variables and the size of the constraints. Along with this numerical improvement, the paper produces a new modeling of periodic discrete-time systems in descriptor form that is believed promising for future research.

Keywords: Periodic systems, LMI, Robust Control, H_2

1. INTRODUCTION

In the context of control of time-varying systems, an important sub-class is the periodic case. Many processes in chemistry, biology and mechanics are indeed conceived to follow a periodic path to achieve their goals. Control of this periodic path following problem formulates then as a control issue for models with periodically time-vaying parameters. As long as small errors are assumed along the trajectory, and in case of sampled-data control, linear periodic time-varying (LPTV) discrete-time models form a first approximation of such systems (see Bittanti and Colaneri [2007], Peaucelle et al. [2007c]). But this is not all, another situation when LPTV discrete-time systems occur is for multi-rate sampled-data systems (see for example Lall and Dullerud [2001], Sagfors et al. [2000]). Development of analysis (and then design) results for such models is therefore of major importance. Moreover, since linear models are only approximations of the original problem, results should be able to prove robust performances with respect to uncertainties.

Beyond the application issues, LPTV discrete-time models have also been intensively studied in the literature because of possible extensions of efficient tools known for linear time-invariant (LTI) systems. Most results therefore have interpretations in terms of *lifted* or *cyclic* LTI representations of the LPTV systems (see Bittanti and Colaneri [2000]). But these LTI representations have the disadvantage of being often complex to manipulate, in particular for uncertain systems, and methods working directly on the original periodic state-space representations have been produced recently for that purpose De Souza and Trofino [2000], Farges et al. [2007a,b]. Compared to results produced by the *lifted* representation these new results have (as for the *cyclic* representation case) the disadvantage to produce conditions of high numerical complexity. This explains by the use of Lyapunov-type variables of size $nN \times nN$ where n is the size of the state a each sample of time and N is the length of the period. This is in contrast with *lifted* representation type results in which the Lyapunov-type variable is of size $n \times n$ utilizing the fact that convergence properties can be assessed by the the behavior of only one representative state in each period.

The goal of the paper is to provide linear matrix inequality (LMI) conditions for robust stability of such systems combining both properties described above: simplicity for deriving robust results and reduced numerical complexity. Two main ideas are used for this purpose. One is descriptor-like modeling which, as illustrated in De Oliveira and Skelton [2001], Ebihara et al. [2005], Peaucelle et al. [2007a], proves efficient for simple derivation of robust conditions with reduced conservatism. The second is a technique taken from Lu et al. [2005] that allows in the LPTV context to reduce significantly the number of LMIs for the H_2 problem.

The outline of the paper is as follows. Section 2 provides the descriptor-like modeling of LPTV systems and performance analysis LMI results are derived based on that representation. H_2 performance is studied in details but stability and H_∞ performance are described as well. Section 3 is dedicated to robustness issues for the case of polytopic uncertainties. Robustness is achieved by means of slack-variables techniques (De Oliveira and Skelton [2001], Ebihara et al. [2005]) allowing parameter-dependent Lyapunov functions and thus reduced conservatism. Section 4 illustrates the theoretical results on a numerical example.

Notations: For two symmetric matrices, A and B, $A > (\geq) B$ means that $A - B$ is positive (semi-) definite. A^T denotes the transpose of A, A^* is the conjugate transpose. $\mathbf{1}_n$ and $\mathbf{0}_{m,n}$ denote the respectively the identity matrix of size n and null matrix of size $m \times n$. If the context allows it the dimensions of these matrices are often omitted. For a given matrix $B \in \mathsf{R}^{m \times n}$ such that $rank(B) = r$, we define $B^\perp \in \mathsf{R}^{n \times (n-r)}$ the right orthogonal complement of B by $BB^\perp = 0$ and $B^\perp B^{\perp T} > 0$. The notation $\langle A \rangle$ stands for the symmetric matrix $A + A^T$. For concatenated vectors the following notation is adopted: $\mathrm{vec}\left(x \; y \right) = \left(x^T \; y^T \right)^T$.

2. DESCRIPTOR MODELS AND LMIS

2.1 Lifted descriptor model

Assume the N-periodic linear system

$$x_{k+1} = A_k x_k + B_k w_k \quad , \quad z_k = C_k x_k + D_k w_k \tag{1}$$

where for all $k \geq 0$ the parameters satisfy the following periodic condition $A_{k+N} = A_k$, $B_{k+N} = B_k$, $C_{k+N} = C_k$, $D_{k+N} = D_k$. $x_k \in \mathsf{R}^{n_k}$ are the instantaneous states, $w_k \in \mathsf{R}^{m_k}$ are disturbance inputs and $z_k \in \mathsf{R}^{p_k}$ is a control output. All vectors are assumed to be possibly of variable length along the period. This is for example the case for periodic models representing multi-rate sampled-data systems, Lall and Dullerud [2001], Sagfors et al. [2000].

Systems such as (1) have two classical LTI representations, Bittanti and Colaneri [2000]. One, called *cyclic*, amounts to the relations between the actual states of the system obtained as the concatenation of all instantaneous states over one period $\mathrm{vec}\left(x_{iN} \cdots x_{(i+1)N-1} \right)$. The other, called *lifted*, considers only one instant of each period to be the representative of the systems' state, the remaining instants being seen as intermediate variables. Our approach is intermediate between these two.

