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On the Characterization of Discrete Mode Uncertainty in Hybrid State Estimation ⋆

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Abstract: This paper addresses the problem of state estimation for hybrid systems, with special emphasis on the uncertainty associated with the discrete mode estimates. The nature of hybrid systems, which are composed by both discrete modes and continuous states, requires a specific description of the uncertainty associated with the computed estimates. The estimation uncertainty is shown to depend both on the estimation algorithm and on the actual trajectory followed by the system. A new definition of observability for the discrete mode of a hybrid system is proposed determining the best accuracy obtainable when estimating the discrete mode. A simple numerical example with a PWA system clarifies the presented concepts.

Keywords: Hybrid systems, Discrete state estimation, Observability, Uncertainty evaluation, Error quantification

1. INTRODUCTION

Hybrid systems are dynamical systems composed by both discrete valued and continuous states. Basically, the dynamics of a hybrid system is governed by a mode selector that determines, at each time instant, the discrete mode from endogenous and/or exogenous variables. The continuous state can then updated through a linear dynamic relation that is selected from a set of linear dynamics according to the value of the discrete mode. Hybrid systems are usually analyzed using tools developed for linear systems, however, extreme care must be taken because the influence of the discrete dynamics is usually very intricate.

There are two main approaches to the hybrid state estimation problem: the simpler one considers that the discrete mode is known in advance and the problem is reduced to the state estimation of a linear time-varying system. This approach has been used by several authors such as Alessandri and Coletta [2001], Böker and Lunze [2002]. The main difficulty lies on ensuring convergence of the estimates for every admissible sequence of discrete modes. If, on the other hand, the discrete mode must also be estimated the estimation problem becomes much more complex and every discrete mode sequence (*dms*) must be checked to choose the one that better fits the observed data. The continuous state estimates are then computed for the estimated *dms*. Several works address this problem, see Balluchi et al. [2002], Ferrari-Trecate et al. [2002], Pina and Botto [2006].

The truly optimal way of simultaneously estimating the discrete mode and the continuous state of a hybrid system was derived in Athans and Chang [1977] using Bayesian methodologies. The objective was to perform simultaneous system identification and state estimation for linear systems but the derivation is quite general and is directly applicable to the hybrid state estimation problem. This method requires the consideration of all admissible *dms* starting from the initial time instant, being obviously unpractical since the number of *dms* grows exponentially in time, and so, suboptimal methods were developed. From the various possibilities, considering all the admissible *dms* of a given length is usually the preferred methodology.

The method of characterizing the discrete mode uncertainty proposed in this paper is rather general and applicable to most of the existing models for hybrid systems subject to disturbances with explicitly known probability density function. In Heemels et al. [2001], PWA systems are proven to be equivalent to many other classes of hybrid systems under some mild assumptions, and so, a PWA system subject to uniformly distributed disturbances will be considered to clarify the proposed concepts.

The remainder of this paper is organized as follows: the considered models of hybrid systems and disturbances are presented in section 2, along with the example system. Section 3 presents the detailed analysis of the uncertainty associated with discrete mode estimates. Finally, in section 4 some conclusions are drawn.

2. SYSTEM DESCRIPTION

PWA systems were introduced in Sontag [1981]. Here, the following stochastic PWA model will be considered:

$$x(k+1) = A_{i(k)}x(k)+B_{i(k)}u(k)+f_{i(k)}+L_{i(k)}w(k) \quad (1a)$$

$$y(k) = C_{i(k)}x(k)+D_{i(k)}u(k)+g_{i(k)}+v(k) \quad (1b)$$

$$\text{iff } \begin{bmatrix} x(k) \\ u(k) \\ w(k) \end{bmatrix} \in \Omega_{i(k)} \quad (1c)$$

where k is the discrete time, $x(k) \in \mathbb{X} \subset \mathbb{R}^{n_x}$ is the continuous state, $u(k) \in \mathbb{U} \subset \mathbb{R}^{n_u}$ is the input, $w(k) \in \mathbb{R}^{n_w}$ is the input disturbance, $y(k) \in \mathbb{R}^{n_y}$ is the output,

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and $v(k) \in \mathbb{R}^{n_v}$ is the measurement noise. $i(k) \in \mathcal{I} = \{1, \ldots, s\} \subset \mathbb{N}^+$ is the discrete mode and s is the total number of discrete modes.

The matrices and vectors A_i, B_i, f_i, L_i, C_i, D_i, g_i depend on the discrete mode i and have appropriate dimensions.

The discrete mode $i(k)$ is a piecewise constant function of the state, input and input disturbance whose value is defined by a given polytopic region Ω_i:

$$\Omega_i \quad : \quad S_i\, x(k) + R_i\, u(k) + Q_i\, w(k) \leq T_i \tag{2}$$

As the subsequent analysis will consider sequences of measurements and discrete modes, some helpful notation regarding the time-compressed representation presented in Kamen [1992] will now be introduced for system (1). The time-compressed representation of a system defines the dynamics of the system over a sequence of time instants in opposition to the single time step state-space representation. Consider the time interval $[k, k+T-1]$, the sequence of discrete modes over this interval is represented as $\mathbf{i}_T(k) \triangleq \{i(k), \ldots, i(k+T-1)\}$. To simplify the notation, the time index k may be removed from the *dms* whenever it is clear from the context. The output sequence over the same interval is given by:

$$Y_T(k) = \mathbf{C}_{\mathbf{i}_T} x(k) + \mathbf{D}_{\mathbf{i}_T} U_T(k) + \mathbf{g}_{\mathbf{i}_T} + \mathbf{L}_{\mathbf{i}_T} W_T(k) + V_T(k) \tag{3}$$

where the input, input disturbance and measurement noise sequences, $U_T(k)$, $W_T(k)$, and $V_T(k)$, respectively, are defined in the same way as the output sequence $Y_T(k) \triangleq [y(k)^{\mathrm{T}}, \ldots, y(k+T-1)^{\mathrm{T}}]^{\mathrm{T}}$. The matrices and vectors $\mathbf{C}_{\mathbf{i}_T}$, $\mathbf{D}_{\mathbf{i}_T}$, $\mathbf{g}_{\mathbf{i}_T}$, and $\mathbf{L}_{\mathbf{i}_T}$ are directly computed from the system dynamics (1a-1b) and will not be presented here for brevity.

