Proceedings of the 17th World Congress
The International Federation of Automatic Control
Seoul, Korea, July 6-11, 2008

Parsimonious Representation of Signals Based on Scattering Transform

Michel Sorine * **Qinghua Zhang** ** **Taous-Meriem Laleg** *
Emmanuelle Crepeau *

* *INRIA-Rocquencourt, Domaine de Voluceau, 78153 Le Chesnay Cedex, France (e-mail: FirstName.FamilyName@inria.fr).*
** *INRIA-Rennes, Campus de Beaulieu, 35042 Rennes Cedex, France, (e-mail: zhang@irisa.fr).*

Abstract. A parsimonious representation of signals is a mathematic model parametrized with a small number of parameters. Such models are useful for analysis, interpolation, filtering, feature extraction, and data compression. A new parsimonious model is presented in this paper based on scattering transforms. It is closely related to the eigenvalues and eigenfunctions of the linear Schrödinger equation. The efficiency of this method is illustrated in this paper with examples of both synthetic and real signals.

Keywords: signal processing, parsimonious model, inverse scattering transform.

1. INTRODUCTION

Representing signals or numerical data sequences with mathematical models is a classical problem. Within a certain chosen model structure, such a model is usually characterized by a set of parameters fitted to the modeled signal. Classical models of this nature are well known, for instance, *interpolating polynomials* (Szego [1992], Seroul [2000]), *splines* (De Boor [1978], Bartels et al. [1989]) and *Fourier series* (Seeley [2006]). These examples have the advantage of being computationally simple and are mostly suitable for data interpolation. However, such models are usually not parsimonious. A model is qualified *parsimonious* if it is characterized by a small number of parameters compared to the amount of data in the signal it represents. Though not necessary for data interpolation, the parsimonious property is useful for the purposes of *analysis*, *filtering*, *feature extraction*, and *data compression*.

The construction of (parsimonious) models can be based on physical knowledge about the origin of the considered signals, if such knowledge is available in some appropriate form. However, such models depend on particular physical situations and it is difficult to develop a general method for their construction. This paper is about a general method for parsimonious modeling, which is, of course, not relied on any particular physical knowledge.

In this paper, a signal is a scalar function of a real variable $x \in \mathbb{R}$, typically representing the time. Many general purpose models, including the above mentioned interpolating polynomials, splines and Fourier series, can be written in the form of

$$V(x) \approx \sum_{n \in S} a_n g_n(x) \tag{1}$$

with a chosen family of functions $g_n : \mathbb{R} \to \mathbb{R}$, linear coefficients $a_n \in \mathbb{R}$, and S being a (finite) countable index set.

If the functions $g_n(x)$ form an orthogonal basis (like Fourier series), it is very easy to fit the coefficients a_n to any signal $V(x)$. However, usually such orthogonal expansions do not lead to parsimonious models. A parsimonious model in this form would mean a small number of terms in the linear expansion. To make an efficient representation of a given signal $V(x)$ with a small number of terms, the (small number of) functions $g_n(x)$ should not be fixed in advance, but be adapted to the particular signal. In practice, this adaptation has to be restricted to some chosen set of functions $\{g_n(x)\}$. The larger is this chosen set, the more likely a given signal can be efficiently represented by a small number of functions selected within this set. In addition, the redundancy within $\{g_n(x)\}$ is also important. For example, there is no redundancy in an orthogonal Fourier basis, but the set $\{g_n(x)\}$ composed of sinusoid functions with tightly discretized frequencies is redundant. The redundancy of $\{g_n(x)\}$ improves model parsimony, but makes the selection of functions within the set more difficult: the optimal solution usually corresponds to a NP-complete problem. Nevertheless, suboptimal algorithms exist for this purpose (Chen et al. [1998], Gribonval and Nielsen [2003], Fuchs [2004]).

Alternatively, nonlinearly parametrized functions can be used to build parsimonious models. Most nonlinearly parametrized models can be written in the form of

$$V(x) \approx \sum_{n \in S} a_n g(x; b_n) \tag{2}$$

where $g(x; b_n)$ is a chosen function of x nonlinearly parametrized by a real vector b_n. Though S is still a countable or finite set, the fact that the nonlinear parameters b_n can take any real values implies that $V(x)$ is modeled with a linear combination of functions taken from an *uncountable* set. Compared to (finite) countable sets $\{g_n(x)\}$, the richness of an uncountable set (or continuously parametrized nonlinear functions) is clearly an advantage for building parsimonious models. This ap-

proach is typically used in neural networks (Haykin [1998], Ablameyko et al. [2003]). The drawback of this approach is the difficulty to fit the nonlinear parameters b_n to given signals. Usually nonlinear optimization algorithms are used, with random initial guesses for the values of b_n.

