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Some Results on Stabilizability of Controlled Lagrangian Systems by Energy Shaping

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Abstract: We provide necessary and sufficient conditions for Lyapunov stabilizability and exponential stabilizability by the energy shaping method for the class of all linear controlled Lagrangian systems of an arbitrary degree of under-actuation, and for the class of all controlled Lagrangian systems of one degree of under-actuation. We give a sufficient condition for asymptotic stabilizability for the class of all controlled Lagrangian systems of one degree of under-actuation. For a general controlled Lagrangian system, we give only necessary conditions for Lyapunov stabilizability and exponential stabilizability by energy shaping. In addition, we make a new derivation of the Euler-Lagrange matching conditions both in a simple tensor form and in a coordinate-dependent form, for which we make effective use of gyroscopic forces.

Keywords: Controlled Lagrangian, Stabilization, Energy Shaping, Dissipation, Gyroscopic Force

1. INTRODUCTION

The energy shaping method for stabilization of mechanical systems has been widely used for the last twenty years both on the Lagrangian side and Hamiltonian side; Acosta *et al.* [2005], Auckly *et al.* [2000], Bloch *et al.* [2001, 1997, 2000], Bloch and Marsden [1990], Chang [2005, 2007*a*], Chang *et al.* [2002], Fantoni *et al.* [2000], Hamberg [2000], Ortega *et al.* [2002], van der Schaft [1986], Woolsey *et al.* [2004], Zenkov [2000]. The basic idea in this method can be summarized from the Lagrangian viewpoint as follows: given a mechanical system, we find a feedback control such that the closed-loop system can be represented by a new mechanical system connected with a dissipative force and a gyroscopic force, and the energy of the second mechanical system attains a non-degenerate minimum at an equilibrium of interest. Despite the wide use of this method, there has been a lack of criteria for its applicability except for controlled Lagrangian systems of two degrees of freedom and one degree of under-actuation Chang [2007*a*]. Hence, the goal of this article is to provide some useful criteria for stabilizability by energy shaping for a larger class of controlled Lagrangian systems.

In this article, we find necessary and sufficient conditions for Lyapunov stabilizability and exponential stabilizability by the energy shaping method for the class of all linear controlled Lagrangian systems of an arbitrary degree of under-actuation, and for the class of all controlled Lagrangian systems of one degree of under-actuation. For the latter class, we also give a sufficient condition for asymptotic stabilizability by the energy shaping method. For a general controlled Lagrangian system, we derive only necessary conditions for Lyapunov stabilizability and exponential stabilizability by energy shaping. In addition, we make a new derivation of the Euler-Lagrange matching conditions both in a simple tensor form and in a coordinate-dependent form, for which we make effective use of gyroscopic forces. We omit most proofs here due to space limit, but proofs in detail will be given in a future journal article.

2. REVIEW OF CONTROLLED LAGRANGIAN SYSTEMS

We review the basic notions on controlled Lagrangian systems from Chang *et al.* [2002], and pose the main questions for this article at the end of this section. We use various indices as follows:

$$\begin{gathered} i, j, k, l, r = 1, \ldots, n, \\ \alpha, \beta, \gamma = 1, \ldots, n_1, \\ a, b = n_1 + 1, \ldots, n \end{gathered}$$

where n and $n_1 < n$ are fixed positive integers.

We introduce the notion of controlled Lagrangian systems, and the Euler-Lagrange matching conditions.

Definition 2.1. A (simple) controlled Lagrangian system on TQ is a triple (L, F, W) where

- the Lagrangian L on TQ is the kinetic minus potential energy given by
$$L(q, \dot{q}) = \frac{1}{2} m(\dot{q}, \dot{q}) - V(q) \quad (1)$$
where $m = m_{ij} \mathbf{d}q^i \otimes \mathbf{d}q^j \in \Gamma(T^*Q \otimes T^*Q)$ is symmetric and non-degenerate,
- the external force F is a T^*Q-valued map,
- the control bundle W is a sub-bundle of T^*Q, and a control for this system is a W-valued map.

We call the dimension of Q the degree of freedom, and the codimension of W the degree of under-actuation.

The equations of motion of a controlled Lagrangian system (L, F, W) with $n = \dim Q$ and $n - n_1 = \dim W$ are given in coordinates by

$$\mathcal{EL}(L)_i = F_i + B_i^a u_a, \quad i = 1, \ldots, n$$

where the Euler-Lagrange operator $\mathcal{EL}$ is given by

$$\mathcal{EL}(L) = \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} \right) \mathbf{d}q^i,$$

and the control bundle is given by $W = \langle B_i^a \mathbf{d}q^i \mid a = n_1 + 1, \ldots, n\rangle$. The equations of motion can be written also as

$$m_{ij}\ddot{q}^j + [jk, i]\dot{q}^j\dot{q}^k + \frac{\partial V}{\partial q^i} = F_i + B_i^a u_a, \qquad (2)$$

where $[ij, l]$ is the Christoffel symbol of the first kind for the mass matrix (m_{ij}) defined by

$$[ij, l] = \frac{1}{2}\left(\frac{\partial m_{il}}{\partial q^j} + \frac{\partial m_{jl}}{\partial q^i} - \frac{\partial m_{ij}}{\partial q^l} \right).$$