The same reasoning can be applied to the constraints $\Omega_{\mathbf{i}_T}$:

$$\mathbf{\Omega}_{\mathbf{i}_T} \quad : \quad \mathbf{S}_{\mathbf{i}_T}\, x(k) + \mathbf{R}_{\mathbf{i}_T}\, U_T(k) + \mathbf{Q}_{\mathbf{i}_T}\, W_T(k) \leq \mathbf{T}_{\mathbf{i}_T} \tag{4}$$

where, once again, the matrices $\mathbf{S}_{\mathbf{i}_T}$, $\mathbf{R}_{\mathbf{i}_T}$, $\mathbf{Q}_{\mathbf{i}_T}$ and $\mathbf{T}_{\mathbf{i}_T}$ can be computed from the system dynamics (1a) and the polytopic region equation (2).

2.1 Disturbances

Despite the possibility of considering any model with explicitly known probability density function for disturbances, the example system considered in this paper is subject to uniformly distributed disturbances defined by polytopes centered at the origin:

$$w(k) \in \mathbb{W} \subset \mathbb{R}^{n_w}\ ,\ \forall_k \qquad ; \qquad \int_{\mathbb{W}} w\, dw = 0 \tag{5a}$$

$$v(k) \in \mathbb{V} \subset \mathbb{R}^{n_v}\ ,\ \forall_k \qquad ; \qquad \int_{\mathbb{V}} v\, dv = 0 \tag{5b}$$

The polytopes $\mathbb{W}$ and $\mathbb{V}$ are defined by a set of linear inequalities:

$$\mathbb{W} \triangleq \{w \in \mathbb{R}^{n_w} : H_w w - h_w \leq 0\} \tag{6a}$$

$$\mathbb{V} \triangleq \{v \in \mathbb{R}^{n_v} : H_v v - h_v \leq 0\} \tag{6b}$$

where H_w is a $(p_w \times n_w)$ matrix and h_w is a p_w-dimensional vector. p_w represents the number of linear constraints defining $\mathbb{W}$. The same applies to H_v and h_v but the dimensions of H_v are $(p_v \times n_y)$.

Remark I: The disturbances properties in a PWA system may depend on the actual mode of the system. Each mode i may have some particular input disturbance and measurement noise distributions associated, resulting in different probability density functions associated.

2.2 Example system

The following PWA system will be used to clarify the presented concepts:

$$\begin{aligned}
x(k+1) &= \begin{cases} -0.5\, x(k) + u(k) & , \quad \text{iff} \quad x(k) \leq 0 \\ \quad - x(k) + u(k) & , \quad \text{iff} \quad x(k) > 0 \end{cases} \\
y(k) &= \begin{cases} x(k) + v(k) & , \quad \text{iff} \quad x(k) \leq 0 \\ x(k) + v(k) - 0.5 & , \quad \text{iff} \quad x(k) > 0 \end{cases} \\
x(k) &\in \mathbb{X} \triangleq \big[-2\ ,\ 2\big] \\
u(k) &\in \mathbb{U} \triangleq \big[-1\ ,\ 1\big] \\
&\begin{cases} v(k) \in \mathbb{V}_1 \triangleq \big[-0.2\ ,\ 0.2\big] & , \quad \text{iff} \quad x(k) \leq 0 \\ v(k) \in \mathbb{V}_2 \triangleq \big[-0.5\ ,\ 0.5\big] & , \quad \text{iff} \quad x(k) > 0 \end{cases}
\end{aligned} \tag{7}$$

The measurement noises are uniformly distributed over the respective intervals. No input disturbances were considered to simplify the analysis. This system will be analyzed considering sequences of 2 time instants, $T = 2$.

3. UNCERTAINTY ANALYSIS

The uncertainty present when the continuous state $x(k)$ and *dms* $\mathbf{i}_T(k)$ of system (1) are estimated from sequences of measurements $Y_T(k)$ and inputs $U_T(k)$ will now be analyzed. First some important concepts must be introduced:

Definition 1. Hybrid Trajectory:
The triplet $\big(x, U_T, \mathbf{i}_T\big) \in \mathbb{X} \times \mathbb{U}^T \times \mathcal{I}^T \subset \mathbb{R}^{(n_x + T.n_u)} \times \mathbb{N^+}^T$ defines a hybrid trajectory for the PWA system (1), where x is the initial state, U_T is the input sequence and $\mathbf{i}_T$ is the discrete mode sequence. The length T of $\mathbf{i}_T$ and U_T is the same and defines the length of the trajectory.
A hybrid trajectory is feasible if and only if:

$$\exists\, W_T \in \mathbb{W}^T_{\mathbf{i}_T} \subset \mathbb{R}^{T.n_w} \qquad \Rightarrow \qquad \begin{bmatrix} x \\ U_T \\ W_T \end{bmatrix} \in \mathbf{\Omega}_{\mathbf{i_T}} \tag{8}$$

As there are several different mathematical formulations for hybrid systems, many other definitions of hybrid trajectories have been proposed in Bemporad and Morari [1999], Hu et al. [2000], Branicky et al. [1998].

