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Robust Stability of Distributed Delay Systems ⋆

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Abstract: We present an LMI condition for robust stability of uncertain distributed delay systems (DDS). It is based on recent results for DDS with rational delay kernels. After the incorporation of uncertain kernels, the new approach is now applicable to any time-varying, uncertain, piecewise continuous delay kernel. The stability analysis is formulated as an LMI and at the same time uses explicitly the information about the nominal delay kernel. This is the main advantage of the new approach compared to existing solutions. The performance of the approach is illustrated in an example.

Keywords: Time-delay systems, distributed delays, uncertain delay kernel, robust stability, parametric matrix inequalities, full-block S-procedure, linear fractional representation.

1. INTRODUCTION

Time-delays appear in many fields of engineering, as for example transportation delays and communication delays. For linear time-delay systems (TDS), two different types of delays are distinguished: *discrete delays* and *distributed delays*. TDS with discrete delays have been studied extensively over the last years, see e.g. Hale and Lunel [1993], Dugard and Verriest [1998], Niculescu [2001], Gu et al. [2003], Niculescu and Gu [2004]. However, the research on *distributed delay systems* (DDS)

$$\dot{x}(t) = Ax(t) + \int_0^r F(\theta)x(t-\theta)d\theta, \tag{1}$$

with *delay kernel* $F : [0, r] \to \mathbb{R}^{n\times n}$ has been far less intense.

The stability of DDS can be analyzed in the frequency domain, see for example Michiels et al. [2005], Breda et al. [2005]. In general, this approach is rather difficult because the characteristic equation (CE) has infinitely many roots and is quite complex. The analysis becomes even more complicated if the stability of uncertain systems is studied [Verriest, 1999]. An exception are distributed delays with a γ-*distributed delay kernel* F. They can be transformed into rational CEs, see Bernard et al. [2001], Morărescu et al. [2007]. However, the assumption of γ-distributed kernels is very restrictive. Summarizing, all frequency domain algorithms are very accurate, however they fail for robust or nonlinear stability analysis as well as for most synthesis problems. For this purpose, Lyapunov-based conditions are more suitable. However, all time-domain conditions from the literature either result in linear matrix inequalities (LMIs) that do not take advantage of the information about the delay kernel F, e.g. Xie et al. [2001], or they are formulated as parametric matrix inequalities that contain the delay kernel F, e.g. Zheng and Frank [2002]. Both approaches have their drawbacks. The first one considers the delay kernel F as an uncertainty and uses algebraic transformations to prove stability. Consequently, the result is conservative if the delay kernel is known at least approximately. The second approach using parametric matrix inequalities of the form $\Xi(\theta) \prec 0, \forall\theta \in [0, r]$, with $\Xi : [0, r] \to \mathbb{R}^{m\times m}$, can only be tested for a finite number of points in the continuous parameter range $[0, r]$. Hence, we may only conclude that LMI $\Xi(\theta) \prec 0$ holds over the whole range by checking a large number of LMIs and using a continuity assumption, but we can never be sure that this assumption holds. As a special case, there are some stability conditions that assume a piecewise constant delay kernel F, e.g. Gu et al. [2001], Santos et al. [2006]. This is of course again a restrictive assumption for the general case. Moreover, this approach may result in large LMIs depending on the number of steps in the approximated piecewise constant delay kernel.

The new approach for the stability analysis of DDS (1) overcomes these drawbacks. The stability condition for the nominal case results in an LMI and includes all the information about the delay kernel, see Münz and Allgöwer [2007]. Therefore, we have to assume that the delay kernel F is a matrix of proper rational functions of θ. This restriction is removed in the work at hand. Therefore, we introduce time-varying additive uncertainties in the delay kernel F and the system matrix A. This allows us to apply the new approach to uncertain systems with time-varying, uncertain, piecewise continuous delay kernel F. We show the improvements due to the new algorithm in comparison

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to results from the literature in an example at the end of the paper.

The paper is structured as follows: We first present the problem statement and some fundamentals on parametric matrix inequalities in Section 2. Our main result, the new stability condition for uncertain DDS, is given in Section 3. An example is presented in Section 4 before the paper is concluded in Section 5.