The model presented in this paper closely follows the idea originally introduced in (Laleg et al. [2007a]) and (Laleg et al. [2007b]). It can be viewed as a linear combination of functions taken from an uncountable set, though initially it was not formulated as such. In addition to the advantage related to uncountable set of functions as previously explained, *there exists an efficient algorithm* for fitting the model to signals, which will be presented in this paper.

The main mathematical tool used in the presented method is *scattering transforms*. In physics, this tool is useful for determining the characteristics of an object (shape, internal constitution, etc.) from the measurement of radiation or particles scattered from the object (Colton and Kress [1992]). In mathematics, it is used to study the solution of some nonlinear differential equations (Ablowitz and Clarkson [1991]). Though this theory itself is not new, the novelty of this paper is to use it as a general tool for parsimonious signal modeling. With respect to the works reported in (Laleg et al. [2007a]) and (Laleg et al. [2007b]), further development is made here on the analytic aspect for signal representation, which is important for parsimonious signal modeling.

This paper is organized as follows. The basic idea of modeling using eigenvalues of linear operators is explained in Section 2. The linear Schrödinger operator which is at the basis of the proposed method is introduced in Section 3. The new model based on the inverse scattering transform is formulated in Section 4. Numerical examples are presented in Section 5. The analytic aspect of the new model is discussed in Section 6. Some concluding remarks are drawn in Section 7.

2. MODELING WITH EIGENFUNCTIONS

Usually for models in the form of (1) or (2), the functions $g_n(x)$ or $g(x;b_n)$ belong to a pre-specified set. To model a given signal $V(x)$, it may be more efficient to use functions intrinsically related to $V(x)$. Of course, these functions should have (sufficiently simple) parametric forms independent of the particular signal $V(x)$, otherwise the best function for representing $V(x)$ would be itself. In order to look for such functions, let us consider the simple linear operator $\mathcal{L}$ related to the given signal $V(x)$ such that, for an arbitrary signal $f(x)$,

$$\mathcal{L}f(x) = V(x)f(x), \quad \forall x \in \mathbb{R}$$

Are the eigenfunctions of this linear operator useful for the previously exposed purpose? It turns out that these eigenfunctions are simply Dirac functions: for any $y \in \mathbb{R}$, there exists a real value $\lambda = V(y)$, such that

$$\mathcal{L}\delta(x-y) = \lambda\delta(x-y), \quad \forall x \in \mathbb{R}$$

The singularity of the Dirac functions is not desirable in most applications. To avoid this problem, it seems useful to introduce a regularization term in the linear operator. Let us consider

$$\mathcal{L}_r = -\xi\frac{d^2}{dx^2} + V(x) \tag{3}$$

where $\xi > 0$ is a weighting coefficient of the regularization term. It means that, for any signal $f(x)$,

$$\mathcal{L}_r f(x) = -\xi\frac{d^2}{dx^2}f(x) + V(x)f(x), \quad \forall x \in \mathbb{R}$$

For reasons that will become clear later, it is more convenient to consider $\mathcal{S} = \chi\mathcal{L}_r$ with $\chi = 1/\xi$ and

$$\tilde{V}(x) = \chi V(x) \tag{4}$$

The rescaled operator,

$$\mathcal{S} = -\frac{d^2}{dx^2} + \tilde{V}(x) \tag{5}$$

is known as the *Schrödinger operator* (Cycon et al. [2007]) where $\tilde{V}(x)$ is usually referred to as a *potential function*. This operator and the associated *scattering transform* provide a surprisingly efficient tool for the purpose of parsimonious modeling, as developed in the following sections.

The eigenfunctions of this linear operator will be used to build a model similar to (2). These eigenfunctions belong to a continuously parameterized family of functions. More details about this family will be given in Section 6.

3. THE LINEAR SCHRÖDINGER EQUATION AND ITS EIGENVALUE PROBLEM

The mathematical facts introduced in this section are at the basis of the method presented in this paper. Though these facts are well known in the literature related to scattering transforms, they are shortly recalled here in order to help the reading of the following sections.