The following sets arise directly from Definition 1:

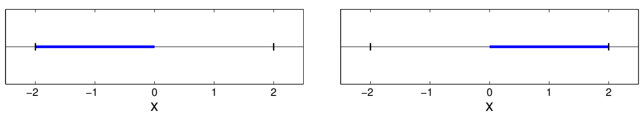
Definition 2. Hybrid Trajectory Feasibility Polytope:
The Hybrid Trajectory Feasibility Polytope (HTFP) of a discrete mode sequence $\mathbf{i}_T$ is defined as the set of all feasible hybrid trajectories of $\mathbf{i}_T$:

$$\mathcal{H}_{\mathbf{i}_T} \triangleq \Big\{ \big(x, U_T, \mathbf{i}_T\big) \in \mathbb{R}^{(n_x + T.n_u)} \times \mathbb{N^+}^T : \\ \exists W_T \in \mathbb{W}_{\mathbf{i}_T} \ \Rightarrow\ \begin{bmatrix} x \\ U_T \\ W_T \end{bmatrix} \in \mathbf{\Omega}_{\mathbf{i_T}} \Big\} \tag{9}$$

The point-to-set map $\mathcal{H}_{\mathbf{i}_T} : \mathbb{R}^{T.n_u} \mapsto 2^{\mathbb{R}^{(n_x+T.n_u)} \times \mathbb{N^+}^T}$ defines the HTFP for a given input sequence U of length T:

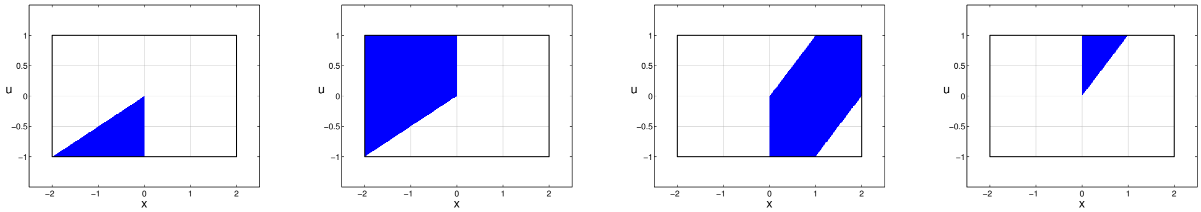
$$\mathcal{H}_{\mathbf{i}_T}(U) \triangleq \Big\{ \big(x, U_T, \mathbf{i}_T\big) \in \mathbb{R}^{(n_x + T.n_u)} \times \mathbb{N^+}^T : \\ U_T = U\ ,\ \exists W_T \in \mathbb{W}_{\mathbf{i}_T} \ \Rightarrow\ \begin{bmatrix} x \\ U_T \\ W_T \end{bmatrix} \in \mathbf{\Omega}_{\mathbf{i_T}} \Big\} \tag{10}$$

Figures 1 and 2 show the HTFP $\mathcal{H}_{\mathbf{i}_T}$ of all *dms*, with length 1 and 2, respectively, from the example system (7).



(a) $\mathbf{i}_T = [\,1\,]$ (b) $\mathbf{i}_T = [\,2\,]$

Fig. 1. Hybrid trajectory feasibility polytopes, $\mathcal{H}_{\mathbf{i}_T}$, for the example system (7), length $T = 1$.



(a) $\mathbf{i}_T = [1\,1]$ (b) $\mathbf{i}_T = [1\,2]$ (c) $\mathbf{i}_T = [2\,1]$ (d) $\mathbf{i}_T = [2\,2]$

Fig. 2. Hybrid trajectory feasibility polytopes, $\mathcal{H}_{\mathbf{i}_T}$, for the example system (7), length $T = 2$.

The hybrid trajectories of length 1 are defined by the initial state alone, while the trajectories of length 2 are defined by the initial state and the initial input. As the last component of the input sequence U_T has no effect on the continuous state evolution, neither on the output sequence or the partitions, it will not be represented. The HTFP for a given input sequence, $\mathcal{H}_{\mathbf{i}_T}(U)$, are obviously horizontal lines in figure 2.

Output trajectories corresponding to a given hybrid trajectory can also be defined as follows:

Definition 3. Output Trajectory:
The map
$Y : \mathbb{X}\times\mathbb{U}^T\times\mathbb{W}^T\times\mathbb{V}^T\times\mathcal{I}^T \subset \mathbb{R}^{(n_x+T(n_u+n_w+n_v))}\times\mathbb{N}^{+T} \mapsto \mathbb{R}^{T.n_y}$
defines the output trajectory $Y\big(x, U_T, W_T, V_T, \mathbf{i}_T\big)$ corresponding to the hybrid trajectory $\big(x, U_T, \mathbf{i}_T\big)$ of the PWA system (1) with input disturbance sequence W_T and measurement noise sequence V_T, and is computed using equation (3).
An output trajectory is feasible if and only if:

$$\begin{bmatrix} x \\ U_T \\ W_T \end{bmatrix} \in \mathbf{\Omega}_{\mathbf{i_T}} \tag{11}$$

$Y\big(x, U_T, 0, 0, \mathbf{i}_T\big)$ is the corresponding nominal output trajectory.