2. PROBLEM STATEMENT AND PRELIMINARIES

2.1 Problem Statement

We consider the following class of *uncertain distributed delay systems* (DDS)

$$\dot{x}(t) = \tilde{A}(t)x(t) + \int_0^r \tilde{F}(t,\theta)x(t-\theta)d\theta, \tag{2}$$

with state $x(t) \in \mathbb{R}^n$ and initial condition $x(\eta) = \phi(\eta)$ for $\eta \in [-r, 0]$. The dynamic matrix $\tilde{A}(t) = A + \Delta A(t)$ consists of a known constant part $A \in \mathbb{R}^{n\times n}$ and a time-varying uncertainty $\Delta A : \mathbb{R} \to \mathbb{R}^{n\times n}$. Accordingly, the delay kernel $\tilde{F} = F(\theta) + \Delta F(t,\theta)$ consists of a known part $F : \Omega \to \mathbb{R}^{n\times n}$ and a time-varying uncertainty $\Delta F : \mathbb{R} \times \Omega \to \mathbb{R}^{n\times n}$ with $\Omega = [0, r]$, $r > 0$.

Assumption 1. The admissible uncertainties are bounded and of the form

$$\Delta A(t) = U_A \Delta A_0(t) V_A, \quad \forall t \in \mathbb{R} \tag{3}$$
$$\Delta F(t,\theta) = U_F \Delta F_0(t,\theta) V_F, \quad \forall t \in \mathbb{R}, \theta \in \Omega, \tag{4}$$

where U_A, V_A, U_F, and V_F are known constant matrices of appropriate dimensions. Without loss of generality, we assume that the induced matrix 2-norm of $\Delta A_0(t)$ and $\Delta F_0(t,\theta)$ satisfy $\|\Delta A_0(t)\| \le 1$ for all $t \in \mathbb{R}$ and $\|\Delta F_0(t,\theta)\| \le 1$ for all $t \in \mathbb{R}$ and $\theta \in \Omega$, respectively.

The new stability condition can be easily extended to systems with additional discrete and distributed delays. We choose this reduced structure to show the principal idea. We assume that the known part of the kernel F satisfies the following:

Assumption 2. The matrix function F can be written as a linear fractional representation *(LFR)*

$$F(\theta) = D_F + C_F(I - \theta A_F)^{-1}\theta B_F, \tag{5}$$

with $A_F \in \mathbb{R}^{n_F\times n_F}$ and B_F, C_F, and D_F of appropriate dimensions.

Note that there exist different LFRs of $F(\theta)$. Here, we assume that LFR (5) is minimal in the sense that there are no pole-zero-cancelations. Since F does not have poles on Ω, $\det(I - \theta A_F) \neq 0$ for all $\theta \in \Omega$. Assumption 2 is not restrictive for F. It is a well-known fact that any continuous function on a closed and bounded interval can be approximated by a polynomial of sufficiently high order. Clearly, the set of polynomial functions is a subset of the set of rational functions. Given a non-rational delay kernel F^*, the best rational approximation is given by the Padé approximation [Baker, 1975]. The error of this approximation can be included in the uncertainty ΔF. Note that we do not assume that F is piecewise constant nor $F(\theta) \ge 0$, $\forall\,\theta \in \Omega$, nor $\|F(\theta)\| = 1$ as in other publications, e.g. Gu et al. [2001], Santos et al. [2006], Bernard et al. [2001].

2.2 Preliminaries on Parametric Matrix Inequalities

In our main theorem, we use the full-block S-procedure. It is a powerful tool to rewrite parametric matrix inequalities. It is used intensively for solving robust analysis and synthesis problems, e.g. Scherer [2000, 2006], Iwasaki and Shibata [2001]. In this subsection, we present some basic results related to this tool.

First, we define a function $G(\delta) : \Delta \to \mathbb{R}^{n_1\times n_2}$ that is rational in $\delta \in \Delta \subseteq \mathbb{R}$. Hence, G can be written as a LFR

$$G(\delta) = D_G + C_G(I - \delta A_G)^{-1}\delta B_G\,, \tag{6}$$

with $A_G \in \mathbb{R}^{n_G\times n_G}$ and B_G, C_G, and D_G of appropriate dimensions. The LFR (6) is well-posed if $\det(I - \delta A_G) \neq 0$ for all $\delta \in \Delta$ (cf. Scherer [2006, 2000]). This is obviously fulfilled if $G(\delta)$ has no poles for $\delta \in \Delta$.