The sequence $\{x_{iN}\}_{i \geq 0}$ is chosen to be the the representative state of the system (chosen as such because includes the initial conditions x_0). Any other choice $\{x_{iN+j}\}_{i \geq 0}$ with $j \in \{0 \ldots N-1\}$ is admissible as well (it has no influence on the results in the nominal case but does modify results for the robust case as illustrated in the examples). Stability of the system is in the following defined with respect to convergence of the representative state. Nevertheless, the other intermediate states are kept in the model and gathered in the following sequence $\{\eta_i = \mathrm{vec}\left(x_{iN+1} \cdots x_{(i+1)N-1} \right)\}_{i \geq 0}$.

Define as well $\hat{w}_i = \mathrm{vec}\left(w_{iN} \cdots w_{(i+1)N-1} \right)$ and $\hat{z}_i = \mathrm{vec}\left(z_{iN} \cdots z_{(i+1)N-1} \right)$ respectively the vectors of disturbances inputs and control outputs over a period. Using these notations, the dynamics of the system involve the signals stacked in a unique vector $q_i = \mathrm{vec}\left(x_{iN} \; \eta_i \; x_{(i+1)N} | \hat{w}_i | \hat{z}_i \right)$ and are fully modeled by the descriptor-like form

$$\hat{M} q_i = 0 \quad , \quad \forall i \geq 0 \tag{2}$$

where $\hat{M} = \begin{bmatrix} \hat{A} & \hat{B} & \mathbf{0}_{n,p} \\ \hat{C} & \hat{D} & -1_p \end{bmatrix} =$

$$\left[\begin{array}{cccc|ccc|ccc} A_0 & -\mathbf{1}_{n_1} & & \mathbf{0} & B_0 & & \mathbf{0} & \mathbf{0} & & \mathbf{0} \\ & \ddots & \ddots & & & \ddots & & & & \\ \mathbf{0} & & A_{N-1} & -\mathbf{1}_{n_N} & \mathbf{0} & & B_{N-1} & \mathbf{0} & & \mathbf{0} \\ \hline C_0 & \mathbf{0} & & \mathbf{0} & D_0 & & \mathbf{0} & -\mathbf{1}_{p_0} & & \mathbf{0} \\ & \ddots & \ddots & & & \ddots & & & \ddots & \\ \mathbf{0} & & C_{N-1} & \mathbf{0} & \mathbf{0} & & D_{N-1} & \mathbf{0} & & -\mathbf{1}_{p_{N-1}} \end{array}\right].$$

$n = \sum_{k=1}^N n_k$, $m = \sum_{k=1}^N m_k$ and $p = \sum_{k=1}^N p_k$ define the overall dimensions of the model.

2.2 Stability analysis

Since all intermediate states of a periodic system are bounded as long as the representative state x_{iN} and the disturbances $\hat{w}_i$ are bounded, stability may be proved in the Lyapunov context as follows.

Theorem 1. Asymptotic stability of the N-periodic system $x_{k+1} = A_k x_k$ where $A_{k+N} = A_k$ is equivalent to the existence of a Lyapunov function $V_i = x_{iN}^T P x_{iN}$ with $P > \mathbf{0}$ such that $V_{i+1} < V_i$, $(\forall x_{iN} \neq 0)$. An LMI formulation of this result is

$$P > \mathbf{0} \quad , \quad \hat{A}^{\perp T} \begin{bmatrix} -P & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0}_{n-n_N} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & P \end{bmatrix} \hat{A}^\perp < \mathbf{0}. \tag{3}$$

Proof. Stability of the linear periodic system is equivalent proving Schur stability of the monodromy matrix $\Phi_{N-1,0} = A_{N-1} \cdots A_1 A_0$ which is such that $x_{(i+1)N} = \Phi_{N-1,0} x_{iN}$. Let $P > \mathbf{0}$ be the Lyapunov matrix that proves the Schur stability of $\Phi_{N-1,0}$ then $\Phi_{N-1,0}^T P \Phi_{N-1,0} < P$. Noticing that

$$\hat{A}^{\perp T} = \left[\mathbf{1} \; A_0^T \; (A_1 A_0)^T \ldots (A_{N-1} \cdots A_1 A_0)^T \right]$$

this last inequality is exactly (3).

Since Schur stability of monodromy matrices is independent of the initial instant used to define it ($\Phi_{N+j,1+j}$ Schur stable for all j), the result is independent of the choice of the representative state.

2.3 H_2 performance analysis

Theorem 2. The H_2 norm of the N-periodic system is the solution to the LMI optimization problem

$$\min \sqrt{\tfrac{1}{N} \mathrm{Trace}(T)}$$

subject to the constraints

$$\hat{M}^{\perp T} \left[\begin{array}{ccc|c|c} -P & \mathbf{0} & \mathbf{0} & W & \mathbf{0} \\ \mathbf{0} & \mathbf{0}_{n-n_N} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & P & \mathbf{0} & \mathbf{0} \\ \hline W^T & \mathbf{0} & \mathbf{0} & -T & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1}_p \end{array}\right] \hat{M}^\perp < \mathbf{0} \tag{4}$$

where $X > \mathbf{0} \in \mathsf{R}^{n_N \times n_N}$, $T \in \mathsf{R}^{m_w \times m_w}$ and $W \in \mathsf{R}^{n_1 \times m_w}$.