The function $V : \mathbb{R} \to \mathbb{R}$ involved in the definition of the Schrödinger operator is assumed to satisfy

$$V(x) \leq 0, \quad \forall x \in \mathbb{R}$$
$$\int_{-\infty}^{\infty}(1+|x|)|V(x)|dx < \infty \tag{6}$$

Notice that the two functions $V(x)$ and $\tilde{V}(x)$ are simply related by a positive factor χ through (4), the above assumptions on $V(x)$ hold equally for $\tilde{V}(x)$.

The linear operator defined in (5) is closely related to the linear Schrödinger equation, for $x \in \mathbb{R}$,

$$-\frac{d^2}{dx^2}\psi(x) + \tilde{V}(x)\psi(x) = \lambda\psi(x) \tag{7}$$

In quantum mechanics, this equation is used to describe the probability distribution of particles under the *potential function* $\tilde{V}(x)$ (Cycon et al. [2007]).

The assumed inequality (6) implies (see Koelink [2008]) that, as $|x| \to \infty$, the solutions $\psi(x)$ of the differential equation (7) are asymptotically similar to those of

$$-\frac{d^2}{dx^2}\psi(x) = \lambda\psi(x)$$

whose behavior depends on the sign of λ. Therefore, as $|x| \to \infty$,

$$\psi(x) \sim \exp(\pm i\sqrt{\lambda}x), \qquad \text{for } \lambda > 0$$
$$\psi(x) \sim \exp(\pm\sqrt{-\lambda}x), \qquad \text{for } \lambda < 0$$

It is thus clear that $\psi(x)$ is bounded for any positive λ, but usually unbounded for negative λ. It may happen that,

for some particular negative values of λ, the corresponding solution $\psi(x)$ of the Schrödinger equation (7) is bounded. The number of such particular negative values of λ, if any, is finite and depends on the given potential function $\tilde{V}(x) \leq 0$.

The negative values of λ corresponding to bounded $\psi(x)$ are called discrete (or negative) eigenvalues (they constitute the discrete spectrum and are associated to $L^2(\mathbb{R})$ eigenfunctions). On the other hand, any positive value of λ is called a continuous eigenvalue (or rather, an element of the continuous spectrum). For each eigenvalue λ, the corresponding solution of $\psi(x)$ is called an eigenfunction.

Given the function $\tilde{V}(x)$, the computation of the eigenvalues and eigenfunctions of the linear operator $\mathcal{S}$ is known as the *scattering problem* related to the linear Schrödinger equation (7). Accordingly, the *inverse scattering problem* amounts to reconstructing the function $\tilde{V}(x)$ from the eigenvalues and eigenfunctions.

The main idea behind the method presented in this paper can be stated as follows. If the function $\tilde{V}(x)$ is a signal to be modeled, then the reconstruction of $\tilde{V}(x)$ from the eigenvalues and eigenfunctions of the linear operator $\mathcal{S}$ can be seen as a model of the signal. When the reconstruction of $\tilde{V}(x)$ is made from the eigenfunctions corresponding to discrete eigenvalues only, this model has a form similar to (2), as shown in the next section.

The assumption (6) may seem restrictive. In practice, the processed signal is defined in a finite interval, it can thus be appropriately extended outside this interval in order to satisfy the inequality (6). The negativeness of $V(x)$ may require a shifting and/or mirroring of the original signal.

4. INVERSE SCATTERING AS A PARSIMONIOUS MODEL

For the inverse scattering problem, the eigenfunctions need to be appropriately normalized. For a given $\tilde{V}(x)$, assume that there are N negative eigenvalues λ_n, $n = 1, \ldots, N$. Define

$$\kappa_n = \sqrt{-\lambda_n}, \quad n = 1, \ldots, N$$

The associated eigenfunctions are normalized such that

$$\int_{-\infty}^{\infty} \psi_n^2(x)dx = 1$$

For any positive eigenvalue λ, define

$$k = \sqrt{\lambda}, \quad \lambda > 0$$

As $|x| \to \infty$, the eigenfunction associated to the positive eigenvalue $\lambda = k^2$, denoted by $\psi(x;k)$, is asymptotically a linear combination of $\exp(\pm ikx)$. Consider a particular solution (known as Jost solution) such that, as $x \to -\infty$,

$$\psi(x;k) \sim \exp(-ikx), \quad \frac{d}{dx}\psi(x;k) \sim -ik\exp(-ikx)$$

then, on the other side, as $x \to \infty$,

$$\psi(x;k) \sim \frac{1}{T(k)}\exp(-ikx) + \frac{R(k)}{T(k)}\exp(ikx)$$

with two uniquely defined complex-valued functions $T(k)$ and $R(k)$, known respectively as *transmission and reflexion coefficients*.