It is possible to simplify some parametric matrix inequalities related to G using the *full-block S-procedure*:

Lemma 3. (Scherer [2006, 2001], Iwasaki and Shibata [2001]). *Suppose $R_p = R_p^T$, $Q_p = Q_p^T$, S_p, $G(\delta)$ according to (6), and a compact set $\Delta \subseteq \mathbb{R}$ are given. Then*

$$\begin{bmatrix} I \\ G(\delta) \end{bmatrix}^T \begin{bmatrix} Q_p & S_p \\ S_p^T & R_p \end{bmatrix} \begin{bmatrix} I \\ G(\delta) \end{bmatrix} \prec 0, \quad \forall \delta \in \Delta \tag{7}$$

if and only if there exist matrices $Q, R, S \in \mathbb{R}^{n_G\times n_G}$ with $Q = Q^T$, $R = R^T$, satisfying

$$\begin{bmatrix} \delta I \\ I \end{bmatrix}^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} \delta I \\ I \end{bmatrix} \succeq 0, \quad \forall \delta \in \Delta \tag{8}$$

and

$$\begin{bmatrix} I & 0 \\ A_G & B_G \end{bmatrix}^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} I & 0 \\ A_G & B_G \end{bmatrix} + \begin{bmatrix} 0 & I \\ C_G & D_G \end{bmatrix}^T \begin{bmatrix} Q_p & S_p \\ S_p^T & R_p \end{bmatrix} \begin{bmatrix} 0 & I \\ C_G & D_G \end{bmatrix} \prec 0\,. \tag{9}$$

Here, $\succ, \succeq, \prec$, and $\preceq$ indicate positive and negative (semi-) definiteness, respectively.

Clearly, we still have a parametric matrix inequality (8). However, using the convex hull relaxation from Scherer [2006, 2000], it is possible to transform this into a finite set of non-parametric matrix inequality.

Lemma 4. (Scherer [2006, 2000]). *Suppose that $Q \prec 0$ and Δ is the convex hull of two points δ_1, $\delta_2 \in \mathbb{R}$ with $\delta_1 < \delta_2$, i.e. $\Delta = Co(\{\delta_1,\ \delta_2\}) = [\delta_1,\ \delta_2]$. Then*

$$\begin{bmatrix} \delta I \\ I \end{bmatrix}^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} \delta I \\ I \end{bmatrix} \succeq 0, \quad \forall \delta \in \Delta \tag{10}$$

if and only if

$$\begin{bmatrix} \delta_i I \\ I \end{bmatrix}^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} \delta_i I \\ I \end{bmatrix} \succeq 0, \quad i = 1, 2\,. \tag{11}$$

Summarizing, the parametric LMI (7) can be replaced by (9) and (11). The only introduced conservatism is $Q \prec 0$.

2.3 Preliminaries on Uncertain LMIs

In order to deal with the model uncertainties ΔA and ΔF, we use the following lemma taken from de Souza and Li [1999]:

$$\begin{bmatrix} Q + A_F S^T + S A_F^T + A_F R A_F^T + B_F Q_2 B_F^T & S C_F^T + A_F R C_F^T + B_F Q_2 D_F^T & 0 & 0 & B_F Q_2 V_F^T \\ * & \Theta & Q_1 & Q_1 V_A^T & D_F Q_2 V_F^T \\ * & * & -Q_2 & 0 & 0 \\ * & * & * & -r\epsilon_1 I & 0 \\ * & * & * & * & V_F Q_2 V_F^T - \epsilon_2 I \end{bmatrix} \prec 0 \quad (12)$$

Lemma 5. Let U, V, W, and Z be real matrices of appropriate dimensions with Z satisfying $\|Z\| \leq 1$, where $\|\cdot\|$ is the induced matrix 2-norm. Then, we have the following:

(1) For any real number $\epsilon > 0$, $UZV + (UZV)^T \preceq \epsilon UU^T + \epsilon^{-1} V^T V$.
(2) For any matrix $P \succ 0$ and scalar $\epsilon > 0$ such that $\epsilon I - VPV^T \succ 0$, we have

$$(W + UZV)P(W + UZV)^T \preceq WPW^T + WPV^T(\epsilon I - VPV^T)^{-1} VPW^T + \epsilon UU^T.$$

3. MAIN RESULT

We are now ready to state our main result: a stability condition for DDS (2).