Before producing the proof, a technical lemma is stated. It is a reformulation of the technique in Lu et al. [2005] that allows to reduce the size of LMIs for H_2 discrete-time analysis. The lemma is proved in Peaucelle et al. [2007b].

Lemma 3. Let $Z_{i=1\ldots5}$, $M_{i=1\ldots4}$ be given matrices of appropriate dimensions with $Z_2 \geq \mathbf{0}$ and $Z_3 \geq \mathbf{0}$ positive semi-definite. The following two conditions hold simultaneously

$$x^T \begin{bmatrix} Z_1 & \mathbf{0} & Z_5 \\ \mathbf{0} & -Z_2 & \mathbf{0} \\ Z_5^T & \mathbf{0} & Z_4 \end{bmatrix} x < 0, \quad \begin{array}{l} \forall x \neq 0 \ : \\ [\, M_1 \ M_2 \ M_4 \,]\, x = 0 \end{array} \tag{5}$$

$$y^T \begin{bmatrix} Z_1 & \mathbf{0} & Z_5 \\ \mathbf{0} & -Z_3 & \mathbf{0} \\ Z_5^T & \mathbf{0} & Z_4 \end{bmatrix} y < 0, \quad \begin{array}{l} \forall y \neq 0 \ : \\ [\, M_1 \ M_3 \ M_4 \,]\, y = 0 \end{array} \tag{6}$$

if and only if there exists a matrix Z_{23} such that

$$z^T \begin{bmatrix} Z_1 & \mathbf{0} & \mathbf{0} & Z_5 \\ \mathbf{0} & -Z_2 & Z_{23} & \mathbf{0} \\ \mathbf{0} & Z_{23}^T & -Z_3 & \mathbf{0} \\ Z_5^T & \mathbf{0} & \mathbf{0} & Z_4 \end{bmatrix} z < 0 \quad \begin{array}{l} \forall z \neq 0 \ : \\ [\, M_1 \ M_2 \ M_3 \ M_4 \,]\, z = 0 \end{array} \tag{7}$$

Proof. The proof of Theorem 2 is made assuming $N = 3$ for simplicity. The general case has no more complexity.

Recall the grammian-based formulation of the H_2 performance problem, Bittanti and Cuzzola [2000]. If $\|\Sigma\|_2$ denotes the H_2 norm of the system, then

$$\|\Sigma\|_2^2 = \tfrac{1}{N} \textstyle\sum_{k=0}^{N-1} \mathrm{Trace}(D_{zk}^T D_{zk} + B_{wk}^T P_{k+1} B_{wk}) \tag{8}$$

where the grammians P_k are N-periodic ($P_{k+N} = P_k$) and solution of

$$A_k^T P_{k+1} A_k - P_k + C_{zk}^T C_{zk} = \mathbf{0}. \tag{9}$$

Combining all equalities (9) for $k = 0 \ldots 2$ gives

$$P_0 = \begin{bmatrix} A_2A_1A_0 \\ C_0 \\ C_1A_0 \\ C_2A_1A_0 \end{bmatrix}^T \begin{bmatrix} P_0 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1}_{p_0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1}_{p_1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1}_{p_2} \end{bmatrix} \begin{bmatrix} A_2A_1A_0 \\ C_0 \\ C_1A_0 \\ C_2A_1A_0 \end{bmatrix} \tag{10}$$

and formula (8) also reads as $\|\Sigma\|_2^2 = \frac{1}{N}\sum_{k=0}^{N-1} \mathrm{Trace}(T_k)$ where

$$T_0 = \begin{bmatrix} A_2A_1B_0 \\ D_0 \\ C_1B_0 \\ C_2A_1B_0 \end{bmatrix}^T \begin{bmatrix} P_0 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1}_{p_0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1}_{p_1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1}_{p_2} \end{bmatrix} \begin{bmatrix} A_2A_1B_0 \\ D_0 \\ C_1B_0 \\ C_2A_1B_0 \end{bmatrix} \tag{11}$$

$$T_1 = \begin{bmatrix} A_2B_1 \\ D_1 \\ C_2B_1 \end{bmatrix}^T \begin{bmatrix} P_0 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1}_{p_1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1}_{p_2} \end{bmatrix} \begin{bmatrix} A_2B_1 \\ D_1 \\ C_2B_1 \end{bmatrix} \tag{12}$$

$$T_2 = \begin{bmatrix} B_2 \\ D_2 \end{bmatrix}^T \begin{bmatrix} P_0 & \mathbf{0} \\ \mathbf{0} & \mathbf{1}_{p_2} \end{bmatrix} \begin{bmatrix} B_2 \\ D_2 \end{bmatrix} \tag{13}$$

All four equality signs of the last equations can be replaced by $>$ inequality signs in which case $\frac{1}{3}\sum_{k=1}^{3} \mathrm{Trace}(T_k)$ defines an upper bound on the H_2 norm. Minimization over the resulting LMI constraints will give the exact H_2 performance. In the following the upper bound on P_0 is denoted P. The remaining of the proof demonstrates the LMIs defined in this way are equivalent to the LMI (4).