With these notations, the solution of the inverse scattering problem is (see Deift and Trubowitz [1979]):

$$\tilde{V}(x) = -4\sum_{n=1}^{N} \kappa_n \psi_n^2(x) + \frac{2i}{\pi}\int_{-\infty}^{\infty} kR(k)\psi^2(x;k)dk \quad (8)$$

If it happens that $R(k) = 0$ for all $k > 0$, the function $\tilde{V}(x)$ is called a *reflectionless potential*. In this particular case the function $\tilde{V}(x)$ can be simply reconstructed from the (finite number of) negative eigenvalues and the associated eigenfunctions. In general, by omitting the second term in (8), the reconstruction with the first term can be used as an approximation:

$$\tilde{V}(x) \approx -4\sum_{n=1}^{N} \kappa_n \psi_n^2(x) \quad (9)$$

When this approximation is used as a model of $\tilde{V}(x)$, apparently it is similar to (1). In fact, this model is more similar to (2), because the eigenfunctions $\psi_n(x)$ are not taken from a pre-specified countable set of functions, but are defined by the eigenvalue problem of the Schrödinger equation (7) for the given function $\tilde{V}(x)$.

Intuitively, the quality of the approximation (9) is related to the number of terms N. This number N is determined by the signal $\tilde{V}(x)$ itself, thus there would be no way to tune the value of N if $\tilde{V}(x)$ was the given signal. Remind that the original signal to be modeled is $V(x)$, and $\tilde{V}(x)$ is in fact a rescaled version of $V(x)$ as introduced in (4). It turns out that, for a given $V(x)$, the coefficient χ allows to tune the value of N for $\tilde{V}(x) = \chi V(x)$: the larger is χ, the larger is N. This convenient way for tuning the quality of approximation was first introduced in (Laleg et al. [2007a]).

To model the original signal $V(x)$, the approximate reconstruction (9) becomes

$$V(x) \approx -\frac{4}{\chi}\sum_{n=1}^{N} \kappa_n \psi_n^2(x) \quad (10)$$

5. NUMERICAL EXAMPLES OF SIGNAL RECONSTRUCTION WITH EIGENFUNCTIONS

The purpose of this section is to illustrate the efficiency of the proposed method with numerical examples. The realization of these examples requires a numerical algorithm for the computation of eigenvalues and eigenfunctions related to the linear Schrödinger equation. This numerical aspect is discussed in Appendix A, since it is not the central issue of this paper.

5.1 A single Gaussian function

Let us first consider a very simple example: the Gaussian function

$$\tilde{V}(x) = -1.2\exp\left(-0.5x^2\right)$$

In Figure 1, $-\tilde{V}(x)$ is plotted in solid line and its reconstruction from the single negative eigenvalue and the associated eigenfunction is plotted in dashed line. This reconstruction is already quite good.

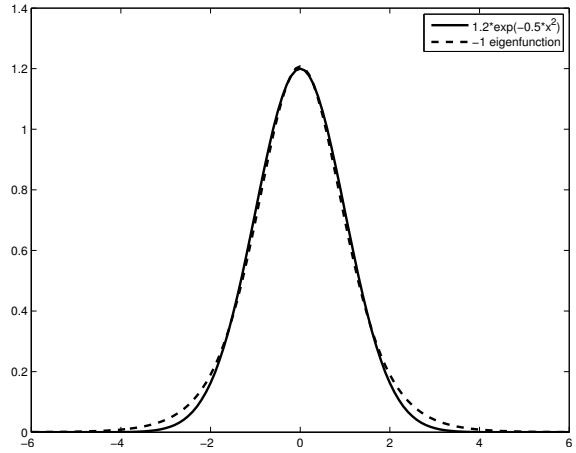

Figure 1. Reconstruction of $1.2\exp\left(-0.5x^2\right)$ with one eigenfunction.