As can be seen from definition 3, a hybrid trajectory may produce various distinct output trajectories depending on the actual input disturbance and measurement noise sequences that acted on the system. The following sets arise directly from definition 3:

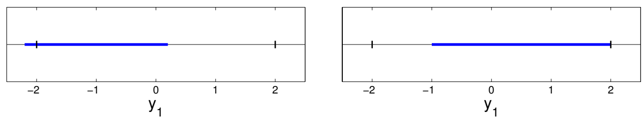
Definition 4. Output Trajectory Feasibility Polytope:
The Output Trajectory Feasibility Polytope (OTFP) of a discrete mode sequence $\mathbf{i}_T$ is defined as the set of all admissible output trajectories of $\mathbf{i}_T$:

$$\mathcal{Y}_{\mathbf{i}_T} \triangleq \Big\{ Y\big(x, U_T, W_T, V_T, \mathbf{i}_T\big) \in \mathbb{R}^{T.n_y} : \\ W_T \in \mathbb{W}_{\mathbf{i}_T} \ , \ V_T \in \mathbb{V}_{\mathbf{i}_T} \ , \ \begin{bmatrix} x \\ U_T \\ W_T \end{bmatrix} \in \mathbf{\Omega}_{\mathbf{i_T}} \Big\} \tag{12}$$

The point-to-set map $\mathcal{Y}_{\mathbf{i}_T} : \mathbb{R}^{T.n_u} \mapsto 2^{\mathbb{R}^{T.n_y}}$ defines the OTFP for a given input sequence U of length T:

$$\mathcal{Y}_{\mathbf{i}_T}(U) \triangleq \Big\{ Y\big(x, U_T, W_T, V_T, \mathbf{i}_T\big) \in \mathbb{R}^{T.n_y} : \\ U_T = U, W_T \in \mathbb{W}_{\mathbf{i}_T}, V_T \in \mathbb{V}_{\mathbf{i}_T}, \begin{bmatrix} x \\ U_T \\ W_T \end{bmatrix} \in \mathbf{\Omega}_{\mathbf{i_T}} \Big\} \tag{13}$$

Figures 3 and 4 show the OTFP $\mathcal{Y}_{\mathbf{i}_T}$ of all *dms*, with length 1 and 2 respectively, from the example system (7). When the length of the output trajectories is 1, they only depend on the initial state.
The output trajectories of length 2, as there is a very important dependence of the OTFP on the input sequences, are represented for several input sequences. To make the figures easier to read, the projection of all $\mathcal{Y}_{\mathbf{i}_T}(U)$ is also presented. Once again, the last element of the input sequences is not relevant and so the input sequences are represented only by the first input.



(a) $\mathbf{i}_T = [\,1\,]$ (b) $\mathbf{i}_T = [\,2\,]$

Fig. 3. Output trajectory feasibility polytopes, $\mathcal{Y}_{\mathbf{i}_T}$, for the example system (7), length $T = 1$.

3.1 Uncertainty in the Discrete Mode Sequence

The uncertainty in the discrete mode sequence arises from the possibility of estimating a *dms* $\hat{\mathbf{i}}_T$ different from the true *dms* $\mathbf{i}_T$. The process of estimating a *dms* from the input and output sequences is of most importance and the following definition is required:

Definition 5. Discrete Mode Sequence Estimator:
The map $\hat{\mathbf{i}}_\tau : \mathbb{R}^{T.n_y}\times\mathbb{R}^{T.n_u}\times 2^{\mathcal{J}_\tau} \mapsto \big\{\mathcal{J}_\tau, \mathbf{0}\big\}$ defines a Discrete Mode Sequence Estimator (DMSE) $\hat{\mathbf{i}}_\tau\big(Y_T, U_T, \mathcal{J}_\tau\big)$ which computes a discrete mode sequence estimate $\hat{\mathbf{i}}_\tau$ of length τ from the set of candidate discrete mode sequences with length τ, $\mathcal{J}_\tau$, as a function of the measurement sequence Y_T and input sequence U_T with lengths T, $(\tau \le T)$. The estimate $\mathbf{0} \notin \mathcal{J}_\tau$ is produced when the estimator rejects all discrete mode sequences of $\mathcal{J}_\tau$.

Definition 5 allows the estimation of *dms* with lengths τ smaller than the length of the measurement and input sequences T. This is related with the delay of the estimation. The following definition states the notation for the *dms* that share the same prefix:

Definition 6. Set of Prefixed Discrete Mode Sequences:
The set of discrete mode sequences of length T that share the same prefix discrete mode sequence $\mathbf{i}_\tau$ of length τ is defined as:

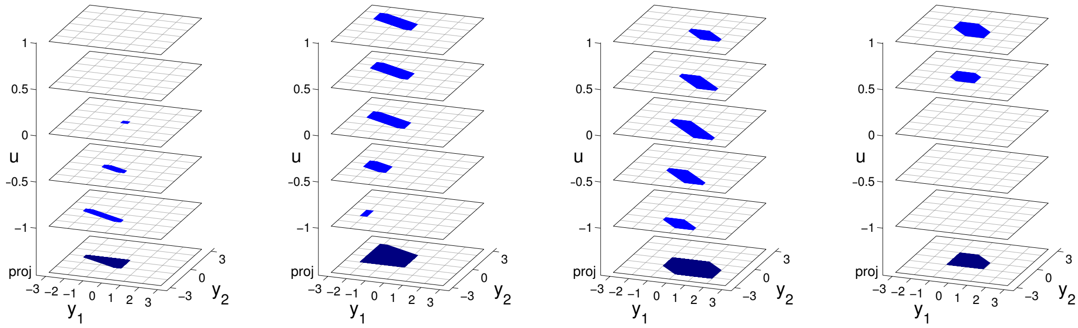
$$\big[\mathbf{i}_\tau, \ldots\big]_T \triangleq \Big\{ \big[\mathbf{j}_\tau, \mathbf{j}_{[\tau+1,T]}\big] \in \mathcal{I}^T \subset \mathbb{N}^{+T} : \mathbf{j}_\tau = \mathbf{i}_\tau \Big\} \tag{14}$$

Definition 5 describes a DMSE as a function whose output is either one of the candidate *dms* represented by $\mathcal{J}_\tau$, or the error output $\mathbf{0}$. This last case happens, for instance, when the measurement sequence Y_T does not belong to the OTFP, $\mathcal{Y}_{[\mathbf{j}_\tau,\ldots]_T}$, of any *dms* $\big[\mathbf{j}_\tau, \ldots\big]_T$, with $\mathbf{j}_\tau \in \mathcal{J}_\tau$.