Theorem 6. Let Assumptions 1 and 2 hold. System (2) is asymptotically stable for all admissible uncertainties ΔA and ΔF if there exist real $\epsilon_1 > 0$, $\epsilon_2 > 0$, positive definite, symmetric matrices Q_1, $Q_2 \in \mathbb{R}^{n\times n}$ and matrices R, Q, $S \in \mathbb{R}^{n_F \times n_F}$ with $R = R^T \succeq 0$ and $Q = Q^T \prec 0$ such that LMI (12) (top of this page) and

$$r^2 Q + r(S + S^T) + R \succeq 0 \quad (13)$$

hold with $\Theta = r^{-1}(Q_1 A^T + AQ_1 + \epsilon_1 U_A U_A^T) + \epsilon_2 U_F U_F^T + C_F R C_F^T + D_F Q_2 D_F^T$.

Proof. Consider the Lyapunov-Krasovskii functional candidate (cf. Gu et al. [2003])

$$V(x_t) = x^T P_1 x + \int_0^r \int_{t-\theta}^t x^T(\xi) P_2 x(\xi) d\xi d\theta, \quad (14)$$

with $P_1 \succ 0$ and $P_2 \succ 0$. The derivative of V along the solutions of (2) is

$$\begin{aligned} \dot{V}(x_t) &= x^T(t)(\tilde{A}^T(t) P_1 + P_1 \tilde{A}(t) + rP_2)x(t) \\ &\quad + 2x^T(t) \int_0^r P_1 \tilde{F}(t,\theta) x(t-\theta) d\theta \\ &\quad - \int_0^r x^T(t-\theta) P_2 x(t-\theta) d\theta \quad (15) \\ &= \int_0^r \begin{bmatrix} x(t) \\ x(t-\theta) \end{bmatrix}^T M(t,\theta) \begin{bmatrix} x(t) \\ x(t-\theta) \end{bmatrix} d\theta \end{aligned}$$

with

$$M(t,\theta) = \begin{bmatrix} r^{-1}\left(\tilde{A}^T(t) P_1 + P_1 \tilde{A}(t)\right) + P_2 & P_1 \tilde{F}(t,\theta) \\ \tilde{F}^T(t,\theta) P_1 & -P_2 \end{bmatrix}.$$

Clearly, $\dot{V} < 0$ and asymptotical stability of system (2) is guaranteed if the parametric matrix inequality $M(t,\theta) \prec 0$ holds for all $\theta \in \Omega$ and all $t \in \mathbb{R}$.

First, we eliminate the uncertainties ΔA and ΔF. Therefore, we apply the Schur complement [Boyd et al., 1994] to $M(t,\theta) \prec 0$ and obtain:

$$\begin{aligned} &r^{-1}\left((A + U_A \Delta A_0(t) V_A)^T P_1 + P_1(A + U_A \Delta A_0(t) V_A)\right) \\ &\quad + P_2 + P_1 \tilde{F}(t,\theta) P_2^{-1} \tilde{F}^T(t,\theta) P_1 \prec 0. \quad (16) \end{aligned}$$

With Lemma 5, we see that

$$\begin{aligned} &(P_1 U_A \Delta A_0(t) V_A)^T + P_1 U_A \Delta A_0(t) V_A \\ &\quad \preceq \epsilon_1 P_1 U_A U_A^T P_1 + \epsilon_1^{-1} V_A^T V_A, \quad (17) \end{aligned}$$

for any $\epsilon_1 > 0$. Moreover, we have

$$\begin{aligned} &(F(\theta) + U_F \Delta F_0(t,\theta) V_F) \\ &\quad \times P_2^{-1} (F(\theta) + U_F \Delta F_0(t,\theta) V_F)^T \\ &\quad \preceq F(\theta) P_2^{-1} F^T(\theta) + \epsilon_2 U_F U_F^T \\ &+ F(\theta) P_2^{-1} V_F^T \left(\epsilon_2 I - V_F P_2^{-1} V_F^T\right)^{-1} V_F P_2^{-1} F^T(\theta), \quad (18) \end{aligned}$$

for any $\epsilon_2 > 0$.