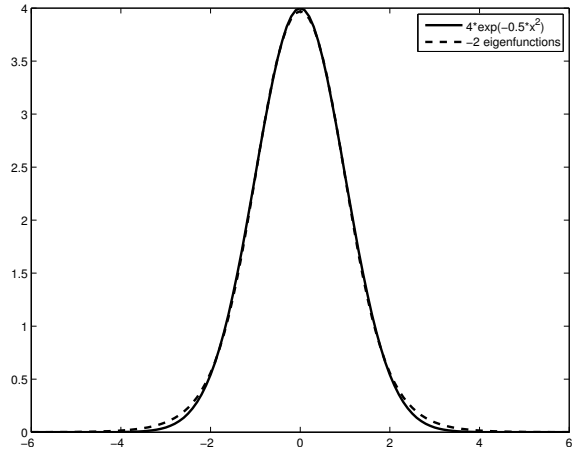

Figure 2. Reconstruction of $4\exp\left(-0.5x^2\right)$ with 2 eigenfunctions.

Notice that, in Figure 1, $-\tilde{V}(x)$ (instead of $\tilde{V}(x)$) and its reconstruction are plotted, because $\tilde{V}(x)$ itself is negative and it is more comfortable to examine "positive" curves. The same remark applies also to the figures of the following examples.

To improve the reconstruction with more eigenfunctions, a larger value of the coefficient χ should be used. The function

$$\tilde{V}(x) = -4\exp\left(-0.5x^2\right)$$

leads to two negative eigenvalues. The reconstruction with the corresponding two eigenfunctions is plotted in Figure 2.

5.2 The mixture of two Gaussian functions

For a less trivial example, let us try with the mixture of two Gaussian functions:

$$\tilde{V}(x) = -5\exp(-0.5(x+1.5)^2) - \exp(-0.5(x-1.5)^2)$$

With only 3 eigenfunctions, the reconstruction shown in Figure 3 is surprisingly good.

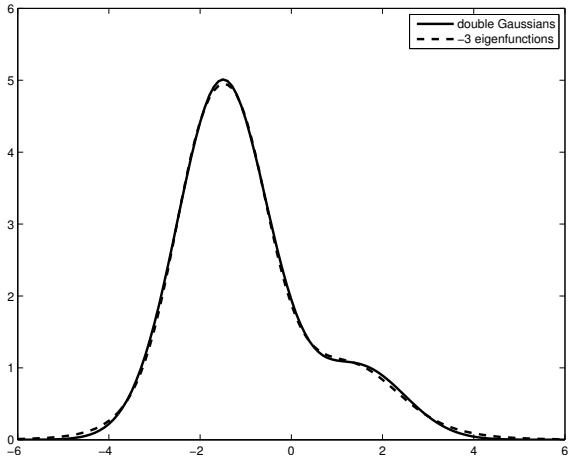

Figure 3. Reconstruction of $5\exp(-0.5(x+1.5)^2) + \exp(-0.5(x-1.5)^2)$ with 3 eigenfunctions.

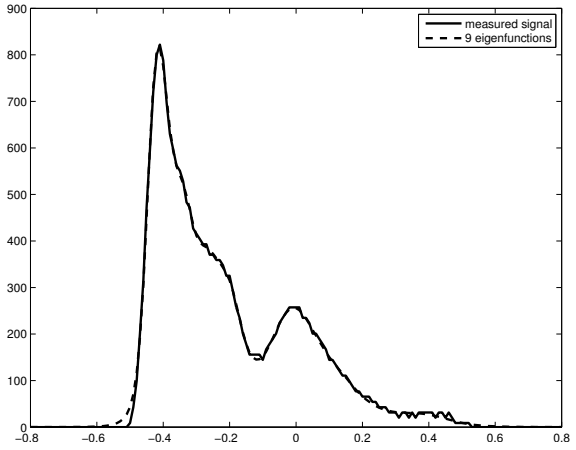

Figure 4. Reconstruction of a blood pressure signal with 9 eigenfunctions.

5.3 A blood pressure signal

After the previous synthetic examples, it is certainly more convincing to examine the case of a real signal. In Figure 4 a blood pressure signal measured at the end of a finger is plotted in solid line and its reconstruction with 9 eigenfunctions is plotted in dashed line.

This example illustrates the efficiency of the proposed method as a general tool for signal approximation. This efficiency relies on the fact that the eigenfunctions capture the essential features of the signal.