Based on the fact that $\left[\begin{array}{c|c|c} -\mathbf{1}_{n_3} & B_2 & \mathbf{0} \\ \hline \mathbf{0} & D_2 & -\mathbf{1}_{p_2} \end{array}\right]^{\perp} = \begin{bmatrix} B_2 \\ \mathbf{1}_{m_2} \\ D_2 \end{bmatrix}$, the matrix inequality issued from (13) may be reformulated as:

$$q_2^T \begin{bmatrix} P & & \mathbf{0} \\ & -T_2 & \\ \mathbf{0} & & \mathbf{1}_{p_2} \end{bmatrix} q_2 < 0, \quad \begin{array}{l} \forall q_2 \neq 0 : \\ \left[\begin{array}{ccc} -\mathbf{1}_{n_3} & B_2 & \mathbf{0} \\ \hline \mathbf{0} & D_2 & -\mathbf{1}_{p_2} \end{array}\right] q_2 = 0 \end{array} \tag{14}$$

where $q_2 = \mathrm{vec}\,(\, x_3 \ w_2 \ z_2 \,)$. Trivially it also reads as:

$$q_2^T \begin{bmatrix} P & & \mathbf{0} \\ & -T_2 & \\ \mathbf{0} & & \mathbf{1}_{p_2} \end{bmatrix} q_2 < -z_0^T z_0 - z_1^T z_1$$

holds for all vectors such that $q_2 \neq 0$ and

$$\begin{array}{l} x_2 = 0 \ \ , \ \ x_1 = 0 \ \ , \\ z_1 = 0 \ \ , \ \ z_0 = 0 \ \ , \end{array} \left[\begin{array}{ccc} -\mathbf{1}_{n_3} & B_2 & \mathbf{0} \\ \hline \mathbf{0} & D_2 & -\mathbf{1}_{p_2} \end{array}\right] q_2 = 0.$$

Whatever A_1, A_2, C_1, C_2, these last equality constraints are equivalent to

$$\left[\begin{array}{ccc|c|ccc} -\mathbf{1}_{n_1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ A_1 & -\mathbf{1}_{n_2} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & A_2 & -\mathbf{1}_{n_3} & B_2 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{1}_{p_0} & \mathbf{0} & \mathbf{0} \\ C_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{1}_{p_1} & \mathbf{0} \\ \mathbf{0} & C_2 & \mathbf{0} & D_2 & \mathbf{0} & \mathbf{0} & -\mathbf{1}_{p_2} \end{array}\right] \hat{q}_2 = 0 \tag{15}$$

where $\hat{q}_2 = \mathrm{vec}\,(\, x_1 \ x_2 \ x_3 \ w_2 \ z_0 \ z_1 \ z_2 \,) \neq 0$. For compactness of next formulas, decompose the matrix in (15) in three bloc-columns with notations

$$M_1^T = \left[\begin{array}{ccc|ccc} -\mathbf{1}_{n_1} & A_1^T & \mathbf{0} & \mathbf{0} & C_1^T & \mathbf{0} \\ \mathbf{0} & -\mathbf{1}_{n_2} & A_2^T & \mathbf{0} & \mathbf{0} & C_2^T \\ \mathbf{0} & \mathbf{0} & -\mathbf{1}_{n_3} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array}\right], \ M_4 = \begin{bmatrix} \mathbf{0}_{n,p} \\ -\mathbf{1}_p \end{bmatrix}$$

and $M_{32} = \left[\, \mathbf{0} \ \mathbf{0} \ B_2^T \middle| \mathbf{0} \ \mathbf{0} \ D_2^T \,\right]^T$. With these, the fact (14) is equivalent to

$$\hat{q}_2^T \left[\begin{array}{c|c|c} \begin{matrix} \mathbf{0}_{n-n_3} & \\ & P \end{matrix} & & \mathbf{0} \\ \hline & -T_2 & \\ \hline \mathbf{0} & & \mathbf{1}_p \end{array}\right] \hat{q}_2 < 0, \quad \begin{array}{l} \forall \hat{q}_2 \neq 0 \ : \\ [\, M_1 | M_{32} | M_4 \,]\, \hat{q}_2 = 0 \end{array} \tag{16}$$

With similar considerations, starting from the fact that

$$\left[\begin{array}{cc|c|cc} -\mathbf{1}_{n_2} & \mathbf{0} & B_1 & \mathbf{0} & \mathbf{0} \\ A_2 & -\mathbf{1}_{n_3} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & D_1 & -\mathbf{1}_{p_1} & \mathbf{0} \\ C_2 & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{1}_{p_2} \end{array}\right]^{\perp} = \begin{bmatrix} B_1 \\ A_2B_1 \\ \mathbf{1}_{m_1} \\ D_1 \\ C_2B_1 \end{bmatrix},$$

define $M_{31} = \left[\, \mathbf{0} \ B_1^T \ \mathbf{0} \middle| \mathbf{0} \ D_1^T \ \mathbf{0} \,\right]^T$ and get that the inequalities issued from (12) can be formulated as:

$$\hat{q}_1^T \left[\begin{array}{c|c|c} \begin{matrix} \mathbf{0}_{n-n_N} & \\ & P \end{matrix} & & \mathbf{0} \\ \hline & -T_1 & \\ \hline \mathbf{0} & & \mathbf{1}_p \end{array}\right] \hat{q}_1 < 0, \quad \begin{array}{l} \forall \hat{q}_1 \neq 0 \ : \\ [\, M_1 | M_{31} | M_4 \,]\, \hat{q}_1 = 0 \end{array} \tag{17}$$