Figure 5 presents the outcome from a DMSE, $\hat{\mathbf{i}}^e_T$, for the example system (7), where e stands for example. $\hat{\mathbf{i}}^e_T$ is only one from the infinite possible DMSE, and its specific algorithm is not relevant for the present development.

$$\begin{aligned}\mathcal{P}^a_{\mathbf{i}_T,\{\mathbf{i}_T,\mathbf{j}_T\}}\Big((x,U_T,\mathbf{i}_T)\Big) &= \Pr\Big(\hat{\mathbf{i}}^a_T\big(Y(x,U_T,W_T,V_T,\mathbf{i}_T),U_T,\{\mathbf{i}_T,\mathbf{j}_T\}\big)=\mathbf{i}_T\big|(x,U_T,\mathbf{i}_T)\Big) = \\ &= \int_{\mathbb{R}^{T.n_w}}\int_{\mathbb{R}^{T.n_v}}\Big[\hat{\mathbf{i}}^a_T\big(Y(x,U_T,W_T,V_T,\mathbf{i}_T),U_T,\{\mathbf{i}_T,\mathbf{j}_T\}\big)=\mathbf{i}_T\Big]\Pr\Big(W_T,V_T\big|(x,U_T,\mathbf{i}_T)\Big)\; dV_T\, dW_T\end{aligned} \tag{15}$$

$$\begin{aligned}\mathcal{P}^a_{\mathbf{j}_T,\{\mathbf{i}_T,\mathbf{j}_T\}}\Big((x,U_T,\mathbf{i}_T)\Big) &= \Pr\Big(\hat{\mathbf{i}}^a_T\big(Y(x,U_T,W_T,V_T,\mathbf{i}_T),U_T,\{\mathbf{i}_T,\mathbf{j}_T\}\big)=\mathbf{j}_T\big|(x,U_T,\mathbf{i}_T)\Big) = \\ &= \int_{\mathbb{R}^{T.n_w}}\int_{\mathbb{R}^{T.n_v}}\Big[\hat{\mathbf{i}}^a_T\big(Y(x,U_T,W_T,V_T,\mathbf{i}_T),U_T,\{\mathbf{i}_T,\mathbf{j}_T\}\big)=\mathbf{j}_T\Big]\Pr\Big(W_T,V_T\big|(x,U_T,\mathbf{i}_T)\Big)\; dV_T\, dW_T\end{aligned} \tag{16}$$



(a) $\mathbf{i}_T = [1\,1]$ (b) $\mathbf{i}_T = [1\,2]$ (c) $\mathbf{i}_T = [2\,1]$ (d) $\mathbf{i}_T = [2\,2]$

Fig. 4. Output trajectory feasibility polytopes, $\mathcal{Y}_{\mathbf{i}_T}$, for the example system (7), length $T = 2$.

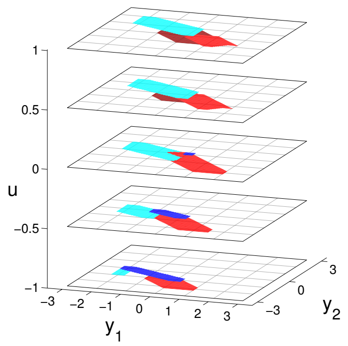


Fig. 5. DMSE $\hat{\mathbf{i}}^e_T$ for the example system (7). (Dark blue - $\hat{\mathbf{i}}^e_T = [1\,1]$, Light blue - $\hat{\mathbf{i}}^e_T = [1\,2]$, Light red - $\hat{\mathbf{i}}^e_T = [2\,1]$, Dark red - $\hat{\mathbf{i}}^e_T = [2\,2]$, White - $\hat{\mathbf{i}}^e_T = \mathbf{0}$.)

From now on, only trajectories of length $T = 2$ will be considered since these can provide better insight on the methodology of defining the uncertainty properties.

From figure 4 is obvious that the same output sequence can be reproduced by different *dms*. For instance $Y_T = \left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]$ can be obtained, for $U_T = \left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]$, from $\mathbf{i}_T = [1\,1]$, from $\mathbf{i}_T = [1\,2]$, or from $\mathbf{i}_T = [2\,1]$. As any DMSE can only provide one *dms* estimate for given Y_T and U_T, the possibility of incorrectly estimating the *dms* is unavoidable. Considering this, a method for determining the probability of correctly estimating the *dms* is required and will now be presented.