6. ANALYTIC FORMULAS OF EIGENFUNCTIONS

The examples of the previous section have shown the efficiency of the inverse scattering method for approximate signal reconstruction. For the example of blood pressure signal shown in Figure 4, it is obvious that the reconstructed signal can be used as a filtered version of the original signal. The eigenvalues used in the signal reconstruction are related to essential features of the signal, and thus convey important information for the analysis of the signal (this fact has been investigated in (Laleg et al. [2007a]), (Laleg et al. [2007b])). However, at this stage,

an important piece is still missing to use this signal reconstruction as a parsimonious model: the eigenfunctions $\psi_n(x)$ associated to negative eigenvalues, which are numerically computed (see Appendix A), are stored as numerical sequences. Analytic formulas of these eigenfunctions are necessary in order to parametrize the reconstruction (10) with a small number of parameters. The purpose of this section is to address this problem.

Remind that the N eigenfunctions $\psi_n(x)$ associated to the negative eigenvalues $\lambda_n = -\kappa_n^2$ are normalized so that

$$\int_{-\infty}^{\infty} \psi_n^2(x)dx = 1$$

It is known that such that

$$\psi_n(x) \sim c_n \exp(-\kappa_n x) \quad \text{as } x \to \infty \tag{11}$$

where $c_n \in \mathbb{R}$ is called *normalization coefficient*.

Now consider the special case of a reflectionless potential function $\tilde{V}(x)$. It is shown in Gardner et al. [1974] that the following equations hold in this case for $m = 1, 2, \ldots, N$:

$$\psi_m(x) + \sum_{n=1}^{N} c_m c_n \frac{\exp\left(-(\kappa_m + \kappa_n)x\right)}{\kappa_m + \kappa_n} \psi_n(x) = c_m \exp(-\kappa_m x) \tag{12}$$

They constitute a system of N linear equations for the N unknowns $\psi_1(x), \psi_1(x), \ldots, \psi_N(x)$. Let $Q = [q_{m,n}]$ be the $N \times N$ matrix filled with the entries

$$q_{m,n} = c_m c_n \frac{\exp\left(-(\kappa_m + \kappa_n)x\right)}{\kappa_m + \kappa_n}$$

then

$$\begin{bmatrix} \psi_1(x) \\ \vdots \\ \psi_N(x) \end{bmatrix} = (Q + I)^{-1} \begin{bmatrix} c_1 \exp(-\kappa_1 x) \\ \vdots \\ c_N \exp(-\kappa_N x) \end{bmatrix} \tag{13}$$

where I is the $N \times N$ identity matrix. It is then clear that each $\psi_m(x)$ is a rational function of $\exp(-\kappa_n x)$ with $n = 1, 2, \ldots, N$. Notice that these functions are parametrized by κ_n, c_n, $n = 1, 2, \ldots, N$.

The $2N$ parameters κ_n, c_n, $n = 1, 2, \ldots, N$, are thus sufficient to fully characterize a reflectionless potential function. In general, the signal $\tilde{V}(x)$ to be modeled is not a reflectionless potential function. In order to use the analytic form (13) of eigenfunctions in the general case, the following approach is adopted. Let $\tilde{V}^{(0)}(x) = \tilde{V}(x)$, then do the iterations, for $l = 1, 2, \ldots, L$,

$$-\frac{d^2}{dx^2}\psi(x) + \tilde{V}^{(l-1)}(x)\psi(x) = \lambda\psi(x) \rightsquigarrow \kappa_n^{(l)}, \psi_n^{(l)}(x)$$

$$\tilde{V}^{(l)} = -4 \sum_{n=1}^{N} \kappa_n^{(l)} \left(\psi_n^{(l)}(x)\right)^2$$

where "$\rightsquigarrow$" means the solution of the corresponding eigenvalue problem. Though the convergence of these iterations is not yet formally proved, empirically $\tilde{V}^{(l)}$ tends to a reflectionless potential function close to $\tilde{V}(x)$. The original signal $\tilde{V}(x)$ is then replaced by $\tilde{V}^{(L)}(x)$ in order to apply the analytic form (13) of eigenfunctions.

The parameters κ_n and c_n characterizing the analytic solution (13), are computed by solving an eigenvalue problem (see Appendix A). The computation of c_n is more troublesome. The attempts using the asymptotic relationship (11) are not fruitful, because computations are made within finite intervals in practice. A better solution is to use equations (12) where the coefficients c_n are seen as unknowns. The values of κ_n and the sampled values of $\psi_n(x)$ come from the numerical solution of the eigenvalue problem, thus the only unknowns in (12) are the coefficients c_n. The drawback of this approach is the nonlinearity due to the cross terms $c_m c_n$ in these equations.