Proposition 2.2. Let two controlled Lagrangian systems (L, F, W) and $(\widehat{L}, \widehat{F}, \widehat{W})$ be given where

$$L = \frac{1}{2}m(\dot{q}, \dot{q}) - V(q), \quad \widehat{L} = \frac{1}{2}\widehat{m}(\dot{q}, \dot{q}) - \widehat{V}(q).$$

They are (affine-)feedback equivalent if and only if the following Euler-Lagrange matching conditions hold:

ELM-1: $m^{-1}W = \widehat{m}^{-1}\widehat{W}$.
ELM-2: $\langle \mathcal{EL}(L) - F - m\widehat{m}^{-1}(\mathcal{EL}(\widehat{L}) - \widehat{F}), W^0\rangle = 0$

where

$$W^0 = \{X \in TQ \mid \langle w, X\rangle = 0 \quad \forall w \in W\}.$$

We introduce the linearization of a (nonlinear) controlled Lagrangian system. We say that a controlled Lagrangian system (L, F, W) has an equilibrium point at $(q_e, \dot{q}_e)$ if the point $(q_e, \dot{q}_e)$ is an equilibrium point in the equations of motion of $(L, F, 0)$, i.e., the equations in (2) with $u_a = 0$. Suppose that a system (L, F, W) with L in (1) has an equilibrium point at $(q, \dot{q}) = (0, 0)$. Its linearized controlled Lagrangian system (L^l, F^l, W^l) at $(q, \dot{q}) = (0, 0)$ is defined by

$$L^l = \frac{1}{2}m_{ij}(0)\dot{q}^i\dot{q}^j - \frac{1}{2}\frac{\partial^2 V}{\partial q^i \partial q^j}(0)q^i q^j,$$
$$F^l = \frac{\partial F}{\partial \dot{q}^i}(0,0)\dot{q}^i + \frac{\partial F}{\partial q^i}(0,0)q^i,$$
$$W^l = W(0),$$

where we have intentionally excluded $\frac{\partial V}{\partial q^i}(0)q^i$ and $F(0, 0)$ since their effects do not appear in the linearization of (2) at the origin.

Lemma 2.3. If two controlled Lagrangian systems (L, F, W) and $(\widehat{L}, \widehat{F}, \widehat{W})$ with a common equilibrium point at $(q, \dot{q}) = (0, 0)$ are feedback equivalent, then their linearized controlled Lagrangian systems (L^l, F^l, W^l) and $(\widehat{L}^l, \widehat{F}^l, \widehat{W}^l)$ at $(q, \dot{q}) = (0, 0)$ are feedback equivalent, too.

The energy corresponding to a Lagrangian $L = \frac{1}{2}m(\dot{q}, \dot{q}) - V(q)$, is the function $E = \frac{1}{2}m(\dot{q}, \dot{q}) + V(q)$.

A dissipative force F is a $\mathcal{F}(Q)$-linear map from $\Gamma(TQ)$ to $\Gamma(T^*Q)$ that satisfies

$$\langle F(X), Y\rangle = \langle F(Y), X\rangle, \quad \langle F(X), X\rangle \leq 0$$

for all $X, Y \in \Gamma(TQ)$.

A (quadratic) gyroscopic force G is a $\mathcal{F}(Q)$-bilinear map from $\Gamma(TQ) \times \Gamma(TQ)$ to $\Gamma(T^*Q)$ that satisfies

$$\langle G(X, Y), Z\rangle = \langle G(Y, X), Z\rangle, \qquad (3)$$
$$\langle G(X, X), X\rangle = 0 \qquad (4)$$

for all $X, Y, Z \in \Gamma(TQ)$. Because $\langle G(X, X), X\rangle = 0$, a gyroscopic force does not change the energy along the trajectories of a controlled Lagrangian system. It is straightforward to show that the set of the two properties in (3) and (4) is equivalent to the following set:

$$\langle G(X, Y), Z\rangle = \langle G(Y, X), Z\rangle, \qquad (5)$$
$$\langle G(X,Y),Z\rangle + \langle G(Y,Z),X\rangle + \langle G(Z,X),Y\rangle = 0 \qquad (6)$$

for all $X, Y, Z \in \Gamma(TQ)$. For notational purposes, let us define the gyroscopic bundle $\mathrm{GS}(Q)$ over Q:

$$\mathrm{GS}(Q) = \{G \in \otimes^3 T^*Q \mid (5) \text{ and } (6) \text{ hold}\},$$
$$= \{(G_{ijk}) \mid G_{ijk} + G_{jki} + G_{kij} = 0, G_{ijk} = G_{jik}\}.$$