For simplicity and without loss of generality, consider that there are only two admissible *dms*, the true one $\mathbf{i}_T$ and the incorrect one $\mathbf{j}_T$. Each hybrid trajectory $(x,U_T,\mathbf{i}_T)$ can produce various different output trajectories $Y(x,U_T,W_T,V_T,\mathbf{i}_T)$ depending on the input disturbance and measurement noise sequences. Each of these measurement sequences has a given probability of being produced, $\Pr\big(L_{\mathbf{i}_T}W_T + V_T = Y(x,U_T,W_T,V_T,\mathbf{i}_T) - Y(x,U_T,0,0,\mathbf{i}_T)\big)$. Considering a specific DMSE, a , $\hat{\mathbf{i}}^a_T\big(Y(x,U_T,W_T,V_T,\mathbf{i}_T),U_T,\{\mathbf{i}_T,\mathbf{j}_T\}\big)$, each output trajectory generates only one discrete mode sequence estimate from the two candidate *dms* $\mathbf{i}_T$ and $\mathbf{j}_T$. Thus, there will be a region of the output trajectories space, $\mathbb{R}^{T.n_y}$, where $\hat{\mathbf{i}}^a_T\big(Y(x,U_T,W_T,V_T,\mathbf{i}_T),U_T,\{\mathbf{i}_T,\mathbf{j}_T\}\big) = \mathbf{i}_T$ and another one where $\hat{\mathbf{i}}^a_T\big(Y(x,U_T,W_T,V_T,\mathbf{i}_T),U_T,\{\mathbf{i}_T,\mathbf{j}_T\}\big) = \mathbf{j}_T$. Integrating the probability of these two regions one arrives at the probability of correctly reconstructing the discrete mode sequence (15), and the probability of incorrectly reconstructing the discrete mode sequence (16), when the system follows the hybrid trajectory $(x,U_T,\mathbf{i}_T)$.

The following function is used in (15) and (16) and compares the *dms* estimate with the desired *dms*:

$$\Big[\hat{\mathbf{i}}^a_T(Y_T,U_T,\mathcal{J}_T)=\mathbf{j}_T\Big] = \begin{cases}1 & \text{iff } \hat{\mathbf{i}}^a_T = \mathbf{j}_T\\ 0 & \text{iff } \hat{\mathbf{i}}^a_T \neq \mathbf{j}_T\end{cases},\quad \hat{\mathbf{i}}^a_T(\cdot) \in \{\mathcal{J}_T, \mathbf{0}\} \tag{17}$$

where the estimate is chosen from the candidate discrete mode sequence set $\mathcal{J}_T, \mathbf{0}$.

Despite the integrals of (15) and (16) being defined in the whole space $\mathbb{R}^{T.n_w}\times\mathbb{R}^{T.n_v}$, the term $\Pr\Big(W_T,V_T\big|(x,U_T,\mathbf{i}_T)\Big)$ will account for existing bounds on the disturbances and feasibility of the hybrid trajectory $(x,U_T,\mathbf{i}_T)$ with W_T.

These introductory concepts lead to the definition of the probability of correct mode estimation for a given algorithm:

Definition 7. Probability of Correct Mode Estimation: For a given PWA system (1), the DMSE a, $\hat{\mathbf{i}}^a_T(\cdot)$, according to definition 5, has Probability of Correct Mode Estimation p_{cme} at τ in $\mathbb{H} \subset \mathbb{R}^{n_x+T.n_u} \times \mathbb{N}^{+T}$ if the measurement sequence Y_T and input sequence U_T with lengths T allow the correct estimation of the initial discrete mode sequence $\mathbf{i}_\tau$, $\hat{\mathbf{i}}^a_\tau(Y_T,U_T,\mathcal{I}_\tau) = \mathbf{i}_\tau$, of any feasible hybrid trajectory $(x_{\mathbf{i}},U_T,[\mathbf{i}_\tau,\mathbf{i}_{[\tau+1,T]}])\in\mathbb{H}$, with no *a priori* knowledge of the initial state $x_{\mathbf{i}}$ and, with probability no smaller than p_{cme}. Equivalently, for any feasible hybrid trajectory $(x_{\mathbf{i}},U_T,[\mathbf{i}_\tau,\mathbf{i}_{[\tau+1,T]}]) \in \mathbb{H}$, it holds:

$$\mathcal{P}^a_{\mathbf{i}_\tau,\mathcal{I}_\tau}\Big((x,U_T,[\mathbf{i}_\tau,\mathbf{i}_{[\tau+1,T]}])\Big) \geq p_{cme} \tag{18}$$

As the *dms* of each hybrid trajectory can only be correctly estimated with a given probability in the interval $[0;1]$, the following sets will group the hybrid trajectories whose probabilities of correct mode estimation are at least p.

Definition 8. Separating in Probability Trajectory Set:
Consider a specific DMSE a, $\hat{\mathbf{i}}^a_T(\cdot)$, according to definition 5. Given two discrete mode sequences $\mathbf{i}_T$ and $\mathbf{j}_T$ of the same length T, the separating in probability trajectory set of $\mathbf{i}_T$ from $\mathbf{j}_T$ is defined as:

$$\mathcal{H}^{S,a}_{\mathbf{i}_T,\mathbf{j}_T}(p) \triangleq \Big\{ (x, U_T, \mathbf{i}_T) \in \mathcal{H}_{\mathbf{i}_T} \subset \mathbb{R}^{(n_x+T.n_u)} \times \mathbb{N}^{+T}: \quad \mathcal{P}^a_{\mathbf{i}_T,\{\mathbf{i}_T,\mathbf{j}_T\}}\big((x,U_T,\mathbf{i}_T)\big) \geq p \Big\} \tag{19}$$

and the non-separating trajectory set of $\mathbf{i}_T$ from $\mathbf{j}_T$ is then defined as:

$$\bar{\mathcal{H}}^{S,a}_{\mathbf{i}_T,\mathbf{j}_T}(p) \triangleq \mathcal{H}_{\mathbf{i}_T} \bigcap \overline{\mathcal{H}^{S,a}_{\mathbf{i}_T,\mathbf{j}_T}(p)} \tag{20}$$

Figures 6 and 7 show the probability of correct mode estimation of each hybrid trajectory of system (1). Figure 6 shows the probability of correctly estimating the whole discrete mode sequence, while figure 7 shows the probability of correctly estimating the initial discrete mode, with one time instant delay. The probabilities of correct mode sequence estimation ranging from 0 to 1 are represented with colors ranging from Red (prob. 0) to Blue (prob.1), respectively.