Let $M_{30}^T = \left[\, B_0^T \ \mathbf{0} \ \mathbf{0} \middle| D_0^T \ \mathbf{0} \ \mathbf{0} \,\right]$, $M_2^T = \left[\, A_0^T \ \mathbf{0} \ \mathbf{0} \middle| C_0^T \ \mathbf{0} \ \mathbf{0} \,\right]$ the same reasoning as above applied to (11) and (10) gives respectively:

$$\hat{q}_0^T \left[\begin{array}{c|c|c} \begin{matrix} \mathbf{0}_{n-n_N} & \\ & P \end{matrix} & & \mathbf{0} \\ \hline & -T_0 & \\ \hline \mathbf{0} & & \mathbf{1}_p \end{array}\right] \hat{q}_0 < 0, \quad \begin{array}{l} \forall \hat{q}_0 \neq 0 \ : \\ [\, M_1 | M_{30} | M_4 \,]\, \hat{q}_0 = 0 \end{array} \tag{18}$$

$$\hat{q}^T \left[\begin{array}{c|c|c} -P & & \mathbf{0} \\ \hline & \begin{matrix} \mathbf{0}_{n-n_N} & \\ & P \end{matrix} & \\ \hline \mathbf{0} & & \mathbf{1}_p \end{array}\right] \hat{q} < 0, \quad \begin{array}{l} \forall \hat{q} \neq 0 \ : \\ [\, M_2 | M_1 | M_4 \,]\, \hat{q} = 0 \end{array} \tag{19}$$

Due to Lemma 3 condition (16) combined with (17) implies the existence of a matrix T_{12} such that

$$\tilde{q}_1^T \left[\begin{array}{c|c|c} \mathbf{0}_{n-n_N} & & \mathbf{0} \\ \quad P & & \\ \hline & -\hat{T}_1 & \\ \hline \mathbf{0} & & \mathbf{1}_p \end{array}\right] \tilde{q}_1 < 0, \quad \begin{array}{c} \forall \tilde{q}_1 \neq 0 \;\; : \\ \left[\, M_1 \,|\, \tilde{M}_1 \,|\, M_4 \,\right] \tilde{q}_1 = 0 \end{array} \tag{20}$$

where $\tilde{M}_1 = [\, M_{31} \;\; M_{32} \,]$, $\hat{T}_1 = \begin{bmatrix} T_1 & T_{12} \\ T_{12}^T & T_2 \end{bmatrix}$ and T_{12} is the matrix variable created when applying the lemma. Lemma 3 is applied again to condition (18) combined with (20) and implies the existence of a matrix $\hat{T}_{02}$ such that

$$\tilde{q}_0^T \left[\begin{array}{c|c|c} \mathbf{0}_{n-n_N} & & \mathbf{0} \\ \quad P & & \\ \hline & -T & \\ \hline \mathbf{0} & & \mathbf{1}_p \end{array}\right] \tilde{q}_0 < 0, \quad \begin{array}{c} \forall \tilde{q}_0 \neq 0 \;\; : \\ \left[\, M_1 \,|\, \tilde{M}_0 \,|\, M_4 \,\right] \tilde{q}_0 = 0 \end{array} \tag{21}$$

where $\tilde{M}_0 = \left[\, M_{30} \;\; \tilde{M}_1 \,\right]$ and $T = \begin{bmatrix} T_0 & \hat{T}_{02} \\ \hat{T}_{02}^T & \hat{T}_1 \end{bmatrix}$. At this point it is clear that taking $N > 3$ would not complexily the proof, it only needs performing the last operation as many times as there are samples in a period.

Lemma 3 (with modified order in the columns) can be applied to condition (19) combined with (21). Noticing that $\hat{M} = \left[\, M_2 \;\; M_1 \;\; \tilde{M}_0 \;\; M_4 \,\right]$, it implies the existence of W such that

$$\tilde{q}^T \left[\begin{array}{cccc|c} -P & \mathbf{0} & \mathbf{0} & W & \mathbf{0} \\ \mathbf{0} & \mathbf{0}_{n-n_N} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & P & \mathbf{0} & \mathbf{0} \\ \hline W^T & \mathbf{0} & \mathbf{0} & -T & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1}_p \end{array}\right] \tilde{q} < \mathbf{0} \;,\; \forall \tilde{q} \neq 0 \;:\; \hat{M}\tilde{q} = 0$$

which is exactly condition (4). The proof ends with the trivial fact $\mathrm{Trace}(T) = \sum_{k=0}^{N-1} \mathrm{Trace}(T_k)$.

A similar result as upper is obtained for H_∞ performance. For space limitation reasons, it is no given here, details can be found in the extended version: Peaucelle et al. [2007b].

3. ROBUSTNESS ISSUES

In the following we concentrate on the H_2 problems but results apply the same to stability and H_∞ analysis. Before entering the core of the robustness issue, note that

$$\hat{M}^\perp = \begin{bmatrix} \mathbf{1}_{n+n_N} & \mathbf{0} & \hat{C}^T \\ \mathbf{0} & \mathbf{1}_m & \hat{D}^T \end{bmatrix} \left[\, \hat{A} \;\; \hat{B} \,\right]^\perp, \tag{22}$$

hence the H_2 cost problem writes also as the minimization of $\mathrm{Trace}(T)$ subject to $\left[\, \hat{A} \;\; \hat{B} \,\right]^{\perp T} \Xi \left[\, \hat{A} \;\; \hat{B} \,\right]^\perp \leq \mathbf{0}$ where

$$\Xi = \left[\begin{array}{ccc|c} -P & \mathbf{0} & \mathbf{0} & W \\ \mathbf{0} & \mathbf{0}_{n-n_N} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & P & \mathbf{0} \\ \hline W^T & \mathbf{0} & \mathbf{0} & -T \end{array}\right] + \left[\begin{array}{c} \hat{C}^T \\ \hline \hat{D}^T \end{array}\right] \left[\begin{array}{c} \hat{C}^T \\ \hline \hat{D}^T \end{array}\right]^T. \tag{23}$$