In our previous articles, Chang [2005, 2007*a*], Chang *et al.* [2002], the gyroscopic force was defined as a section of the bundle

$$B(Q) = \{G \in \otimes^3 T^*Q \mid \langle G(X, Y), Z\rangle = -\langle G(X, Z), Y\rangle \quad \forall X, Y, Z \in TQ\}.$$

Since any section of $B(Q)$ satisfies $\langle G(X, X), X\rangle = 0$, the bundle $B(Q)$ also deserves the name of gyroscopic force. However, along a trajectory $q(t)$ of a Lagrangian system, G appears as $G(\dot{q}(t), \dot{q}(t))$ and its effect on the rate of energy change in time appears as $\langle G(\dot{q}(t), \dot{q}(t)), \dot{q}(t)\rangle$. Hence, the new definition captures the essence of gyroscopic forces. The relation between $B(Q)$ and $\mathrm{GS}(Q)$ is as follows: the map $\psi : B(Q) \to \otimes^3 T^*Q$ defined by $(\psi(G))(X, Y)Z = \frac{1}{2}G(X, Y)Z + \frac{1}{2}G(Y, X)Z$ for all $G \in B(Q)$ and $X, Y, Z \in TQ$, is onto $\mathrm{GS}(Q)$. We can thus recover our old results by choosing a $G_{\text{old}} \in \psi^{-1}(G_{\text{new}}) \in \Gamma(B(Q))$ where $G_{\text{new}} \in \Gamma(\mathrm{GS}(Q))$ is the one used in this article.

We now pose the main questions for this article as follows:

- Q1. For a given controlled Lagrangian system $(L, F = 0, W)$, can we find a system $(\widehat{L}, \widehat{G}, \widehat{W})$ with $\widehat{G} \in \Gamma(\mathrm{GS}(Q))$ that is feedback-equivalent to $(L, 0, W)$?
- Q2. Can we make the energy $\widehat{E} = \frac{1}{2}\widehat{m}(\dot{q}, \dot{q}) + \widehat{V}(q)$ for $(\widehat{L}, \widehat{G}, \widehat{W})$ attain a non-degenerate minimum at an equilibrium point of interest?
- Q3. Does the equilibrium become Lyapunov/ asymptotically/ exponentially stabilized by a dissipative control force?

3. LINEAR CONTROLLED LAGRANGIAN SYSTEMS

In this section, we provide a complete answer to the three questions posed in § 1 for the class of all linear controlled Lagrangian systems. This is not only important by itself, but also useful for stabilization of nonlinear controlled Lagrangian systems. Our result is an extension of Zenkov [2000].

Definition 3.1. The linear system

$$\dot{x} = Ax$$

on $\mathbb{R}^{2n}$ is called oscillatory if A is semi-simple and all eigenvalues of A are non-zero pure imaginary numbers. We say that a matrix A is oscillatory if its associated system $\dot{x} = Ax$ is oscillatory.

Lemma 3.2. The second-order system

$$\ddot{x} = Ax + Bu \tag{7}$$

can be made oscillatory by a positional feedback $u = -Kx$ if and only if one of the following holds:

(a) it is controllable,
(b) it is uncontrollable, and the uncontrollable dynamics are oscillatory.

Lemma 3.3. Consider a linear controlled Lagrangian system

$$M\ddot{q} + Vq = Bu \tag{8}$$

where $q \in \mathbb{R}^n$, $u \in \mathbb{R}^l$ with $l \leq n$, M and V are both $n \times n$ positive definite matrices, and B is an $n \times l$ matrix of rank l. Choose a dissipative feedback force of the form

$$u = -DB^T\dot{q} \tag{9}$$

where D is an $l \times l$ positive definite matrix. Then, $(q, \dot{q}) = (0, 0)$ is an exponentially (or, asymptotically) stable equilibrium in the closed-loop system if and only if (8) is a controllable system.

We now give necessary and sufficient conditions for Lyapunov stabilizability and exponential stabilizability via energy shaping plus dissipation for the class of all linear controlled Lagrangian systems.

Theorem 3.4. Consider a linear controlled Lagrangian system $(L, F = 0, W)$ with $L = \frac{1}{2}m_{ij}\dot{q}^i\dot{q}^j - \frac{1}{2}V_{ij}q^iq^j$ of n degrees of freedom and n_1 $(< n)$ degrees of under-actuation. In order that there exists another linear controlled Lagrangian system $(\widehat{L}, \widehat{G} = 0, \widehat{W})$ with $\widehat{L} = \frac{1}{2}\widehat{m}_{ij}\dot{q}^i\dot{q}^j - \frac{1}{2}\widehat{V}_{ij}q^iq^j$, that is feedback-equivalent to $(L, 0, W)$ such that both $(\widehat{m}_{ij})$ and $(\widehat{V}_{ij})$ are positive definite, it is necessary and sufficient that the given linear system $(L, 0, W)$ satisfies one of the following two conditions:

(a) it is controllable,
(b) it is uncontrollable, and the uncontrollable dynamics is oscillatory.