Next, we pre- and post-multiply P_1^{-1} to the left hand side of (16). We introduce $Q_1 = P_1^{-1}$ and $Q_2 = P_2^{-1}$. With (17) and (18), we see that

$$\begin{aligned} &r^{-1}\left(Q_1 A^T + AQ_1 + \epsilon_1 U_A U_A^T + \epsilon_1^{-1} Q_1 V_A^T V_A Q_1\right) \\ &\quad + F(\theta) Q_2 V_F^T \left(\epsilon_2 I - V_F Q_2 V_F^T\right)^{-1} V_F Q_2 F^T(\theta) \\ &\quad + Q_1 Q_2^{-1} Q_1 + F(\theta) Q_2 F^T(\theta) + \epsilon_2 U_F U_F^T \prec 0 \quad (19) \end{aligned}$$

guarantees $M(t,\theta) \prec 0$. We apply again the Schur complement to the left hand side of (19) and obtain

$$\overline{M}(\theta) = \begin{bmatrix} \tilde{\Phi} & Q_1 & Q_1 V_A^T & F(\theta) Q_2 V_F^T \\ * & -Q_2 & 0 & 0 \\ * & * & -r\epsilon_1 I & 0 \\ * & * & * & V_F Q_2 V_F^T - \epsilon_2 I \end{bmatrix}, \quad (20)$$

with $\Phi = r^{-1}\left(Q_1 A^T + AQ_1 + \epsilon_1 U_A U_A^T\right) + \epsilon_2 U_F U_F^T$ and $\tilde{\Phi} = \Phi + F(\theta) Q_2 F^T(\theta)$.

So far, $\overline{M}(\theta) \prec 0$ for all $\theta \in \Omega$ guarantees the robust stability of system (2). Following the ideas of Münz and Allgöwer [2007], we apply now the full-block S-procedure and the convex hull relaxation from Section 2.2 in order to convert the parametric matrix inequality $\overline{M}(\theta) \prec 0$, $\forall \theta \in \Omega$ into an LMI.

We rewrite $\overline{M}(\theta)$ in the following way

$$\overline{M}(\theta) = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ F^T(\theta) & 0 & 0 & 0 \end{bmatrix}^T \tilde{M} \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ F^T(\theta) & 0 & 0 & 0 \end{bmatrix}$$

$$\tilde{M} = \begin{bmatrix} \Phi & Q_1 & Q_1 V_A^T & 0 & 0 \\ Q_1 & -Q_2 & 0 & 0 & 0 \\ V_A Q_1 & 0 & -r\epsilon_1 I & 0 & 0 \\ 0 & 0 & 0 & V_F Q_2 V_F^T - \epsilon_2 I & V_F Q_2 \\ 0 & 0 & 0 & Q_2 V_F^T & Q_2 \end{bmatrix}. \quad (21)$$

Next, we use Assumption 2 and the full-block S-procedure (Lemma 3) and see that $\overline{M}(\theta) \prec 0$, $\forall \theta \in \Omega$ if and only if there exist $Q, R, S \in \mathbb{R}^{n_F \times n_F}$ with $Q = Q^T$ and $R = R^T$ satisfying

$$\begin{bmatrix} \theta I \\ I \end{bmatrix}^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} \theta I \\ I \end{bmatrix} \succeq 0, \quad \forall \theta \in \Omega, \tag{22}$$

and

$$\begin{bmatrix} I & 0 & 0\ 0\ 0 \\ A_F^T & C_F^T & 0\ 0\ 0 \end{bmatrix}^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} I & 0 & 0\ 0\ 0 \\ A_F^T & C_F^T & 0\ 0\ 0 \end{bmatrix} + \begin{bmatrix} 0 & I & 0\ 0\ 0 \\ 0 & 0 & I\ 0\ 0 \\ 0 & 0 & 0\ I\ 0 \\ 0 & 0 & 0\ 0\ I \\ B_F^T & D_F^T & 0\ 0\ 0 \end{bmatrix}^T \tilde{M} \begin{bmatrix} 0 & I & 0\ 0\ 0 \\ 0 & 0 & I\ 0\ 0 \\ 0 & 0 & 0\ I\ 0 \\ 0 & 0 & 0\ 0\ I \\ B_F^T & D_F^T & 0\ 0\ 0 \end{bmatrix} \prec 0. \tag{23}$$