Polytopic uncertainty is considered. The system data is assumed unknown and bounded in the convex set generated by a finite number of vertices such that

$$\hat{M}(\zeta) = \textstyle\sum_{v=1}^{\bar{v}} \zeta_v \hat{M}^{[v]} \;:\; \sum_{v=1}^{\bar{v}} \zeta_v = 1 \;,\; \zeta_v \geq 0 \tag{24}$$

where the vertices $\hat{M}^{[v=1\ldots\bar{v}]}$ have the structure defined in section 2.1. Given $P^{[v=1\ldots\bar{v}]}$, $W^{[v=1\ldots\bar{v}]}$ and $T^{[v=1\ldots\bar{v}]}$, let the following parameter-dependent matrices

$$P(\zeta) = \sum_{v=1}^{\bar{v}} \zeta_v P^{[v]}, W(\zeta) = \sum_{v=1}^{\bar{v}} \zeta_v W^{[v]}, T(\zeta) = \sum_{v=1}^{\bar{v}} \zeta_v T^{[v]}.$$

Robustness is proved in the following using these parameter-dependent matrices. To this end denote $\Xi^{[v]}$ the matrix defined as in (23) with vertex matrices $P^{[v]}$, $W^{[v]}$, $T^{[v]}$, $\hat{C}^{[v]}$, $\hat{D}^{[v]}$ and denote $\Xi(\zeta)$ the one with parameter-dependent matrices $P(\zeta)$, $W(\zeta)$, $T(\zeta)$, $\hat{C}(\zeta)$, $\hat{D}(\zeta)$.

Theorem 4. If ρ^* is the solution to the following LMI optimization problem

$$\begin{array}{l} \min \rho \;\; : \quad \forall v \in \{1 \ldots \bar{v}\} \\ \mathrm{Trace}(T^{[v]}) \leq \rho \;\;,\;\; \Xi^{[v]} + \left\langle F \left[\, \hat{A}^{[v]} \;\; \hat{B}^{[v]} \,\right] \right\rangle < \mathbf{0} \end{array} \tag{25}$$

with decision variables ρ, $P^{[v=1\ldots\bar{v}]}$, $W^{[v=1\ldots\bar{v}]}$, $T^{[v=1\ldots\bar{v}]}$ and F, then $\gamma_{nc} = \sqrt{\rho^*/N}$ is an upper bound on the H_2 norm of all systems in the uncertainty set.

Proof. The inequalities in (25) are convex with respect to the vertex matrices with notations $^{[v]}$, therefore if they hold for all vertices they also hold for all values in their convex hull:

$$\begin{array}{c} \forall \zeta \in \{\; \zeta_v \geq \mathbf{0} \;\;,\;\; \sum_{v=1}^{\bar{v}} \zeta_v = 1 \;\} \\ \mathrm{Trace}(T(\zeta)) \leq \rho \;,\; \Xi(\zeta) + \left\langle F \left[\, \hat{A}(\zeta) \;\; \hat{B}(\zeta) \,\right] \right\rangle < \mathbf{0} \end{array}$$

Post and pre-multiply the last inequality by $\left[\, \hat{A}(\zeta) \;\; \hat{B}(\zeta) \,\right]^\perp$ and by its transpose respectively to get

$$\left[\, \hat{A}(\zeta) \;\; \hat{B}(\zeta) \,\right]^{\perp T} \Xi(\zeta) \left[\, \hat{A}(\zeta) \;\; \hat{B}(\zeta) \,\right]^\perp \leq \mathbf{0},$$

which is the parameter-dependent version of (4) when utilizing fact (22).

The matrix F introduced in Theorem 4 is a slack variable that makes tractable the robust parameter-dependent problem. Yet, it has the disadvantage to add many additional variables in the LMI optimization. These should therefore be studied in details: without conservatism the matrix F factorizes as $F^T = \hat{F} \left[\, \hat{G} \;\; \hat{H} \,\right]$ where

$$\hat{F} = \begin{bmatrix} F_0 & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & F_{N-1} \end{bmatrix}, \hat{H} = \begin{bmatrix} H_{0,0} & & H_{0,N-1} \\ & \ddots & \\ H_{N-1,0} & & H_{N-1,N-1} \end{bmatrix},$$

$$\hat{G} = \begin{bmatrix} G_{0,0} & -\mathbf{1}_{n_1} & & G_{0,N} \\ & \ddots & \ddots & \\ G_{N-1,0} & & G_{N-1,N-1} & -\mathbf{1}_{n_N} \end{bmatrix}.$$

For the same reasons as in the proof of Theorem 4, the slack variable F needs to satisfy inequality

$$\left[\, \hat{G} \;\; \hat{H} \,\right]^{\perp T} \Xi(\zeta) \left[\, \hat{G} \;\; \hat{H} \,\right]^\perp \leq \mathbf{0}$$

which, according to the exposed results, implies that $\sqrt{\rho/N}$ is an upper bound on the H_2 norm of the virtual system $\left[\begin{array}{c|c|c} \hat{G} & \hat{H} & \mathbf{0}_{n,p} \\ \hat{C}(\zeta) & \hat{D}(\zeta) & -\mathbf{1}_p \end{array}\right] q_i = 0 \;\;,\;\; \forall i \geq 0.$