Moreover, if a $\widehat{W}$-valued dissipative control force $\widehat{u}$ with $\operatorname{rank}\widehat{u} = \dim\widehat{W}$ is chosen for $(\widehat{L}, \widehat{G}, \widehat{W})$, then the origin is Lyapunov stable in the closed-loop system. In particular, the origin becomes exponentially stable if and only if the given linear system $(L, 0, W)$ is controllable.

We can utilize the result in Theorem 3.4 for nonlinear controlled Lagrangian systems via linearization.

Corollary 3.5. Consider a (nonlinear) controlled Lagrangian system $(L, 0, W)$ for which the origin $(q, \dot{q}) = (0, 0)$ is an equilibrium point. Suppose that there exists a controlled Lagrangian system $(\widehat{L}, \widehat{G}, \widehat{W})$ with $\widehat{G} \in \Gamma(\mathrm{GS}(Q))$ that is feedback-equivalent to $(L, 0, W)$ such that the energy $\widehat{E}(q, \dot{q}) = \frac{1}{2}\widehat{m}_{ij}(q)\dot{q}^i\dot{q}^j + \widehat{V}(q)$ attains a non-degenerate minimum at the origin. Then, it is necessary that the linearization $(L^l, 0, W^l)$ of $(L, 0, W)$ at the origin satisfies condition (a) or (b) in Theorem 3.4. Moreover, if a $\widehat{W}$-valued dissipative control force $\widehat{u}$ with $\operatorname{rank}\widehat{u} = \dim\widehat{W}$ is chosen for $(\widehat{L}, \widehat{G}, \widehat{W})$, then the origin is Lyapunov stable in the closed-loop system. In particular, the origin becomes exponentially stable if and only if the linearized system $(L^l, 0, W^l)$ is controllable.

4. NONLINEAR CONTROLLED LAGRANGIAN SYSTEMS

In this section we put the matching conditions in Proposition 2.2 into a simple tensor form, and apply it so as to give an almost complete answer to the main questions posed in § 1 for the class of all controlled Lagrangian systems of one degree of under-actuation: a necessary and sufficient condition for Lyapunov/exponential stabilizability and a sufficient condition for asymptotic stabilizability, all by the energy shaping method.

4.1 Matching Conditions.

The goal of this section is to put the second Euler-Lagrange matching condition, **ELM-2**, in Proposition 2.2 into a simple form for two controlled Lagrangian systems $(L, 0, W)$ and $(\widehat{L}, \widehat{G}, \widehat{W})$ with $\widehat{G} \in \Gamma(\mathrm{GS}(Q))$.

We consider two controlled Lagrangian systems $(L, F = 0, W)$ and $(\widehat{L}, \widehat{G}, \widehat{W})$ where $\widehat{G} \in \Gamma(\mathrm{GS}(Q))$ and

$$L = \frac{1}{2}m(\dot{q}, \dot{q}) - V(q), \quad \widehat{L} = \frac{1}{2}\widehat{m}(\dot{q}, \dot{q}) - \widehat{V}(q).$$

Let us define $\lambda \in \Gamma(\mathrm{Aut}(TQ))$ by

$$\lambda = \widehat{m}^{-1}m.$$

The variable λ was first introduced in Auckly *et al.* [2000]. Let $P \in \Gamma(\mathrm{End}(TQ))$ with $\operatorname{Im} P = W^0$. Let ∇ be the connection of the metric m.

Proposition 4.1. Let $(L, 0, W)$ and $(\widehat{L}, \widehat{G}, \widehat{W})$ be given as above. They are feedback equivalent to each other if and only if the following hold: for all $X, Y, Z \in TQ$

$$\begin{aligned} 0 &= (\nabla_{\lambda PZ}\widehat{m})(X, Y) - (\nabla_X\widehat{m})(Y, \lambda PZ) \\ &\quad - (\nabla_Y\widehat{m})(X, \lambda PZ) + 2\widehat{G}(X, Y)\lambda PZ, \qquad (10) \\ 0 &= (\lambda PZ).\widehat{V} - (PZ).V, \end{aligned}$$

and $\widehat{W} = \widehat{m}m^{-1}W$.

The gyroscopic term $\widehat{G}$ allows us to transform (10) into a simpler form.