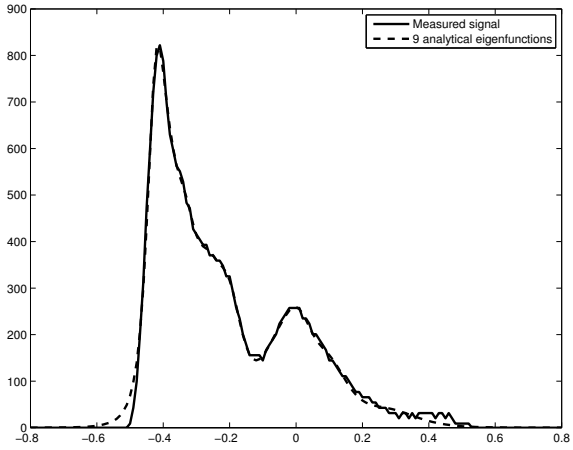

Figure 5. Reconstruction of a blood pressure signal with 9 analytic eigenfunctions, completely parametrized by 18 real coefficients ($\kappa_n, c_n,\ n = 1, \ldots, 9$).

Currently the best known solution is obtained by using the relation

$$\int_x^{\infty} \tilde{V}(y)dy = -2 \sum_{n=1}^{N} c_n \psi_n(x) \exp(-\kappa_n x) \tag{14}$$

which holds for reflectionless potential functions. Its advantage is its linearity in c_n. By evaluating both sides at different sampled values of x, the values of c_n can be estimated by the least squares method.

The estimation of c_n based on (14) is still a numerically difficult problem. Because of the exponential factors, the terms at the right hand side of (14) have very different numeric values (though in theory $\psi_n(x)\exp(-\kappa_n x)$ tends to finite values as $|x| \to \infty$). Moreover, the values of $\psi_n(x)$ computed by solving the eigenvalue problem of the linear operator (5) are subject to numerical errors. For these reasons, currently the values of c_n can be reasonably estimated only when the number of negative eigenvalues is small, say $N < 10$.

Let us consider again the example of the blood pressure signal for which $N = 9$. After the estimation of the coefficients c_n, the reconstruction of the signal with 9 analytic eigenfunctions is illustrated in Figure 5. Obviously, due to the previously explained numerical difficulty, this reconstruction is not as good as the one shown in Figure 4 which was obtained with numerically stored eigenfunctions. Efforts are being undertaken to further improve this result.

Now let us go back to the general formulation (2). For the presented new method, the functions $g(x; b_n)$ correspond to the eigenfunctions $\psi_n(x)$ which are rational functions of exponential functions parametrized by κ_n and c_n. Because κ_n and c_n are not taken from a pre-specified countable set,

the eigenfunctions $\psi_n(x)$ belong to an uncountable set of nonlinear functions. This richness is essential for building parsimonious models.

7. CONCLUSION

A new method has been presented in this paper for parsimonious representation of signals, based on scattering transforms. Its parsimony relies on two facts: the richness of the functions used in the model, and the efficiency of the associated algorithm for fitting the model to signals. This method has been illustrated in this paper with examples of both synthetic and real signals.

As a final remark, the relationship between the presented method and the *quantum neural networks* (see Rigatos and Tzafestas [2006]) is worth mentioning. The models used in these two methods can both be written in the form of (2), and they both use functions $g(x; b_n)$ related to solutions of a linear Schrödinger equation. However, there is an important difference between the two methods: for the method presented in this paper, the functions $g(x; b_n)$ are obtained by solving an eigenvalue problem related to the signal to be modeled, whereas for quantum neural networks, the functions $g(x; b_n)$ are usually estimated by solving a non convex optimization problem.