A feature of this virtual system is that it contains more dynamics than the original system (x_{iN+k} depends of all previous states down to x_{iN}) and it is non-causal (x_{iN+k} depends of the future states up to $x_{(i+1)N}$). Theorem 4 is therefore referred to in the following as the *non-causal* slack variable result. To explore the effects of introducing non-causal virtual system, three other optimization problems are defined as well

• *Dynamic causal* slack variable result (the virtual system is causal): $\gamma_d = \sqrt{\rho^*/N}$ where ρ^* is the solution of the optimization problem (25) with F constrained as $F^T =$

$$\left[\begin{array}{cccc|cc} F_{0,0} & F_{0,1} & & \mathbf{0} & F_{0,N+1} & \mathbf{0} \\ & \ddots & \ddots & & & \ddots \\ F_{N-1,0} & & F_{N-1,N-1} & F_{N-1,N} & F_{N-1,N+1} & F_{N-1,2N} \end{array}\right]$$

• *Static* slack variable result (the virtual system is periodic with no memory of previous intermediate states): $\gamma_s = \sqrt{\rho^*/N}$ where ρ^* is the solution of the optimization problem (25) with F constrained as $F^T =$

$$\left[\begin{array}{cccc|cc} F_{0,0} & F_{0,1} & & \mathbf{0} & F_{0,N+1} & \mathbf{0} \\ & \ddots & \ddots & & & \ddots \\ \mathbf{0} & & F_{N-1,N-1} & F_{N-1,N} & \mathbf{0} & F_{N-1,2N} \end{array}\right]$$

• *Zero* slack variable result (the virtual system has all its dynamics and perturbation inputs equal to zero): $\gamma_z = \sqrt{\rho^*/N}$ where ρ^* is the solution of the optimization problem (25) with F constrained as

$$F^T = \left[\begin{array}{cccc|cc} \mathbf{0} & F_{0,1} & & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ & \ddots & \ddots & & & \ddots \\ \mathbf{0} & & \mathbf{0} & F_{N-1,N} & \mathbf{0} & \mathbf{0} \end{array}\right]$$

Trivially one has $\gamma_{nc} \leq \gamma_d \leq \gamma_s \leq \gamma_z$. Conservatism reduction due to non-causal, dynamic slack variables, goes along with an increasing complexity of the LMIs. This increasing complexity is due to increased number of variables (the matrices F_{ij}). It depends only on the order nN of the periodic system, but does not depend on the number of vertices of the polytopic set. The dimension of LMIs is identical for all four results.

This contrasts with the complexity of other existing conservative formulations of the same problem. These are recalled now (results are partially improved compared to their original formulation by allowing the T_k matrices to be parameter-dependent).

Proposition 5. (Quadratic stability). If ρ^* is the solution of the optimization problem

$$\min \rho \ : \ \sum_{k=0}^{N-1} \mathrm{Trace}(T_k^{[v]}) \leq \rho \qquad (26)$$

constrained by $P_N = P_0$ and for all $v = 1 \ldots \bar{v}$, $k = 0 \ldots N$:

$$A_k^{[v]^T} P_{k+1} A_k^{[v]} - P_k + C_k^{[v]^T} C_k^{[v]} < \mathbf{0}$$
$$B_k^{[v]^T} P_{k+1} B_k^{[v]} - T_k^{[v]} + D_k^{[v]^T} D_k^{[v]} < \mathbf{0}$$

then $\gamma_q = \sqrt{\rho^*/N}$ is an upper bound on the H_2 norm of all systems in the uncertainty set.

Proposition 6. (Farges et al. [2007a]). If ρ^* is the solution of the optimization problem (26) constrained by $P_N = P_0$ and for all $v = 1 \ldots \bar{v}$, $k = 0 \ldots N$:

$$\begin{bmatrix} -P_k^{[v]} + C_k^{[v]^T} C_k^{[v]} & \mathbf{0} \\ \mathbf{0} & P_{k+1}^{[v]} \end{bmatrix} + \left\langle \begin{bmatrix} \tilde{F}_{1k} \\ \tilde{F}_{2k} \end{bmatrix} \begin{bmatrix} A_k^{[v]} & -\mathbf{1} \end{bmatrix} \right\rangle < \mathbf{0}$$
$$\begin{bmatrix} -T_k^{[v]} + D_k^{[v]^T} D_k^{[v]} & \mathbf{0} \\ \mathbf{0} & P_{k+1}^{[v]} \end{bmatrix} + \left\langle \begin{bmatrix} \tilde{F}_{3k} \\ \tilde{F}_{4k} \end{bmatrix} \begin{bmatrix} B_k^{[v]} & -\mathbf{1} \end{bmatrix} \right\rangle < \mathbf{0}$$

then $\gamma_f = \sqrt{\rho^*/N}$ is an upper bound on the H_2 norm of all systems in the uncertainty set.