Theorem 4.2. Let $(L, 0, W)$ be a controlled Lagrangian system with n_1 degrees of under-actuation. Then, a controlled Lagrangian system $(\widehat{L}, \widehat{G}, \widehat{W})$ is feedback equivalent to $(L, 0, W)$ if and only if there exists an $\widehat{A} \in \Gamma(\mathrm{GS}(Q))$ such that for all $X, Y, Z \in TQ$

$$0 = (\nabla_{\lambda PZ}\widehat{m})(X, Y) - \widehat{A}(X, Y)\lambda PZ, \tag{11}$$

$$0 = (\lambda PZ).\widehat{V} - (PZ).V \tag{12}$$

for all $X, Y, Z \in TQ$, and

$$\widehat{W} = \widehat{m}m^{-1}W \tag{13}$$

where

$$\lambda P = \widehat{m}^{-1}mP. \tag{14}$$

The relationship between $\widehat{G}$ and $\widehat{A}$ is given by

$$\begin{aligned} \widehat{G}(X, Y)Z &= \frac{1}{2}\widehat{A}(X, Y)Z + \frac{1}{2}(\nabla_X\widehat{m})(Y, Z) \qquad (15) \\ &\quad + \frac{1}{2}(\nabla_Y\widehat{m})(X, Z) - (\nabla_Z\widehat{m})(X, Y) \end{aligned}$$

If W is integrable, i.e., $W = \langle \mathbf{d}q^{n_1+1}, \ldots, \mathbf{d}q^n \rangle$ in some coordinates, $q = (q^1, \ldots, q^n)$, then (11) and (12) are given by

$$0 = \lambda^k_\alpha \frac{\partial \widehat{m}_{ij}}{\partial q^k} - \Gamma^l_{ik}\lambda^k_\alpha \widehat{m}_{lj} - \Gamma^l_{jk}\lambda^k_\alpha \widehat{m}_{li} - \widehat{A}_{ijk}\lambda^k_\alpha \tag{16}$$

$$0 = \lambda^k_\alpha \frac{\partial \widehat{V}}{\partial q^k} - \frac{\partial V}{\partial q^\alpha} \tag{17}$$

with $\Gamma^i_{jk} = m^{il}[jk, l]$ and

$$\lambda^k_\alpha = m_{\alpha r}\widehat{m}^{rk} \tag{18}$$

where $\alpha = 1, \ldots, n_1$ and $i, j, k, l, r = 1, \ldots, n$. The control bundle $\widehat{W}$ is given by $\widehat{W} = \widehat{m}m^{-1}W$.

Let us derive a set of equations that are equivalent to but simpler than (16). The main idea is to make the derivation in a fixed set of coordinates, and make use of the gyroscopic term $\widehat{A}_{ijk}$.

Corollary 4.3. Let $(L, 0, W)$ and $(\widehat{L}, \widehat{G}, \widehat{W})$ be given where $\widehat{G} \in \Gamma(\mathrm{GS}(Q))$. Suppose that W is integrable such that $W = \langle \mathbf{d}q^{n_1+1}, \ldots, \mathbf{d}q^n\rangle$ in a fixed set of coordinates $(q^1, \ldots, q^n)$ of an open set U. Then, the two systems are feedback-equivalent (on U) if and only if there exists $(\widehat{B}_{ijk}) \in \Gamma(\mathrm{GS}(U))$ such that

$$0 = m_{\alpha k}\widehat{m}^{kl}\left(\frac{\partial \widehat{m}_{ij}}{\partial q^l} - \widehat{B}_{ijl}\right) - 2[ij, \alpha], \tag{19}$$

$$0 = m_{\alpha k}\widehat{m}^{kl}\frac{\partial \widehat{V}}{\partial q^l} - \frac{\partial V}{\partial q^\alpha}, \tag{20}$$

and $\widehat{m}^{-1}\widehat{W} = m^{-1}W$. The relationship between $(\widehat{A}_{ijk})$ and $(\widehat{B}_{ijk})$ is given by

$$\widehat{A}_{ijk} = \widehat{B}_{ijk} + 2\Gamma^l_{ij}\widehat{m}_{lk} - \Gamma^l_{ik}\widehat{m}_{lj} - \Gamma^l_{jk}\widehat{m}_{li}.$$

4.2 Controlled Lagrangian Systems of One Degree of Under-Actuation

We now consider the class of all controlled Lagrangian systems of one degree of under-actuation on an *analytic* manifold Q. Suppose that a given *analytic* controlled Lagrangian system $(L, 0, W)$ with $\operatorname{codim} W = 1$ has an equilibrium at the origin. We want to find a feedback equivalent system $(\widehat{L}, \widehat{G}, \widehat{W})$ with $\widehat{G} \in \Gamma(\mathrm{GS}(Q))$ such that the energy $\widehat{E} = \frac{1}{2}\widehat{m}(\dot{q}, \dot{q}) + \widehat{V}(q)$ attains a non-degenerate minimum at the origin. By Corollary 3.5, it is necessary to assume that the linearization $(L^l, 0, W^l)$ of $(L, 0, W)$ at the origin satisfies condition (a) or (b) in Theorem 3.4. We will show that the condition (a) or (b) is sufficient as well. Suppose that $(L^l, 0, W^l)$ satisfies condition (a) or (b) in Theorem 3.4.