We can simplify LMI (23) and obtain (12). Moreover, we apply the convex hull relaxation (Lemma 4) to the parametric matrix inequality (22) with $\delta_1 = 0$ and $\delta_2 = r$. It results $Q \prec 0, R \succeq 0$, and (13). □

As mentioned in the introduction, the condition in Theorem 6 is formulated as an LMI and takes advantage of the information about the delay kernel F. Note that Equation (12) contains the LFR matrices A_F, B_F, C_F, and D_F of Assumption 2. Moreover, we see that $\Theta \prec 0$, i.e. $A \prec 0$, is necessary for (12) to hold.

A straightforward extension of this stability result for $\mathcal{L}_2$-based controller design is presented in Münz and Allgöwer [2007] for systems without uncertainties. The same controller design is possible for uncertain systems combining the results of the two papers.

4. NUMERICAL EXAMPLE

We consider the following simple example for a numerical assessment of the new stability condition:

$$\dot{x}(t) = \tilde{a}x(t) + \int_0^r \tilde{f}(\theta)x(t-\theta)d\theta, \tag{24}$$

with $x(t) \in \mathbb{R}$, initial condition $x(\eta) = \phi(\eta)$ for $\eta \in [-r, 0]$, $r \in (0, \infty)$, $\tilde{a} = (a + \Delta a)$, and $\tilde{f}(\theta) = f(\theta) + \Delta f(\theta)$. Thereby, we assume that $f(\theta) = \frac{0.6\,\theta}{1+\theta^2}$, $\|\Delta a\| \leq 0.1$, and $\|\Delta f(\theta)\| \leq 0.1$ for all $\theta \in \Omega = [0, r]$. The parameters of the delay kernel have been chosen such that a gamma-distributed kernel $\gamma(\alpha, \beta)$ with $\alpha = 2$ and $\beta = 1$ is covered by the uncertainties, see Figure 1. For completeness, we give the following matrices:

$$A_F = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad B_F = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C_F = [\,0\ 0.6\,], \quad D_F = 0\,.$$

For a given integral range r, we are looking for the maximal stability region $\Gamma \subset \mathbb{R}$ such that (24) is stable for all $a \in \Gamma$ and all admissible uncertainties. We compare the result of the new condition with an parametric matrix inequality condition from Zheng and Frank [2002], Theorem 1.

Note that Θ of (12) can be separated for any $\hat{a} \in \mathbb{R}$ as follows:

$$\begin{aligned} \Theta &= r^{-1}(2aQ_1 + \epsilon_1 U_A U_A^T) + \epsilon_2 U_F U_F^T \\ &\quad + C_F R C_F^T + D_F Q_2 D_F^T \\ &= r^{-1}(2\hat{a}Q_1 + \epsilon_1 U_A U_A^T) + r^{-1}2(a-\hat{a})Q_1 + \epsilon_2 U_F U_F^T \\ &\quad + C_F R C_F^T + D_F Q_2 D_F^T \\ &= \hat{\Theta} + r^{-1}2(a-\hat{a})Q_1. \end{aligned}$$

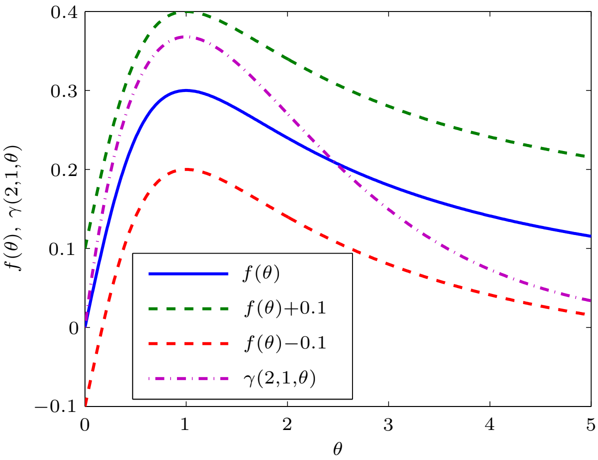

Fig. 1. Uncertain delay kernel $\tilde{f} = f + \Delta f$ of the example system (24) as well as gamma-distribution $\gamma(2, 1, \theta)$ for $\theta \in [0, 5]$.