Remark II: As the integrals of equations (15) and (16) can not be computed exactly in the general case, conservative approximate numerical integration methods must be used when computing probabilities of correct mode estimation. Notice that conservative approximation for these probabilities must respect the following relations:

$$\widetilde{\mathcal{P}}^a_{\mathbf{i}_T,\{\mathbf{i}_T,\mathbf{j}_T\}}\Big((x,U_T,\mathbf{i}_T)\Big) \leq \mathcal{P}^a_{\mathbf{i}_T,\{\mathbf{i}_T,\mathbf{j}_T\}}\Big((x,U_T,\mathbf{i}_T)\Big) \tag{21}$$

$$\widetilde{\mathcal{P}}^a_{\mathbf{j}_T,\{\mathbf{i}_T,\mathbf{j}_T\}}\Big((x,U_T,\mathbf{i}_T)\Big) \geq \mathcal{P}^a_{\mathbf{j}_T,\{\mathbf{i}_T,\mathbf{j}_T\}}\Big((x,U_T,\mathbf{i}_T)\Big) \tag{22}$$

The separating in probability trajectory sets can be easily derived from the maps of figures 6 and 7.

Comparing figures 6 and 7 it can easily be seen that the probability of correct mode estimation at τ increases for decreasing values of τ. The following proposition states this monotonic relation:

Proposition 1. Monotonicity of the Probability of Correct Mode Estimation with the estimation delay:
For a given PWA system (1) and DMSE a, $\hat{\mathbf{i}}^a_T(\cdot)$, according to definition 5, the probability of correct mode estimation p_{cme} at τ in $\mathbb{H} \subset \mathbb{R}^{(n_x+T.n_u)} \times \mathbb{N}^{+T}$ is monotonically non increasing with τ:

$$p_{cme}(\tau+1) \geq p_{cme}(\tau) \tag{23}$$

Proof: The proof arises directly from the following relations:

$$\begin{aligned} &\mathcal{P}^a_{\mathbf{i}_\tau,\mathcal{I}_\tau}\Big((x,U_T,[\mathbf{i}_\tau,\mathbf{i}_{[\tau+1,T]}])\Big) \triangleq \\ &\triangleq \sum_{j\in\mathcal{I}} \mathcal{P}^a_{[\mathbf{i}_\tau,j],\mathcal{I}_{\tau+1}}\Big((x,U_T,[[\mathbf{i}_\tau,j],\mathbf{i}_{[\tau+2,T]}])\Big) \geq \\ &\geq \mathcal{P}^a_{[\mathbf{i}_\tau,l],\mathcal{I}_{\tau+1}}\Big((x,U_T,[[\mathbf{i}_\tau,l],\mathbf{i}_{[\tau+2,T]}])\Big), \forall\, l \in \mathcal{I} \end{aligned} \tag{24}$$

□

Proposition 1 states that the probability of correctly estimating the initial discrete mode sequence $\mathbf{i}_\tau$ of a hybrid trajectory $\big(x,U_T,[\mathbf{i}_\tau,\mathbf{i}_{[\tau+1,T]}]\big)$ is given by the probability of correctly estimating the discrete mode sequence $\mathbf{i}_{\tau+1}$ plus the probability of incorrectly estimating the discrete mode sequences of length $\tau+1$ that start with $\mathbf{i}_\tau$. So, it must increase or remain constant with the considered delay.

The following proposition states the conditions that determine the probability of correct mode estimation of a given algorithm in a given region of the hybrid trajectories space.

Proposition 2. Probability of Correct Mode Estimation of a DMSE:
For a given PWA system (1), the DMSE a, $\hat{\mathbf{i}}^a_T(\cdot)$, according to definition 5, has probability of correct mode estimation p_{cme} at τ in $\mathbb{H} \subset \mathbb{R}^{(n_x+T.n_u)} \times \mathbb{N}^{+T}$ if it holds:

$$\mathbb{H} \bigcap \bar{\mathcal{H}}^{S,a}_{\mathbf{i}_\tau,\mathcal{I}_\tau}(p_{cme}) = \emptyset \quad , \quad \forall\, \mathbf{i}_T \in \mathcal{I}_{T\mathbb{H}} \tag{25}$$

Proof: If the DMSE a has probability of correct mode estimation p_{cme} at τ in $\mathbb{H}$ then it holds:

$$\mathcal{P}^a_{\mathbf{i}_\tau,\mathcal{I}_\tau}\Big((x,U_T,\mathbf{i}_T)\Big) \geq p_{cme}\ ,\ \forall\ \big(x,U_T,\mathbf{i}_T\big) \in \mathbb{H} \bigcap \mathcal{H}_{\mathbf{i}_T} \tag{26}$$

The proof continues by contradiction.
Consider that there exists a hybrid trajectory $\big(x_{\mathbf{i}},U_T,\mathbf{i}_T\big) \in \mathbb{H} \bigcap \mathcal{H}_{\mathbf{i}_T}$ with $\mathbf{i}_T = [\mathbf{i}_\tau, \mathbf{i}_{[\tau+1,T]}]$ such that:

$$\mathcal{P}^a_{\mathbf{i}_\tau,\mathcal{I}_\tau}\Big((x,U_T,\mathbf{i}_T)\Big) < p_{cme} \tag{27}$$

Then, according to definition 8, $\big(x_{\mathbf{i}},U_T,\mathbf{i}_T\big) \notin \mathcal{H}^{S,a}_{\mathbf{i}_\tau,\mathcal{I}_\tau}(p_{cme})$ implying that $\big(x_{\mathbf{i}},U_T,\mathbf{i}_T\big) \in \bar{\mathcal{H}}^{S,a}_{\mathbf{i}_\tau,\mathcal{I}_\tau}(p_{cme})$.