Since $W \subset T^*Q$ with codimension 1 is integrable, there are coordinates $q = (q^1, q^2, \ldots, q^n)$ such that $W = \langle \mathbf{d}q^2, \ldots, \mathbf{d}q^n\rangle$. In these coordinates, equations (16) and (17) become

$$0 = m_{1k}\widehat{m}^{kl}\frac{\partial \widehat{m}_{ij}}{\partial q^l} - \Gamma^k_{il}m_{1r}\widehat{m}^{rl}\widehat{m}_{kj} - \Gamma^k_{lj}m_{1r}\widehat{m}^{rl}\widehat{m}_{ki} - \widehat{A}_{ijl}m_{1r}\widehat{m}^{rl}, \tag{21}$$

$$0 = m_{1k}\widehat{m}^{kl}\frac{\partial \widehat{V}}{\partial q^l} - \frac{\partial V}{\partial q^1}, \tag{22}$$

instead of which one could use (19) and (20). The equations of motion for $(L^l, 0, W^l)$ are given by

$$m_{1k}(0)\ddot{q}^k + \frac{\partial^2 V}{\partial q^1 \partial q^k}(0)q^k = 0,$$

$$m_{ak}(0)\ddot{q}^k + \frac{\partial^2 V}{\partial q^a \partial q^k}(0)q^k = u_a$$

where $a = 2, \ldots, n$. Since $(L^l, 0, W^l)$ satisfies condition (a) or (b) in Theorem 3.4, there exist constant matrices $(M_{ij}) \succ 0$ and $(U_{ij}) \succ 0$ such that

$$m_{1k}(0)M^{kl}U_{lj} - \frac{\partial^2 V}{\partial q^1 \partial q^j}(0) = 0$$

by Theorem 3.4.

Let us search for $\widehat{m}$, $\widehat{V}$ and $\widehat{A}$ that satisfy (21), (22) and the following initial condition

$$\widehat{m}_{ij}(0) = M_{ij}, \quad \frac{\partial \widehat{V}}{\partial q^i}(0) = 0, \quad \frac{\partial^2 \widehat{V}}{\partial q^i \partial q^j}(0) = U_{ij}. \tag{23}$$

Since $(m_{1k}(0))$ has rank 1, there exists an index $i_1 \in \{1, \ldots, n\}$ such that the i_1-th coordinate of $(m_{1k}(0)M^{kl})$ is not zero. Hence, locally around $q = 0$, equations (21) and (22) can be transformed into the following form

$$\frac{\partial Y}{\partial q^{i_1}} = \sum_{1 \le i \le n,\, i \ne i_1} B_i(q, Y)\frac{\partial Y}{\partial q^i} + C(q, Y, \widehat{A}) \tag{24}$$

where $Y(q) = (\widehat{m}_{ij}(q), \widehat{V}(q)) \in \mathbb{R}^N$, $B_i(q, Y) \in \mathbb{R}^{N\times N}$ and $C(q, Y, \widehat{A}) \in \mathbb{R}^N$ with $N = \frac{n(n+1)}{2} + 1$. The initial condition in (23) becomes

$$Y|_{q^{i_1}=0} = (\Phi_{ij}(q^1, \ldots, q^{i_1-1}, q^{i_1+1}, \ldots, q^n), \Psi(q^1, \ldots, q^{i_1-1}, q^{i_1+1}, \ldots, q^n)) \tag{25}$$

where Φ_{ij} and Ψ are arbitrary analytic functions on $\mathbb{R}^{n-1}$ satisfying

$$\Phi_{ij}(0) = M_{ij}, \quad \frac{\partial \Psi}{\partial q^c}(0) = 0, \quad \frac{\partial^2 \Psi}{\partial q^c \partial q^d}(0) = U_{cd} \tag{26}$$

where $c, d \in \{1, \ldots, i_1 - 1, i_1 + 1, \ldots, n\}$. By the Cauchy-Kowalevski Theorem, for any analytic choice of $\widehat{A}(q) \in \Gamma(\mathrm{GS}(Q))$, there exists a unique analytic solution $Y(q)$ to (24) – (26). In other words, there exist unique analytic functions $\widehat{m}_{ij}$ and $\widehat{V}$ that satisfy (21), (22) and (23). By Theorem 4.2, the system $(\widehat{L}, \widehat{G}, \widehat{W})$ with $\widehat{G}$ and $\widehat{W}$ in (15) and (13), is feedback-equivalent to $(L, 0, W)$, and its energy $\widehat{E} = \frac{1}{2}\widehat{m}(\dot{q}, \dot{q}) + \widehat{V}(q)$ attains a non-degenerate minimum at the origin. Combining this result with Corollary 3.5 and Lyapunov's first method, we obtain the following:

Theorem 4.4. Consider a controlled Lagrangian system $(L, 0, W)$ of one degree of under-actuation for which the origin $(q, \dot{q}) = (0, 0)$ is an equilibrium point. In order that there exists a controlled Lagrangian system $(\widehat{L}, \widehat{G}, \widehat{W})$ feedback equivalent to $(L, 0, W)$ where $\widehat{G} \in \Gamma(\mathrm{GS}(Q))$ and the energy $\widehat{E} = \frac{1}{2}\widehat{m}(\dot{q}, \dot{q}) + \widehat{V}(q)$ has a non-degenerate minimum at the origin, it is necessary and sufficient that the linearization $(L^l, 0, W^l)$ of the original system $(L, 0, W)$ at the origin satisfies condition (a) or (b) given in Theorem 3.4. Moreover, The origin is a Lyapunov stable equilibrium in the closed-loop system with any $\widehat{W}$-valued dissipation $\widehat{u}$ for $(\widehat{L}, \widehat{G}, \widehat{W})$ with $\operatorname{rank}\widehat{u} = \dim \widehat{W}$. In particular, the equilibrium is exponentially stable if and only if $(L^l, 0, W^l)$ is controllable.

Example. Systems such as an inverted pendulum on a cart, the Furuta pendulum, the ball and beam system and the Pendubot system, have been asymptotically stabilized by the energy shaping method; see Acosta *et al.* [2005], Bloch *et al.* [2001], Chang *et al.* [2002], Fantoni *et al.* [2000], Nair and Leonard [2002], Ortega *et al.* [2002], Woolsey *et al.* [2004]. Since the linearization of each of them at the equilibrium of interest is controllable, these old results agree with Theorem 4.4. Moreover, the convergence is not only asymptotic but also exponential by Theorem 4.4. In this case, the region of exponential convergence is practically as large as that of asymptotic convergence due to the following lemma:

Lemma 4.5. Consider a differential equation on $\mathbb{R}^n$:

$$\dot{x} = f(x), \quad f(0) = 0.$$

Suppose that the origin is asymptotically stable with a compact region of (asymptotic) convergence Ω. If the origin is exponentially stable, then Ω is a region of exponential convergence.

Example. The dynamics of a vertical takeoff and landing (VTOL) aircraft is given by

$$\ddot{q} = \begin{bmatrix} 0 \\ 0 \\ \dfrac{g}{c}\sin q^3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ \dfrac{1}{\epsilon}\cos q^3 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 1 \\ \dfrac{1}{\epsilon}\sin q^3 \end{bmatrix} u_2 \qquad (27)$$

where $q = (q^1, q^2, q^3)$; see Acosta *et al.* [2005] and references therein for more detail on this dynamics. Suppose that a point $(\overline{q}, 0_3) = (\overline{q}^1, \overline{q}^2, 0, 0, 0, 0)$ is an equilibrium that we want to stabilize. This system can be viewed as the controlled Lagrangian system $(L, 0, W)$ with

$$L = \frac{1}{2}((\dot{q}^1)^2 + (\dot{q}^2)^2 + (\dot{q}^3)^2) - \frac{g}{c}\cos q^3,$$
$$W = \left\langle \left(1, 0, \frac{1}{\epsilon}\cos q^3\right)^T, \left(0, 1, \frac{1}{\epsilon}\sin q^3\right)^T \right\rangle.$$

It has one degree of under-actuation, and its linearization at the equilibrium, $(\overline{q}, 0_3)$,

$$\ddot{q} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \dfrac{g}{c} \end{bmatrix} q + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \dfrac{1}{\epsilon} & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

is controllable. Hence, by Theorem 4.4 the equilibrium, $(\overline{q}, 0_3)$ for the VTOL system in (27) can be exponentially stabilized by the energy shaping method. This result is consistent with that in Acosta *et al.* [2005] where asymptotic stabilizability by energy shaping was shown by designing a concrete family of energy shaping controllers.