For $a \leq \hat{a}$, we see that $\Theta \prec 0$ if $\hat{\Theta} \prec 0$ because $Q_1 \succ 0$. Hence, we can perform a line search for the new algorithm in order to find the maximal $\hat{a}$ that satisfies the stability condition in Theorem 6. Then, the stability region Γ must contain $(-\infty, \hat{a}]$, i.e. $\Gamma \supseteq (-\infty, \hat{a}]$. For the parametric matrix inequality in Zheng and Frank [2002], there is no such argument. With this algorithm, we only compute the robust stability of (24) for individual values of $a = a_{\text{ref}}$.

Note that Theorem 1 in Zheng and Frank [2002] contains a typo. Both terms of the matrix Δ that contain θ have to be subtracted from the other matrices. The correct formulation of the parametric LMI is given at the end of the proof as $S(\theta) \prec 0$ for all $\theta \in [-r, 0]$, where $S(\theta)$ contains the correct signs. Moreover, this theorem requires to search for suitable scalar positive functions $\epsilon_i(\theta), i \in \{2, 3, 4, 6\}$. Since the authors do not specify how to choose these functions, we assumed these functions to be constant.

The LMIs (12) and (13) as well as the parametric matrix inequality have been solved using Matlab, YALMIP [Löfberg, 2004], and the SeDuMi solver [Sturm, 1999]. The results are presented in Table 1, where $\hat{a}$ refers to the value calculated with Theorem 6, i.e. (24) is stable for all $a \leq \hat{a}$. The individual values analyzed with Zheng and Frank [2002] are denoted a_{ref}. The column *stable* indicates if the parametric matrix inequality could be solved, i.e. if the system is stable (+) or not (−) according to the algorithm in Zheng and Frank [2002]. As discussed in the introduction, the solution of the parametric matrix inequality in Zheng and Frank [2002] depends usually on the number of points in the interval Ω for which it is checked. This number is given in the last but one column of Table 1. If the parametric LMI is not feasible for two discretization points, i.e. $\theta = 0$ and $\theta = r$, than it is also not feasible for more points. On the contrary, if it is feasible for two points, this does not guarantee that it is feasible for more points. Therefore, we checked more points if the LMI was feasible.

For this example, we see that the new stability condition always obtains a better result than the reference from the literature. Only for the case with a very small integral

Table 1. Stability of system (24) for different values of r.

r	$\hat{a}$	a_{ref}	no. of param.	stable
0.01	-0.102	-0.102	2 to 11	+
0.01		-0.101	2	–
0.2	-0.144	-0.144	2	–
0.2		-0.145	2 to 11	+
1	-0.501	-0.501	2	–
1		-0.894	2	-
1		-0.895	2 to 11	+
3	-1.301	-1.301	2	–
3		-100	2	–
5	-2.101	-2.101	2	–
5		-100	2	–

range $r = 0.01$, both algorithms obtain the same result. On the other hand, the new algorithm performs much better for integral ranges $r \geq 1$. Note that for the last two cases $r = 3$ and $r = 5$, no feasible value of a_{ref} was found with the parametric matrix inequality.

Summarizing, this example indicates that the new stability condition for uncertain distributed delay systems can improve and simplify considerably the analysis of this system class.

5. CONCLUSIONS

We presented a new stability condition for uncertain distributed delay systems based on a recently proposed stability condition for DDS with rational delay kernel F. Due to the uncertain delay kernel, it is now possible to apply this approach to any distributed delay system with piecewise continuous kernel. In comparison to similar results from the literature, the new approach is the first one that takes advantage of the knowledge of the delay kernel and is formulated as an LMI. Hence, we expect that less restrictive and easy computable results can be obtained with this condition. This has been shown for a simple example. Our ongoing work aims at further reducing the conservatism of this condition.

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