$\big(x_{\mathbf{i}},U_T,\mathbf{i}_T\big) \in \mathbb{H}$ and $\big(x_{\mathbf{i}},U_T,\mathbf{i}_T\big) \in \bar{\mathcal{H}}^{S,a}_{\mathbf{i}_\tau,\mathcal{I}_\tau}(p_{cme})$, which is in contradiction with (25). □

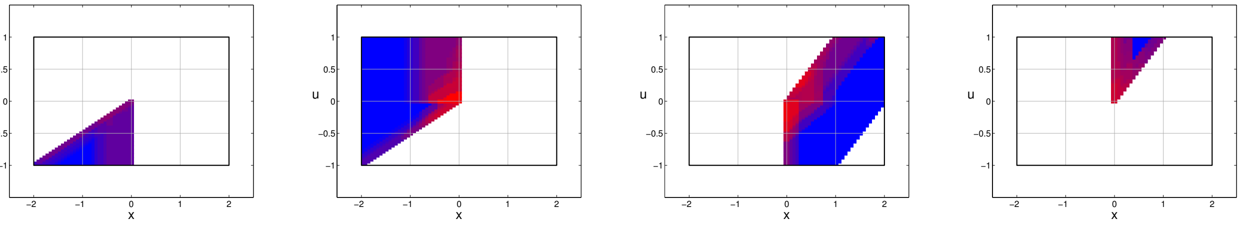
Finally, a notion of observability of the *dms* will be given. As the observability properties of a given system can not depend on the actual estimating procedures, the following discrete mode observability definition is based on the characteristics of the best DMSE that can be designed.

Definition 9. Discrete Mode Observability in Probability:
A PWA system (1) is mode observable with probability p at τ in $\mathbb{H} \subset \mathbb{R}^{n_x+T.n_u} \times \mathbb{N}^{+T}$ if there exists a DMSE a, $\hat{\mathbf{i}}^a_\tau(\cdot)$, according to definition 5, with probability of correct mode estimation p at τ in $\mathbb{H}$. The higher such probability p is the probability of mode observability p_{MO}.

The probability of mode observability p_{MO} is then the highest probability of correct mode estimation that can be achieved by any DMSE defined in accordance with definition 5.

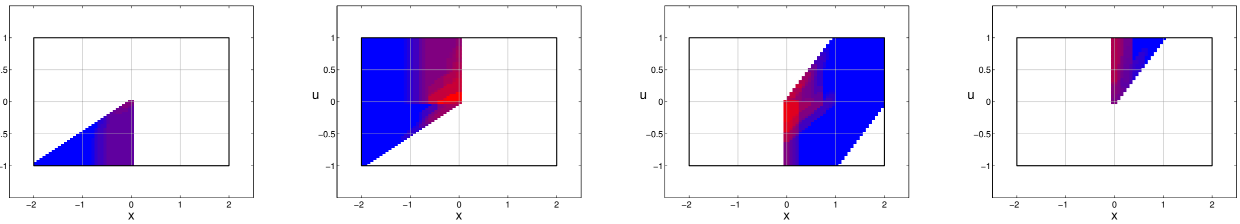
3.2 Uncertainty in the Continuous State

As was stated in the Introduction, the continuous state is estimated assuming that the discrete mode sequence has already been estimated. The associated uncertainty can then be determined using the techniques available for linear time-varying systems, being represented by a probability density function (or a conservative approximation) Simon and Chia [2002], or by a set in the state space as in Lin et al. [2003]. These uncertainties, one for each *dms*, must then be gathered with the probabilities of equations (15) and (16). The process of gathering the uncertainties from the continuous state and *dms* estimates depends on the specific application and falls out of the scope of this paper.



(a) $\mathbf{i}_T = [1\,1]$ (b) $\mathbf{i}_T = [1\,2]$ (c) $\mathbf{i}_T = [2\,1]$ (d) $\mathbf{i}_T = [2\,2]$

Fig. 6. Maps of probability of correct mode estimation at $\tau = 2$ for the example system (7) using the DMSE $\hat{\mathbf{i}}_T^e$. (Red - probability of correct mode estimation 0, Blue - probability of correct mode estimation 1.)



(a) $\mathbf{i}_T = [1\,1]$ (b) $\mathbf{i}_T = [1\,2]$ (c) $\mathbf{i}_T = [2\,1]$ (d) $\mathbf{i}_T = [2\,2]$

Fig. 7. Maps of probability of correct mode estimation at $\tau = 1$ for the example system (7) using the DMSE $\hat{\mathbf{i}}_T^e$. (Red - probability of correct mode estimation 0, Blue - probability of correct mode estimation 1.)

4. CONCLUSIONS

The present paper provided a thorough insight on the uncertainty associated with the estimation of the discrete mode of a hybrid system. It was shown that the discrete mode can not, in general, be uniquely reconstructed from the knowledge of the applied inputs and measured outputs.

The accuracy of the estimates was shown to depend both on the estimation algorithm and on the trajectory followed by the system and, increases with the delay of the estimates.

The uncertainty in the estimated discrete mode can be represented in the form of maps of probability of correct mode estimation, which can only be determined, for the general case, in a conservative approximate way using approximate numerical integration methods.

A new definition of observability for the discrete mode of a hybrid system was proposed. This definition determines the best accuracy obtainable by any DMSE.

Future research should focus on algebraic or efficient numerical methods to determine the maps of probability of correct mode estimation and to include recursiveness in the DMSE algorithms.

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