4.3 A Sufficient Condition for Asymptotic Stabilizability for Controlled Lagrangian Systems

We first introduce the notion of Condition C for a general affine control system (recall that a controlled Lagrangian system can be put into a form of an affine control system). Consider an affine control system

$$\dot{x} = X(x) + \sum_{i=1}^{r} u_i Y_i(x), \quad x \in \mathbb{R}^s. \qquad (28)$$

Associated with this system, let us define the following distributions:

$$\Delta = \langle X, \mathrm{ad}_X^k Y_i \mid 1 \le i \le r; k \in \mathbb{N} \cup \{0\}\rangle,$$
$$\Delta_k = \langle \mathrm{ad}_X^k Y_i \mid 1 \le i \le r\rangle, \quad k = 0, 1, 2, \ldots.$$
$$N_0 = 0$$
$$N_1 = \langle Y_i, \mathrm{ad}_{Y_i} Y_j \mid 1 \le i, j \le r\rangle$$

and inductively for $k = 1, 2, \ldots$

$$N_{k+1} = N_k + \langle \mathrm{ad}_X Z, \mathrm{ad}_{Y_i} Z, \mathrm{ad}_{Y_i} \mathrm{ad}_X^k Y_j \mid 1 \le i, j \le r; Z \in N_k\rangle.$$

For each $k \in \mathbb{N} \bigcup \{0\}$ and each point $x \in \mathbb{R}^s$, let $\Pi_k(x)$ be a maximal subspace of $\Delta_k(x)$ such that $\Pi_k(x) \bigcap N_k(x) = \{0\}$, or equivalently a complement of $\Delta_k(x) \bigcap N_k(x)$ in $\Delta_k(x)$ so that $\Pi_k(x) \bigoplus (\Delta_k(x) \bigcap N_k(x)) = \Delta_k(x)$. Let

$$\Pi_k = \bigcup_{x \in \mathbb{R}^s} \Pi_k(x), \quad k = 0, 1, 2, \ldots.$$

Lemma 4.6. The following hold:

(i) $\Pi_i \subset \Delta_i \subset N_j$ for all $i < j$.
(ii) $\Pi_k \bigcap \Pi_l = \{0\}$ for all $k \neq l$.

In general, $\bigoplus_{k=0}^{\infty} \Pi_k(x) \subset \Delta(x) \subset \mathbb{R}^s$ for every $x \in \mathbb{R}^s$.

Definition 4.7. The affine control system in (28) is said to satisfy condition C around a point $x_0 \in \mathbb{R}^s$ if there is an open neighborhood U of x_0 such that

$$\bigoplus_{k=0}^{\infty} \Pi_k(x) = \Delta(x) = \mathbb{R}^s \qquad (29)$$

for every $x \in U \backslash \{x_0\}$.

The following is the key lemma:

Lemma 4.8. Condition C is invariant under any invertible affine feedback control transformation. Namely, if the system (28) satisfies condition C around a point x_0, so does the system

$$\dot{x} = \widetilde{X}(x) + \sum_{i=1}^{r} v_i \widetilde{Y}_i(x) \qquad (30)$$

with

$$\widetilde{X} = X + \sum_{i=1}^{r} \alpha_i Y_i, \quad \widetilde{Y}_i = \sum_{j=1}^{r} \beta_{ij} Y_j,$$

where α_i, β_{ij} $(1 \le i, j \le r)$ are arbitrary (smooth or analytic) functions on $\mathbb{R}^s$ with the $r \times r$ matrix $(\beta_{ij}(x))$ being invertible for every x.

Combining Lemma 4.8 and the LaSalle invariance principle we obtain the following:

Theorem 4.9. Suppose that a given controlled Lagrangian system $(L, 0, W)$ satisfies condition C around the equilibrium at the origin. If there is another controlled Lagrangian system $(\widehat{L}, \widehat{G}, \widehat{W})$ with $\widehat{G} \in \mathrm{GS}(Q)$ that is feedback equivalent to $(L, 0, W)$ and whose energy has a strict (not necessarily non-degenerate) minimum at the origin, then the origin is asymptotically stable in the closed-loop system with any $\widehat{W}$-valued dissipation $\widehat{u}$ for $(\widehat{L}, \widehat{G}, \widehat{W})$ with $\mathrm{rank}\, \widehat{u} = \dim \widehat{W}$.

We apply this theorem and Theorem 4.4 to obtain a sufficient condition for asymptotic stabilizability of controlled Lagrangian systems of one degree of under-actuation.

Theorem 4.10. Consider a controlled Lagrangian system $(L, 0, W)$ of one degree of under-actuation for which the origin $(q, \dot{q}) = (0, 0)$ is an equilibrium point. Suppose that it satisfies condition C around the origin and its linearization $(L^l, 0, W^l)$ at the origin satisfies condition (a) or (b) given in Theorem 3.4. Then, there exists a controlled Lagrangian system $(\widehat{L}, \widehat{G}, \widehat{W})$ with $\widehat{G} \in \Gamma(\mathrm{GS}(Q))$ that is feedback equivalent to $(L, 0, W)$ and whose energy $\widehat{E} = \frac{1}{2}\widehat{m}(\dot{q}, \dot{q}) + \widehat{V}(q)$ has a non-degenerate minimum at the origin, such that the origin is asymptotically stable in the closed-loop system with any $\widehat{W}$-valued dissipation $\widehat{u}$ for $(\widehat{L}, \widehat{G}, \widehat{W})$ satisfying $\operatorname{rank} \widehat{u} = \dim \widehat{W}$.

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