Appendix A. NUMERICAL COMPUTATION OF EIGENVALUES AND EIGENFUNCTIONS OF THE LINEAR SCHRÖDINGER OPERATOR

The problem considered in this appendix is to find, numerically, the eigenvalues λ and the associated eigenfunctions ψ such that

$$-\frac{d^2}{dx^2}\psi(x) + \tilde{V}(x)\psi(x) = \lambda\psi(x)$$

where $\tilde{V}(x)$ is a signal given in the form of a sequence at sampled values of x. There exist methods more or less general for linear operators, such as the one presented in (Nunez and Izquierdo [1993]). The method used for realizing the examples of this paper consists in simply replacing the second order derivative operator by a finite difference approximation. For example, when using the minimum order approximation at $x = l\Delta$,

$$\frac{d^2}{dx^2}\psi(x) \approx \frac{\psi((l+1)\Delta) + \psi((l-1)\Delta) - 2\psi(l\Delta)}{\Delta^2}$$

where Δ is the discretization step size, it amounts to solve the numerical matrix eigenvalue problem

$$A\psi = \lambda\psi$$

for

$$A = \frac{1}{\Delta^2}\begin{bmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ 0 & -1 & 2 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -1 \\ 0 & \cdots & \cdots & -1 & 2 \end{bmatrix} + \begin{bmatrix} \tilde{V}(\Delta) & 0 & \cdots & \cdots & 0 \\ 0 & \tilde{V}(2\Delta) & 0 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & \tilde{V}(M\Delta) \end{bmatrix}$$

where M is the sample length. In this paper, the highest order approximation allowed by the available data sample length is used.

REFERENCES

S. Ablameyko, L. Goras, M. Gori, and V. Piuri, editors. *Neural Networks for Instrumentation, Measurement and Related Industrial Applications*. IOS Press, 2003.

Mark J. Ablowitz and Peter A. Clarkson. *Solitons, nonlinear evolution equations and inverse scattering*. Cambridge University Press, 1991.

Richard Bartels, John Beatty, and Brian Barsky. *An Introduction to Splines for Use in Computer Graphics and Geometric Modeling*. Morgan Kaufmann, 1989.

S. Chen, D. Donoho, and M. Saunders. Atomic decomposition by basis pursuit. *SIAM Journal on Scientific Computing*, 20(1):33–61, 1998.

David L. Colton and Rainer Kress. *Inverse acoustic and electromagnetic scattering theory*. Springer-Verlag, 1992.

Hans L. Cycon, Richard Froese, Werner Kirsch, and Barry Simon. *Schrödinger Operators: with Applications to Quantum Mechanics and Global Geometry*. Springer, 2007. 2nd edition.

C. De Boor. *Practical guide to splines*. Applied mathematical sciences. Heidelberg, Springer, New York, Berlin, 1978.

P. A. Deift and E. Trubowitz. Inverse scattering on the line. *Communications on Pure and Applied Mathematics*, 32:121–251, 1979.

J.J. Fuchs. On sparse representations in arbitrary redundant basis. *IEEE Trans. on Information Theory*, 50(6): 1341–1344, 2004.

C.S. Gardner, J.M. Greene, M.D. Kruskal, and R.M. Miura. Korteweg-de vries equation and generalizations vi. methods for exact solution. *Communications on pure and applied mathematics*, 27:97–133, 1974.

Rémi Gribonval and Morten Nielsen. Sparse representations in unions of bases. *IEEE TRANSACTIONS ON INFORMATION THEORY*, 49(12):3320–3325, 2003.

Simon Haykin. *Neural Networks: A Comprehensive Foundation*. Prentice Hall, 1998. 2nd edition.

Erik Koelink. Scattering theory. Lecture notes, 2008. www.math.ru.nl/ koelink/edu/ScatteringTheory.html.

T.M. Laleg, E. Crépeau, and M. Sorine. Travelling wave analysis and identification. a scattering theory framework. In *European Control Conference (ECC)*, Kos, Greece, 2007a.

T.M. Laleg, E. Crépeau, Y. Papelier, and M. Sorine. Arterial blood pressure analysis based on scattering transform I. In *EMBC Sciences and Technologies for Health*, Lyon, France, 2007b.

M. A. Nunez and B. G. Izquierdo. Accurate computation of eigenfunctions for schrödinger operators in one dimension. *International journal of quantum chemistry*, 47(6):405–423, 1993.

Gerasimos Rigatos and Spyros Tzafestas. Neural structures using the eigenstates of a quantum harmonic oscillator. *Open Systems & Information Dynamics*, 13(1): 27–41, 2006.

Robert T. Seeley. *An introduction to Fourier series and integrals*. Dover Publications, 2006.

Raymond Seroul. *Programming for Mathematicians*. Springer, 2000.

G.P. Szego. *Orthogonal Polynomials*. American Mathematical Society, 1992.