Moreover, similarly to the above, γ_{fz} is defined as resulting of the same optimization problem but constraining the slack variables as $F_{1k} = \mathbf{0}$, $F_{2k} = F_{4k}$ and $F_{3k} = \mathbf{0}$. Trivially $\gamma_{fz} \leq \gamma_f$. Both γ_z and γ_{fz} are defined because these restrictions on slack variables allow simple derivation of control design results (see Farges et al. [2007a]).

The "quadratic stability" type result (a unique quadratic Lyapunov function is used for all uncertainties) proves easily to be always more conservative than the other results ($\gamma_q \geq \gamma_z$ and $\gamma_q \geq \gamma_{fz}$). It is also the formulation with lower numerical complexity. Unfortunately, there is no such possibility to order the results of Farges et al. [2007a] with those of the current paper. But it is possible to compare them in terms of numerical complexity.

- γ_{nc}: $\bar{v}(n_0+m)(n_0+m+1)/2+n(n_0+n+m)$ variables and $\bar{v}(n+n_0+m)$ rows in the LMIs;
- γ_f: $\sum_{k=0}^{N-1} \bar{v}\,(n_k(n_k+1)/2 + m_k(m_k+1)/2) + (n_k + 2n_{k+1} + m_k)n_k$ variables and $\bar{v}(3n+m)$ rows in the LMIs.

The second formulation is more complex in terms of size of the constraints and, if the number of vertices $\bar{v}$ is large, it is also more demanding in terms of number of variables. The new formulation proposed in the paper is we believe promising in terms of compromise between conservatism reduction and increased numerical complexity.

4. NUMERICAL EXAMPLE

The same numerical example as in Farges et al. [2007a] is considered. Its dimensions are such that $n_k = 2$, $p_k = m_k = 1$ for all $k \geq 0$ and $N = 3$. It contains two uncertain parameters which are constrained as $|\alpha| \leq 0.01$ and $0 \leq \beta \leq 1$. The system is open-loop unstable but the following periodic state-feedback control ($u_k = K_k x_k$, $K_{k+N} = K_k$) is applied

$$K_1 = [0.0167 \ -0.0175] \ , \ K_2 = [0.8495 \ -2.6782] \ ,$$
$$K_3 = [-4.9538 \ -3.6797] \ .$$

Robust H_2 performance analysis is performed on the closed-loop system. Results obtained when applying Propositions 5 and 6 are given in Table 1. The last row in the table gives the number of decision variables in each LMI problem as well as the size of the LMI constraints. The last column gives as well an estimation of the worst-case H_2 cost. It is obtained by performing a fine grid over the parameter space and computing the H_2 norm of each system.

The results obtained when applying Theorem 4 are given in Table 2. Recall that the formulas depend on a choice of the representative state. It was chosen as being the signal $\{x_{iN+0}\}_{i\geq 0}$ in the theoretical part of the paper.

Table 1. Quadratic stability and results based of Farges et al. [2007a]

	γ_{qs}	γ_{fz}	γ_f	γ_{wc}
	19.8482	9.7155	7.7257	6.8430
nb vars/rows	25/45	73/99	103/99	∞

Yet, one can also choose any other sequence $\{x_{iN+j}\}_{i\geq 0}$ where $j \in \{0 \dots N-1\}$. The rows of Table 2 indicate the numerical results obtained for these various choices of j. The choice of the representative state has as expected an influence on the results in the robust case.

Table 2. Influence of the structure of slack variables and of representative state choice

j	γ_{nc}	γ_d	γ_s	γ_z
0	7.1730	7.4646	8.3482	13.2156
1	7.4100	8.1173	8.2347	11.8943
2	7.4003	8.1236	8.2607	10.4779
nb vars/rows	127/57	109/57	91/57	73/57

The results illustrate the improvements due to *non-causal* and *dynamic* slack variables. Results may be further improved by artificially increasing the period of the system. Table 3 gives the values of the H_2 guaranteed costs γ_{nc} for $N = 6$, $N = 9$ and $N = 12$ (the period is respectively repeated two, three and four times in the descriptor model of Section 2.1). For these tests $\{x_{iN+2}\}_{i\geq 0}$ is selected to stand for the representative state.

Table 3. Influence of repeated period

N	3	6	9	12
γ_{nc}	7.4003	6.9470	6.8837	6.8641
nb vars/rows	127/57	385/93	787/129	1333/165
γ_d	8.1236	7.9398	8.1027	8.1610
nb vars/rows	109/57	295/93	571/129	937/165

As the period of the system is artificially augmented, the results obtained by Theorem 4 become less conservative and get closer to the worst case H_2 cost $\gamma_{wc} = 6.8430$. Unfortunately this is only the case for the *non causal* version of the slack variables. For the *dynamic causal* version there is no such decreasing behavior (neither for the *static* and *zero* versions).

5. CONCLUSION

The paper provides a new formulation of LMI-based robust analysis of LPTV systems. As illustrated on examples, the results are promising both in terms of reduction of numerical complexity and in terms of conservatism reduction. With respect to this last issue, the use of non-causal slack-variables merits deeper studies for example to relate these to non-causal scalings of Hagiwara [2006]. The values γ_z which where obtained on the numerical examples are not convincing. As said, the formulas related to γ_z computation are those for which extensions for state-feedback design are the most trivial. Thus prospective work devoted to state-feedback design is needed. A first attempt is done in Ebihara et al. [2008].

ACKNOWLEDGEMENTS

Results were produced during the second author's short visit to LAAS-CNRS France. He is grateful for the members of the MAC group for their kind and warm hospitality.

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